

## SUBORDINATION CRITERIA FOR STARLIKENESS AND CONVEXITY

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Received 21 May 2002

For functions  $p$  analytic in the open unit disc  $U = \{z : |z| < 1\}$  with the normalization  $p(0) = 1$ , we consider the families  $\mathcal{P}[A, -1]$ ,  $-1 < A \leq 1$ , consisting of  $p$  such that  $p(z)$  is subordinate to  $(1 + Az)/(1 - z)$  in  $U$  and  $\mathcal{P}(1, b)$ ,  $b > 0$ , consisting of  $p$ , which have the disc formulation  $|p - 1| < b$  in  $U$ . We then introduce subordination criteria for the choice of  $p(z) = zf'(z)/f(z)$ , where  $f$  is analytic in  $U$  and normalized by  $f(0) = f'(0) - 1 = 0$ . We also obtain starlikeness and convexity conditions for such functions  $f$  and consequently extend and, in some cases, improve the corresponding previously known results.

2000 Mathematics Subject Classification: 30C45, 30C50.

**1. Introduction.** Let  $\mathcal{A}$  denote the class of functions that are analytic in the open unit disc  $U = \{z : |z| < 1\}$ . In the sequel, we assume that  $p$  in  $\mathcal{A}$  is normalized by  $p(0) = 1$  and  $f$  in  $\mathcal{A}$  is normalized by  $f(0) = f'(0) - 1 = 0$ .

For  $0 < b \leq a$ , the function  $p \in \mathcal{A}$  is said to be in  $\mathcal{P}(a, b)$  if and only if

$$|p(z) - a| < b, \quad z \in U. \quad (1.1)$$

Without loss of generality, we omit the trivial case  $p(z) = 1$  and assume that  $|1 - a| < b$ .

For  $-1 \leq B < A \leq 1$ , the function  $p \in \mathcal{A}$  is said to be in  $\mathcal{P}[A, B]$  if and only if

$$p(z) < \frac{1 + Az}{1 + Bz}, \quad z \in U. \quad (1.2)$$

Here the symbol “ $<$ ” stands for *subordination*. For the functions  $f$  and  $g$  in  $\mathcal{A}$ , we say that  $f$  is subordinate to  $g$  in  $U$ , denoted by  $f < g$ , if there exists a Schwarz function  $w$  in  $\mathcal{A}$  with  $|w(z)| < 1$  and  $w(0) = 0$  such that  $f(z) = g(w(z))$  in  $U$ .

For  $0 < b \leq a$ , there is a correspondence between  $\mathcal{P}(a, b)$  and  $\mathcal{P}[A, B]$ ; namely,

$$\mathcal{P}(a, b) \equiv \mathcal{P}\left[\frac{b^2 - a^2 + a}{b}, \frac{1 - a}{b}\right]. \quad (1.3)$$

Two subclasses that have been studied extensively (e.g., see [2, 10]) are  $\mathcal{P}(1, b)$  and  $\mathcal{P}[A, -1]$ . The class  $\mathcal{P}(1, b)$ , which is defined using the disc formulation, has an alternative characterization in terms of subordination, where

$$p \in \mathcal{P}(1, b) \iff p(z) < 1 + bz. \tag{1.4}$$

In this paper, we study the subordination criteria for functions  $p(z) = zf'(z)/f(z)$  in  $\mathcal{A}$ , where  $f \in \mathcal{A}$ . We also obtain starlikeness and convexity conditions for such functions  $f \in \mathcal{A}$  and consequently extend and, in some cases, improve the corresponding previously known results. The significance of the above choice for  $p$  is evident if we recall that  $f \in \mathcal{A}$  is said to be starlike of order  $\alpha$ ,  $0 \leq \alpha \leq 1$  if  $(zf'(z)/f(z)) \in \mathcal{P}(1, 1 - \alpha)$ , and  $f \in \mathcal{A}$  is said to be convex of order  $\alpha$ ,  $0 \leq \alpha \leq 1$  if  $(1 + zf''/f') \in \mathcal{P}(1, 1 - \alpha)$ . Finally, we note that all functions, starlike or convex, of order  $\alpha_2$  are, respectively, starlike or convex of order  $\alpha_1$  if  $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ .

**2. Main results.** First, we introduce a subordination criterion for  $p(z) = zf'(z)/f(z)$  in  $\mathcal{P}[A, -1]$ . To prove our first theorem, we need the following celebrated result, which is due to Miller and Mocanu [3].

**LEMMA 2.1.** *Let  $q$  be univalent in the unit disc  $U$  and let  $\phi$  and  $\psi$  be analytic in a domain  $\mathcal{C}$  containing  $q(U)$  with  $\psi(\omega) \neq 0$  for  $\omega \in q(U)$ . Set  $Q(z) = zq'(z)\psi(q(z))$  and  $h(z) = \phi(q(z)) + Q(z)$ . Also, suppose that  $Q$  is starlike univalent in  $U$  and  $\Re(zh'(z)/Q(z)) = \Re[\phi'(q(z))/\psi(q(z)) + zQ'(z)/Q(z)] > 0$  in  $U$ . If  $p$  is analytic in  $U$ ,  $p(0) = q(0)$ ,  $q(U) \in \mathcal{C}$ , and  $\phi(p(z)) + zp'(z)\psi(p(z)) < h(z)$ , then  $p < q$ , and  $q$  is the best dominant of the subordination.*

**THEOREM 2.2.** *Let  $f$  in  $\mathcal{A}$  be so that  $f(z)/z \neq 0$  in  $U$ . Also, let  $\alpha > 0$ ,  $|\beta| \leq 1$ , and  $-1 < A \leq 1$  be so that*

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{1}{2}(1 + \beta)(1 - A) + \frac{(1 - \beta)(1 - A)}{2(1 + A)} \geq 0. \tag{2.1}$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) < h(z), \tag{2.2}$$

where

$$h(z) = \left(\frac{1 + Az}{1 - z}\right)^{\beta-1} \left[ (1 - \alpha) \frac{1 + Az}{1 - z} + \frac{\alpha(1 + Az)^2 + \alpha(1 + A)z}{(1 - z)^2} \right], \tag{2.3}$$

then

$$\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 - z}. \tag{2.4}$$

**PROOF.** Setting  $zf'(z)/f(z) = p(z)$ , condition (2.2) can be written as

$$(p(z))^\beta [(1 - \alpha) + \alpha p(z)] + \alpha zp'(z)^{\beta-1} < h(z). \tag{2.5}$$

For  $q(z) = (1 + Az)/(1 - z)$ , it is clear that  $q$  is univalent in  $U$  and  $q(U)$  is the region  $\Re z > (1 - A)/2$ . Also, for  $\psi(z) = \alpha z^{\beta-1}$  and  $\phi(z) = z^\beta(1 - \alpha + \alpha z)$ , we observe that  $\psi$  and  $\phi$  satisfy the conditions required by Lemma 2.1. Therefore,

$$\begin{aligned} Q(z) &= zq'(z)\psi(q(z)) = \frac{\alpha(1+A)z(1+Az)^{\beta-1}}{(1-z)^{\beta+1}}, \\ h(z) &= \phi(q(z)) + Q(z) = \left(\frac{1+Az}{1-z}\right)^\beta \left[1 - \alpha + \alpha \frac{1+Az}{1-z}\right] \\ &\quad + \frac{\alpha(1+A)z(1+Az)^{\beta-1}}{(1-z)^{\beta+1}}. \end{aligned} \tag{2.6}$$

Now, the above assumptions yield

$$\begin{aligned} \Re \frac{zQ'(z)}{Q(z)} &= \Re \left[ 1 + (\beta - 1) \frac{Az}{1+Az} + (1 + \beta) \frac{z}{1-z} \right] \\ &> -1 + (1 - \beta) \frac{1}{1+|A|} + (1 + \beta) \frac{1}{2} \\ &= \frac{(1 - |A|)(1 - \beta)}{2(1 + |A|)} > 0, \\ \Re \frac{zh'(z)}{Q(z)} &= \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta) \Re \left( \frac{1+Az}{1-z} \right) + \Re \frac{zQ'(z)}{Q(z)} \\ &> \frac{\beta(1 - \alpha)}{\alpha} + \frac{1}{2}(1 + \beta)(1 - |A|) + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0. \end{aligned} \tag{2.7}$$

This completes the proof since all the conditions required by Lemma 2.1 are satisfied. □

We remark that for  $\beta = A = 0$  and  $\alpha = 1$ , the above theorem reinstates the fact that every convex function is starlike of order  $1/2$ . Also, for  $\beta = A = 1$ , we obtain [8, Theorem 1], and for  $\alpha = \beta = 1$  and  $A = 0$  we obtain [8, Theorem 3]. Furthermore, letting  $\alpha = -\beta = 1$  in the above theorem, yields the following corollary.

**COROLLARY 2.3.** *Let  $f \in \mathcal{A}$  and  $f(z)/z \neq 0$  in  $U$ . If  $-1 < |A| \leq 1$  and*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{(1+A)z}{(1+Az)^2}, \tag{2.8}$$

then

$$\frac{zf'(z)}{f(z)} < \frac{1+Az}{1-z}. \tag{2.9}$$

**REMARK 2.4.** The function  $h(z) = 1 + (1 + A)z/(1 + Az)^2$  has interesting mapping properties. Note that  $h$  takes real values for real values of  $z$  with  $h(0) = 1$  and  $h(U)$  is symmetric with respect to the real axis. Now, for  $D = \{h(e^{i\theta}) : 0 \leq \theta < 2\pi\}$  and  $d = (1, 0)$ , observe that

$$\text{mindist}(D, d) = \frac{1}{1 + A}. \tag{2.10}$$

Consequently,  $h$  maps the unit circle onto the region, which properly contains the region  $|\omega - 1| < (1 + A)/(1 - A)^2$ . This is an extension to [9, Theorem 1] which does not extend as for the sharpness. (Also see Obradović and Tuneski [7].)

Our next theorem is on the subordination criterion for  $zf'(z)/f(z) \in \mathcal{P}(1, b)$ .

**THEOREM 2.5.** Let  $f \in \mathcal{A}$  and  $f(z)/z \neq 0$  in  $U$ . Also, let  $\alpha > 0$ ,  $|\beta| \leq 1$ , and  $0 < b \leq 1$  be so that  $2\beta + \alpha(1 - \beta) + (1 - b)(1 + b + b\beta) \geq 0$ . If

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec \frac{1 + (1 + 2\alpha)bz + \alpha b^2 z^2}{(1 + bz)^{1-\beta}} = h(z), \tag{2.11}$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + bz. \tag{2.12}$$

**PROOF.** Setting  $p(z) = zf'(z)/f(z)$ , condition (2.11) may be written as

$$(p(z))^\beta [(1 - \alpha) + \alpha p(z)] + \alpha zp'(z)(p(z))^{\beta-1} \prec h(z). \tag{2.13}$$

Here, we need once again to make use of Lemma 2.1. Set  $q(z) = 1 + bz$ ,  $\psi(z) = \alpha z^{\beta-1}$ , and  $\phi(z) = z^\beta(1 - \alpha + \alpha z)$ . We observe that  $q$  is univalent and  $q(U)$  is a region, so that its boundary is the circle with radius  $b$  and center at  $(1, 0)$ . Using an argument similar to that used to prove Theorem 2.2, we write  $Q(z) = \alpha bz(1 + bz)^{\beta-1}$  and  $h(z) = \phi(q(z)) + Q(z)$ . Therefore,

$$\begin{aligned} \Re \frac{zQ'(z)}{Q(z)} &= \beta + (1 - \beta)\Re \frac{1}{1 + bz} > \beta + \frac{1 - \beta}{1 + b} = \frac{1 + \beta b}{1 + b} \geq 0, \\ \Re \frac{zh'(z)}{Q(z)} &= \Re \left[ \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta)(1 + bz) \right] + \Re \frac{zQ'(z)}{Q(z)} \\ &> \frac{\beta(1 - \alpha)}{\alpha} + (1 + \alpha)(1 - b) + \frac{1 + \beta b}{1 + b} \geq 0. \end{aligned} \tag{2.14}$$

Thus, the proof is complete since all the conditions required by Lemma 2.1 are satisfied. □

By letting  $\beta = 1$  in Theorem 2.5, we obtain the following corollary, which is an improvement in a result obtained in [6]. For an alternative proof of the following corollary, see Mocanu and Oros [4]. Another generalization of this result is contained in Mocanu and Oros [5].

**COROLLARY 2.6.** *Let  $f \in \mathcal{A}$  and  $f(z)/z \neq 0$  in  $U$ . Also, let  $\alpha > 0$  and  $0 < b \leq 1$ . If*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < 1 + (1 + 2\alpha)bz + \alpha b^2 z^2, \tag{2.15}$$

then

$$\frac{zf'(z)}{f(z)} < 1 + bz. \tag{2.16}$$

Recalling Remark 2.4 after Corollary 2.3 for  $h(z) = 1 + (1 + 2\alpha)bz + \alpha b^2 z^2$ , observe that  $h$  takes real values for real values of  $z$  with  $h(0) = 1$  and  $h(U)$  is symmetric with respect to the real axis. Now, for  $D = \{h(e^{i\theta}) : 0 \leq \theta < 2\pi\}$  and  $d = (1, 0)$ , it can be shown that

$$\begin{aligned} \text{mindist}(D, d) &= (1 + 2\alpha)b - \alpha b^2, \\ \text{Maxdist}(D, d) &= (1 + 2\alpha)b + \alpha b^2. \end{aligned} \tag{2.17}$$

Therefore,  $h$  maps the unit disc  $U$  onto a region, which properly contains the region  $\{z : |z - 1| < (1 + \alpha)b\}$ . This improves [6, Theorem 1] obtained by Obradović et al.

For  $0 \leq \rho < 1$ , define  $\Omega = \{w : |w - 1| \leq 1 - 2\rho + \Re w\}$  and let  $\mathcal{F}(\rho)$  consist of functions  $f \in \mathcal{A}$  satisfying the condition  $zf'/f \in \Omega$ . Note that the class  $\mathcal{F}(\rho)$  consists of starlike functions. Also, we let  $\mathcal{H}(\rho)$  consist of convex functions  $f \in \mathcal{A}$  for which  $zf' \in \mathcal{F}(\rho)$ .

For  $0 \leq \rho < \beta \leq 1$ , let  $\mathcal{M}_\beta(\rho)$  be the largest number for which the disc  $\mathcal{D}(\beta, \mathcal{M}_\beta(\rho)) = \{w : |w - \beta| < \mathcal{M}_\beta(\rho)\}$  lies inside the region  $\Omega$ . A direct calculation yields

$$\mathcal{M}_\beta(\rho) = \begin{cases} \beta - \rho & \text{if } \rho < \beta < 2 - \rho, \\ 2\sqrt{(1 - \rho)(\beta - 1)} & \text{if } \beta \geq 2 - \rho, \end{cases} \tag{2.18}$$

Therefore, the disc contains the point 1 for

$$\frac{1 + \rho}{2} < \beta < (2 - \rho) + \sqrt{\frac{\rho^2 - \rho + 5}{2}} \tag{2.19}$$

and we have justified the following lemma.

**LEMMA 2.7.** *Let  $f \in \mathcal{A}$  and  $(1 + \rho)/2 < \beta < (2 - \rho) + \sqrt{\rho^2 - \rho + 5/2}$ . If*

$$\left| \frac{zf'(z)}{f(z)} - \beta \right| < \mathcal{M}_\beta(\rho), \tag{2.20}$$

then  $f \in \mathcal{F}(\rho)$ .

The above lemma in conjunction with Corollary 2.6 yields the following theorem.

**THEOREM 2.8.** *Let  $f \in \mathcal{A}$  and  $f(z)/z \neq 0$  in  $U$ . Also, let  $\alpha > 0$  and  $0 < b \leq 1$ . If*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < 1 + (1 + 2\alpha)bz + \alpha b^2 z^2, \tag{2.21}$$

then  $f \in \mathcal{F}(1 - b)$ .

With some restrictions on  $\rho$  and  $b$ , we show that we can do even better than the above theorem in terms of classification of the function  $f$ . First, we need the following result due to Jack [1].

**LEMMA 2.9.** *Let  $\omega$  be a nonconstant analytic function in  $U$  with  $\omega(0) = 0$ . If  $|\omega|$  attains its maximum value on the circle  $|z| = r$  at some point  $z_0$ , then  $z_0 \omega'(z_0) = k\omega(z_0)$ , where  $k \geq 1$ .*

**THEOREM 2.10.** *For  $\alpha > 0$ , let  $\rho = (\alpha - b(2 + 3\alpha + \alpha b))/\alpha(1 - b)$  and  $0 < b \leq -(3 + 2\alpha) + \sqrt{9 + 12\alpha + 8\alpha^2}/2\alpha$ . If  $f \in \mathcal{A}$ ,  $f(z)/z \neq 0$ , and*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} < 1 + (1 + 2\alpha)bz + \alpha b^2 z^2, \tag{2.22}$$

then  $f \in \mathcal{K}(\rho)$ .

**PROOF.** Setting  $p(z) = zf'(z)/f(z)$  and  $\omega(z) = \alpha z f''(z)/f'(z)$ , condition (2.22) may be written as

$$p(z)(1 + \omega(z)) < 1 + (1 + 2\alpha)bz + \alpha b^2 z^2 \tag{2.23}$$

or

$$|p(z)(1 + \omega(z)) - 1| < (1 + 2\alpha)b + \alpha b^2, \quad z \in U. \tag{2.24}$$

Therefore,  $|p(z) - 1| < b$  and so, by Corollary 2.6, we only need to show that

$$|\omega(z)| < \frac{2(1 + \alpha)b + \alpha b^2}{1 - b} = T. \tag{2.25}$$

Define  $g(z) = \omega(z)/T$ . Since  $g(0) = 0$  and  $g$  is analytic in  $U$ , it suffices to show that  $|g| < 1$  in  $U$ . On the contrary, suppose that there exists  $z_0 \in U$ , so that  $|g(z_0)| = 1$ . Then, by Lemma 2.9, there exists  $k \geq 1$ , so that  $z_0 g'(z_0) = kg(z_0)$ . Consequently,

$$\begin{aligned} |p(z_0)(1 + \omega(z_0)) - 1| &= |p(z_0)(1 + Tg(z_0)) - 1| \\ &= |(p(z_0) - 1)(1 + Tg(z_0)) + Tg(z_0)| \\ &\geq T|g(z_0)| - b(1 + T|g(z_0)|) \\ &= (1 + 2\alpha)T + \alpha T^2. \end{aligned} \tag{2.26}$$

This is a contradiction to the required condition (2.24), and so the proof is complete. □

As a corollary to the above theorem we obtain the following corollary.

**COROLLARY 2.11.** *Let  $f \in \mathcal{A}$  be so that  $f(z)/z \neq 0$  and*

$$\frac{zf'(z)}{f(z)} + \frac{z^2f''(z)}{f(z)} < 1 + 0.5777z + 0.037z^2, \quad z \in U. \quad (2.27)$$

*Then,  $|zf''(z)/f'(z)| < 0.99987$ , and so  $f$  is convex.*

We note that our [Corollary 2.11](#) is an improvement to [[6](#), Corollary 2(b)].

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