

# Segre decomposition of spacetimes

José M. M. Senovilla\* and Raül Vera\*

Departament de Física Fonamental, Universitat de Barcelona,  
Diagonal 647, E-08028 Barcelona, Catalonia.

February 7, 2008

## Abstract

Following a recent work in which it is shown that a spacetime admitting Lie-group actions may be disjointly decomposed into a closed subset with no interior plus a dense finite union of open sets in each of which the character and dimension of the group orbits as well as the Petrov type are constant, the aim of this work is to include the Segre types of the Ricci tensor (and hence of the Einstein tensor) into the decomposition. We also show how this type of decomposition can be carried out for any type of property of the spacetime depending on the existence of a continuous endomorphism.

## 1 Introduction

In a recent work [1] it has been shown that a spacetime  $\mathcal{M}$  admitting Lie-group actions can be decomposed into a finite disjoint union of open subsets of  $\mathcal{M}$  in each of which the orbit type, the dimension of the finite Lie algebra and the Petrov type are constant, together with a closed subset that has no interior. A first theorem deals with the Petrov type, while a second one treats the Lie-group admitted by  $\mathcal{M}$ . Then, both theorems are combined to give rise to a more refined decomposition.

The motivation in [1] was to ascertain whether special solutions solve the local problem ‘almost everywhere’. By special solutions it was meant those exact solutions with a prescribed form of the energy-momentum tensor to which some other additional simplifying assumptions involving the Petrov types and/or existence of symmetries were imposed. The answer is affirmative, because the additional assumptions (Petrov and orbit types) hold constantly into open subsets that almost cover  $\mathcal{M}$ , leaving aside only the points belonging to the closed subset without interior of the decomposition, which is of zero measure.

---

\*Also at Laboratori de Física Matemàtica, Institut d’Estudis Catalans, Catalonia.

In the same sense, and with the idea of completing the above results, one can wonder whether or not it is possible to find open neighbourhoods in which the algebraic type of the energy-momentum tensor [2, 3] does not change, and further, whether or not the prescriptions on the Segre types [4] solve this local problem ‘almost everywhere’. As in the previous cases shown in [1], an affirmative answer would assure us that choosing *a priori* any algebraic type of the energy-momentum tensor for a spacetime is licit, because given a point with a particular Segre type there would almost everywhere be an open neighbourhood (any connected part of which can be taken as our spacetime) in which the algebraic type of the energy-momentum tensor is constant<sup>1</sup>. Thus, the main aim of the present work consists in adding the Segre types of the Ricci (and the energy-momentum) tensor to the decompositions shown in [1]. To this end we will follow [1] in a very close way, but making extensive use of results and notations of the related papers [5]-[7]. Even though important physical reasons may lead us to consider only those algebraic types of the energy-momentum tensor which satisfy the so-called energy conditions, see for instance [8, 9], no imposition of such kind will be made in the present work mainly because this sort of assumption does not affect the final result and can be easily added afterwards. Thus, we have preferred to keep the mathematical generality.

The importance of these results is not only theoretical, because spacetimes with varying Segre type do appear many times in practice. For example, although the standard approach for finding exact solutions of the Einstein field equations usually starts with the prescription of the algebraic form of the energy-momentum tensor, the resulting spacetimes can be extendible to larger ones such that the extension must be performed through regions that cannot keep the same algebraic type (see e.g. section 3 in [10]). More importantly, many other simplifying assumptions can be used as the starting point (such as the existence of groups of motions or generalized symmetries, Kerr-Schild transformations, specific Petrov types, etc.) and they may lead to completely *explicit* spacetimes having *subregions* with an appropriate or desired energy-momentum tensor (vacuum, perfect-fluid, electromagnetic), but such that, on the whole spacetime, the Segre type of the energy-momentum tensor varies from point to point. Many illustrative examples can be found in [10]-[19], and some considerations on an explicit case concerning  $G_2$  spacetimes will also be given here in Section 4.

The plan of the paper is as follows. Section 2 is devoted to a brief review on the Segre types and related matters concerning their characterizations. These will be used in Section 3 in order to proof the decomposition of any spacetime according to the algebraic type of the Ricci tensor. Finally, in Section 4 some conclusions and examples are presented. In particular, we devise a completely general method (similar to that used in Section 3) which allow to decompose any spacetime in a similar manner according to the different algebraic types of any continuous endomorphism which can act on tangent vector spaces of the spacetime. These general decompositions may be of some importance if the given endomorphism is related to some relevant properties

---

<sup>1</sup>Here we have used the term *a priori* because the imposition of further specializations may restrict the existence of such open sets.

of mathematical or physical interest.

The spacetime  $\mathcal{M}$  is assumed to be of class  $C^r$  with  $r \geq 2$  in order to assure a continuous Ricci tensor. If the energy-momentum tensor were not continuous, the results on this paper would no longer hold, and the corresponding generalized results should be obtained by using the adequate junction conditions and their consequences. The interior of a subset  $A \subset \mathcal{M}$  will be denoted by  $A^\circ$ .

## 2 Segre types

The algebraic classification of a second-order symmetric tensor [3] like the Ricci tensor can be obtained by various methods, and we refer to the introduction of [7] for a complete set of references on this topic and other classification schemes, and to [9, 20] for reviews. We simply note in passing that the classification of the energy-momentum tensor is equivalent to that of the Ricci tensor via the Einstein field equations. In fact, it is more convenient to work on the trace-free Ricci tensor, as the Segre type for both the Ricci tensor and its trace-free part coincide. Considering then the traceless part of the Ricci tensor as a linear map on a four-dimensional real vector space, its algebraic classification consists in the resolution of the eigenvalue problem. The different types and their corresponding subtypes will be denoted by the usual Segre notation [4, 9]. We shall also use the notation introduced in [6] and [7] for the *characteristic polynomial* (CP), the *minimal polynomial* (MP) and the invariants used in the determination of the form of the CP (see [7]): given the trace-free part of the Ricci tensor  $N$  by

$$N^\alpha_\beta \equiv R^\alpha_\beta - \frac{1}{4}\delta^\alpha_\beta R, \quad (1)$$

the CP of  $N$  is  $P(\lambda) \equiv \det(N^\alpha_\beta - \lambda\delta^\alpha_\beta)$  and takes the form

$$P(\lambda) = \lambda^4 - \frac{a}{2}\lambda^2 - \frac{b}{3}\lambda + \frac{1}{4}\left(\frac{a^2}{2} - c\right), \quad (2)$$

where  $a \equiv \text{tr}N^2$ ,  $b \equiv \text{tr}N^3$  and  $c \equiv \text{tr}N^4$ . In general,  $P(\lambda)$  factorizes as

$$(\lambda - \lambda_1)^{d_1}(\lambda - \lambda_2)^{d_2} \dots (\lambda - \lambda_r)^{d_r}, \quad (3)$$

where  $\lambda_i \in \mathbb{C}$  and  $d_i$  ( $i = 1, \dots, r$ ) are the eigenvalues and their corresponding degeneracies respectively. The MP of  $N$  is given by the lowest-degree monic matrix polynomial in  $N^\alpha_\beta$  vanishing identically, which is unique and can always be factorized as

$$(N - \lambda_1\mathbb{I})^{m_1}(N - \lambda_2\mathbb{I})^{m_2} \dots (N - \lambda_r\mathbb{I})^{m_r}, \quad (4)$$

where  $m_i(\leq d_i)$  ( $i = 1, \dots, r$ ) are the maximal dimensions of the Jordan matrices for each eigenvalue. The CP is denoted by the list of the degeneracies  $\{d_1 d_2 \dots d_r\}$ , using in particular the notation  $\{z\bar{z} \dots\}$  when the pair of complex conjugate eigenvalues appear (real second-order symmetric tensors in a Lorentzian spacetime have at most

a pair of complex conjugate eigenvalues [3]). The MP is indicated through the list  $\|m_1 m_2 \cdots m_r\|$ , and finally, the standard Segre symbols are used to denote the Segre types as can be seen in Table 1 (see [6]).

Following [7] one can define the invariants

$$I_1 \equiv I_3^3 - [3aI_3 + 4(3b^2 - a^3)]^2, \quad I_2 \equiv 2a - \sqrt{|I_3|}, \quad I_3 \equiv 7a^2 - 12c, \quad (5)$$

whose signs will determine the form of the CP [7], (see also [3, 21, 22, 23] for other sets of invariants), according to the values shown in Table 1. Some sign combinations cannot occur, and this implies, in fact, the existence of at most a pair of complex conjugate eigenvalues.

In what follows, the set of points of the manifold  $\mathcal{M}$  in which the Ricci tensor has a specific Segre type will be called ‘‘Segre set’’ and denoted by the corresponding Segre symbol. For instance, the Segre set  $[1, (111)]$  will denote the set of points in  $\mathcal{M}$  where the Ricci tensor has that Segre type. Of course, all the Segre sets constitute a collection of disjoint sets whose union covers the whole spacetime. Furthermore, the notation for the CP will be also used to indicate the union of all the Segre sets lying in the same column of Table 1. For instance, we have that  $\{22\} = [2(11)] \cup [(1, 1)(11)]$ .

Another important feature that we will need later is the character of the eigenspaces associated to the simple and the double eigenvalues of  $N$ . We will again use the results in [7] where there is a section treating the covariant determination of the character of these eigenspaces by means of a simple straightforward method which will be very useful for our purposes, and which allow us to avoid the more involved equivalent splittings such as that in [21]. Thus, following [7], the main object to deal with the determination of the character of the eigenspaces is the so-called *eigntensor* of  $N$  corresponding to the eigenvalue  $\lambda$ , denoted by  $N_\lambda$ , which is merely the MP of  $N$  without a  $\lambda$ -factor, that is to say, the MP of  $N$  at a given point can be decomposed in terms of the  $\lambda$ -eigntensor  $N_\lambda$  as follows :  $(N - \lambda\mathbb{I})N_\lambda$ . The eigntensor for simple eigenvalues is, in fact, a projector onto the corresponding one-dimensional eigenspace, that is  $N_\lambda(\vec{w}) = \alpha_{\vec{w}}\vec{v}_\lambda$ , being  $\vec{w}$  an arbitrary vector field and  $\vec{v}_\lambda$  the eigndirection corresponding to  $\lambda$ . In addition, it can be shown [7] that

$$(\vec{v}_\lambda \cdot \vec{v}_\lambda) = \frac{1}{\alpha_{\vec{w}}^2} (\text{tr} N_\lambda) N_\lambda(\vec{w}, \vec{w}), \quad (6)$$

whenever  $\vec{w}$  is not orthogonal to  $\vec{v}_\lambda$ , which ensures that  $\alpha_{\vec{w}} \neq 0$ , and where we have used the notation  $N_\lambda(\vec{w}, \vec{w}) \equiv (N_\lambda(\vec{w}) \cdot \vec{w})$ . When  $\lambda$  is not a simple eigenvalue it is necessary to consider the following double 2-form constructed from the eigntensor  $N_\lambda$ :  $(\mathcal{N}_\lambda)_{\alpha\beta\mu\nu} \equiv (N_\lambda)_{\alpha\mu} (N_\lambda)_{\beta\nu} - (N_\lambda)_{\alpha\nu} (N_\lambda)_{\beta\mu}$ . Let us denote by  $E_\lambda$  the 2-eigenspace associated with  $\lambda$ , and let  $\vec{v}$  and  $\vec{w}$  be two linearly independent vectors in  $E_\lambda$ . Analogously to the previous case,  $\mathcal{N}_\lambda$  projects any 2-form  $F$  to  $E_\lambda$  as follows  $(\mathcal{N}_\lambda)_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu} = \alpha_F F_{\alpha\beta}^{(\lambda)}$ , where  $F_{\alpha\beta}^{(\lambda)} \propto v_{[\alpha} w_{\beta]}$ . If  $\alpha_F \neq 0$ , it can be shown [7] that

$$\mathcal{G}(F^{(\lambda)}, F^{(\lambda)}) = \frac{1}{\alpha_F^2} (\text{tr} N_\lambda)^2 \mathcal{N}_\lambda(F, F), \quad (7)$$

CP $\rightarrow$	$\{1111\}$	$\{z\bar{z}11\}$	$\{z\bar{z}2\}$	$\{211\}$	$\{31\}$	$\{22\}$	$\{4\}$
INV $\rightarrow$	$I_1 > 0$ $I_2 > 0$	$I_1 < 0$	$I_1 = 0$ $I_2 < 0$ $I_3 > 0$	$I_1 = 0$ $I_2 > 0$ $I_3 > 0$	$I_1 = 0$ $I_2 > 0$ $I_3 = 0$	$I_1 = 0$ $I_2 = 0$ $I_3 > 0$	$I_1 = 0$ $I_2 = 0$ $I_3 = 0$
MP $\downarrow$							
$\ 1111\ $	$[1,111]$ rank 4,3	$[z\bar{z}11]$ rank 4,3					
$\ 211\ $				$[211]$ rank 4,3			
$\ 31\ $					$[31]$ rank 4		
$\ 111\ $			$[z\bar{z}(11)]$ rank 4,2	$[1,1(11)]$ rank 4,3,2 $[(1,1)11]$ rank 4,3,2			
$\ 21\ $					$[(21)1]$ rank 4	$[2(11)]$ rank 4,1	
$\ 3\ $							$[(31)]$ rank 2
$\ 11\ $					$[(1,1)1]$ rank 4 $[1,(111)]$ rank 4	$[(1,1)(11)]$ rank 4	
$\ 2\ $							$[(211)]$ rank 1
$\ 1\ $							$[(1,111)]$ rank 0

Table 1: Segre types of the traceless part of the Ricci tensor. The columns indicate the different characteristic polynomial (CP) forms together with their corresponding characterization through the signs of the invariants (INV) introduced in (5), while the rows share the same minimal polynomials (MP).

where  $\mathcal{G}_{\alpha\beta\mu\nu} = g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}$  is the usual bivector metric so that  $\mathcal{G}(F^{(\lambda)}, F^{(\lambda)})$  is positive (resp. negative) when  $E_\lambda$  is spacelike (resp. timelike).

### 3 The decomposition

Let us introduce first some preliminary results. Regarding the continuity and differentiability of the eigenvalues of mixed second-order tensors we have that (see e.g. [5, 24] for proofs):

(i) The simple roots of a polynomial depend differentiably on the polynomial coefficients.

Thus, the simple eigenvalues ( $\lambda_i$ ;  $d_i = 1$ ) of  $N$  at a given point  $p$  give rise to differentiable functions (continuous if  $N$  is only continuous) in some open neighbourhood  $V_p$  of  $p$ , such that  $\lambda_i(q)$  are the simple eigenvalues of  $N$  at every  $q \in V_p$ .

(ii) If a continuous mixed second-order tensor  $T$  has the same form of its CP in an open set  $U \subseteq \mathcal{M}$ , then there is an open neighbourhood  $V_p \subset U$  for each  $p \in U$  and continuous maps  $\lambda_1, \dots, \lambda_r : V_p \rightarrow \mathbb{C}$  giving the eigenvalues of  $T$  at each point in  $V_p$ .

By the ‘same form of the CP’ we mean the same number of different roots and the same set of degeneracies ( $d_r$ ), that is, the same notation  $\{\dots\}$ . In [5] statement (ii) assumed ‘same algebraic type’ (that is, same Segre type), but in fact the proof given there does not need this restriction. On the other hand, the continuity of the eigenvalues is sufficient for what follows, so we have preferred not to demand extra differentiability assumptions on  $N$ .

The continuity of  $N$  has some immediate consequences. First of all, consider the subset  $[1, 111] \subseteq \mathcal{M}$ , whose points  $p \in [1, 111]$  are invariantly determined by  $I_1(p) > 0$  (see Table 1). Since the invariants are continuous functions, it follows that  $[1, 111]^\circ = [1, 111]$ . The same reasoning works for  $[z\bar{z}11]$ , where  $I_1 < 0$ . Thus,  $[1, 111]$  and  $[z\bar{z}11]$  are two open subsets of  $\mathcal{M}$ .

Another preliminary result comes from the rank theorem applied to  $N$ . This theorem [24] states that the rank of a continuous linear map in some open neighbourhood of a point  $p$  is equal or greater than it is at  $p$ . The possible ranks that  $N$  can have for each Segre type are also shown in Table 1. Let us define

$$Y \equiv [1, 111] \cup [z\bar{z}11] \quad \Longrightarrow \quad Y = Y^\circ,$$

$$Z \equiv \{z\bar{z}2\} \cup \{211\} \cup \{31\} \cup \{22\} \cup [(31)] \cup [(211)].$$

Since the ranks in the set  $Y \cup Z$  are greater than zero, the rank theorem implies that this union constitutes an open set. We are now ready to prove the following intermediate step.

**Proposition 1:** *Any spacetime  $\mathcal{M}$  can be decomposed in the following union of disjoint subsets*

$$\mathcal{M} = [1, 111] \cup \left( [1, 1(11)] \cup [(1, 1)11] \right)^\circ \cup [(1, 1)(11)]^\circ \cup \left( [(1, 1)1] \cup [1, (11)] \right)^\circ$$

$$\begin{aligned} & \cup[(1, 111)]^\circ \cup [211]^\circ \cup [2(11)]^\circ \cup [(21)1]^\circ \cup [(211)]^\circ \\ & \cup [31]^\circ \cup [(31)]^\circ \cup [z\bar{z}11] \cup [z\bar{z}(11)]^\circ \cup X. \end{aligned} \tag{8}$$

with  $X^\circ = \emptyset$ , where  $X$  is the necessarily closed subset defined from the decomposition.

*Proof:* From the disjointness of the decomposition we know that  $X \cap [1, 111] = X \cap [z\bar{z}11] = X \cap Y = \emptyset$ , so that  $I_1(p) = 0$  for every  $p \in X$ :

$$I_1(X) = 0.$$

The only thing we have to prove is that the set  $X$ , called the *remainder*, has no interior. Suppose then, on the contrary, that there existed an open subset  $W$  such that  $\emptyset \neq W \subseteq X$ . From the definition of the remainder and the above results, and since the union  $Y \cup Z$  is open, we would have that  $W \cap (Y \cup Z) = W \cap Z$  is open as well. If the open subset  $W \cap Z$  were non-empty, several possibilities would appear.

Possibility 1: Suppose that  $W \cap \{z\bar{z}2\} = W \cap Z \cap \{z\bar{z}2\} \neq \emptyset$ . Let  $p$  be a point in the intersection. As in particular  $p \in \{z\bar{z}2\}$ , this would mean that  $I_1(p) = 0, I_2(p) < 0$  (see Table 1), and therefore, since the invariants are continuous functions, there would exist an open neighbourhood  $U$  of  $p$  such that  $I_2(U) < 0$ . Furthermore, we could choose  $U$  such that  $U \subset W \cap Z$  because  $p$  belongs also to the open set  $W \cap Z$ . This would also imply  $I_1(U) = 0$ . Thus, we would have that  $U \subset W \cap \{z\bar{z}2\}$ , which immediately implies  $W \cap \{z\bar{z}2\}^\circ \neq \emptyset$ , in contradiction with the disjointness in the definition of  $X$ . Thus, possibility 1 is not valid and

$$W \cap \{z\bar{z}2\} = \emptyset.$$

Possibility 2: Suppose then that  $W \cap \{211\} = W \cap Z \cap \{211\} \neq \emptyset$ . Following the same reasoning as above, there would be a point  $p$  in this intersection, so  $p \in \{211\} \Leftrightarrow \{I_1(p) = 0, I_2(p) > 0, I_3(p) > 0\}$ , and therefore, there would be an open neighbourhood  $U$  of  $p$  such that  $I_2(U) > 0, I_3(U) > 0$ . And again, as  $p \in W \cap Z$ , we could choose  $U \subset W \cap Z$  so that  $I_1(U) = 0$ , and we would have then  $U \subset W \cap \{211\}$ .

Now, from the definition of the remainder we have that  $W \cap ([1, 1(11)] \cup [(1, 1)11])^\circ = \emptyset$ , which implies that  $U \subset W$  cannot intersect  $([1, 1(11)] \cup [(1, 1)11])^\circ$ , and in particular  $U$  cannot be entirely contained in  $[1, 1(11)] \cup [(1, 1)11]$ . This would mean that  $U \cap [211] \neq \emptyset$ . There should then be a point  $m \in U$  such that  $m \in [211]$ . The fact that, in particular,  $U \subset \{211\}$  would assure that the CP of  $N$  will have the same form all over  $U$ . Therefore, statement (ii) could be applied to  $N$  in  $U$  in order to state that for every point in  $U$  we could choose an open neighbourhood  $V \subset U \subset W \cap \{211\}$  where three continuous functions  $\lambda_1, \lambda_2, \lambda_3 : V \rightarrow \mathbb{R}$  would give the values of the three different eigenvalues of  $N$  in  $V$ . The linear map

$$\mathcal{Q} \equiv (N - \lambda_1 \mathbb{I})(N - \lambda_2 \mathbb{I})(N - \lambda_3 \mathbb{I}),$$

would be continuous in  $V$  from where we could deduce the existence of an open neighbourhood  $B \subset V$  of  $m$  where  $\mathcal{Q}(B) \neq 0$ , given that  $\mathcal{Q}(m) \neq 0$ . This would in turn

imply that  $m \in B \subset [211]^\circ$ , and hence  $m \in W \cap [211]^\circ \Rightarrow W \cap [211]^\circ \neq \emptyset$ , in contradiction with the disjointness in the definition of  $X$ . Thus, possibility 2 is not valid and

$$W \cap \{211\} = \emptyset.$$

Possibility 3: Let us assume then that  $p \in W \cap \{31\} = W \cap Z \cap \{31\}$ . Since  $p \in \{31\} \Leftrightarrow \{I_1(p) = I_3(p) = 0, I_2(p) > 0\}$ , there would be an open neighbourhood  $U'$  of  $p$  where  $I_2(U') > 0$  and one could take the necessarily open set  $U \equiv U' \cap (W \cap Z)$  containing  $p$ , which would have  $I_1(U) = 0$  and  $I_2(U) > 0$ . But this would further imply  $I_3(U) = 0$  because  $W \cap \{211\} = \emptyset$  (see Table 1), so that  $U \subset W \cap \{31\}$ .

As the open set  $U$  cannot be entirely contained in  $([(1, 11)1] \cup [1, (111)])^\circ$  because  $W \cap ([(1, 11)1] \cup [1, (111)])^\circ = \emptyset$ , there should be at least a point  $m \in U$  such that  $m \in [31] \cup [(21)1]$ . Since  $U \subset \{31\}$  the linear map  $N$  would have the same form of its CP everywhere in  $U$ , and thus result (ii) could be applied. Let  $\lambda_1, \lambda_2 : V \rightarrow \mathbb{R}$  be the two continuous functions defined in an open neighbourhood  $V \subset U$  of  $m$  giving the eigenvalues of  $N$  in  $V$ . In  $V$  we could consider the two linear maps defined by

$$\begin{aligned} \mathcal{Q}_1 &\equiv (N - \lambda_1 \mathbb{I})^2 (N - \lambda_2 \mathbb{I}), \\ \mathcal{Q}_2 &\equiv (N - \lambda_1 \mathbb{I}) (N - \lambda_2 \mathbb{I}). \end{aligned}$$

If  $m$  were in  $[31]$ , we would have that  $\mathcal{Q}_1(m) \neq 0$  and thus there would exist an open neighbourhood  $B \subset V$  of  $m$  such that  $\mathcal{Q}_1(B) \neq 0$ , which would mean that  $m \in B \subset [31]^\circ$  and therefore that  $m \in W \cap [31]^\circ$ , in contradiction with the definition of the remainder. Hence  $U \cap [31] = \emptyset$ . Then, it would necessarily follow that  $m \in U \cap [(21)1]$ . But a similar reasoning, using now  $\mathcal{Q}_2(m) \neq 0$ , would provide  $W \cap [(21)1]^\circ \neq \emptyset$ , in contradiction with the disjointness of the decomposition defining  $X$ . Thus, possibility 3 cannot hold and

$$W \cap \{31\} = \emptyset.$$

Possibility 4: Assume now there were a  $p \in W \cap \{22\} = W \cap Z \cap \{22\}$ . Summing up the results up to now, we know that  $I_1(W) = I_2(W) = 0$ , and since  $p \in \{22\} \Leftrightarrow \{I_1(p) = I_2(p) = 0, I_3(p) > 0\}$ , there would be an open neighbourhood  $U'$  of  $p$  where  $I_3(U') > 0$ . One could then consider the necessarily open set  $U \equiv U' \cap (W \cap Z)$ , so that  $I_1(U) = I_2(U) = 0$  and  $I_3(U) > 0$ . This would mean  $p \in U \subset W \cap \{22\}$ .

As always, the disjointness of the decomposition would imply that the open set  $U$  cannot be entirely contained in  $[(1, 1)(11)]$ . Therefore, there should be a point  $m \in U \cap [2(11)]$ . Since in particular  $U \subset \{22\}$ , we could again apply result (ii) to assure the existence of an open neighbourhood  $V \subset U$  of  $m$  where the two continuous functions  $\lambda_1, \lambda_2 : V \rightarrow \mathbb{R}$  would be the two double eigenvalues of  $N$  at each point in  $V$ . Using the continuous linear map defined in  $V$  and given by

$$\mathcal{Q}_3 \equiv (N - \lambda_1 \mathbb{I}) (N - \lambda_2 \mathbb{I}),$$

and as  $\mathcal{Q}_3(m) \neq 0$ , there would be an open neighbourhood  $B \subset V$  of  $m$  such that  $\mathcal{Q}_3(B) \neq 0 \Rightarrow B \subset [2(11)]^\circ \Rightarrow m \in W \cap [2(11)]^\circ$ , against the disjointness in the



definition of  $X$ . Thus, possibility 4 is, in fact, not possible and

$$W \cap \{22\} = \emptyset.$$

Possibility 5: The only way to keep a non-empty  $W \cap Z$  is that there were a point  $p \in W \cap ((31) \cup (211)) = W \cap Z$ , so that  $\{I_1(p) = I_2(p) = 0, I_3(p) = 0\}$ . Hence, there would be an open neighbourhood  $U$  of  $p$  such that  $U \subset W \cap Z \Rightarrow U \subset (\{4\} - [(1, 111)])$ , and  $U$  could not be entirely contained in  $[(211)]$  in order to keep the disjointness of the decomposition. Then, there should be a point  $m \in U \cap [(31)]$  and, as in particular  $U \subset \{4\}$ , applying again (ii) one could construct an open neighbourhood  $V \subset U$  of  $m$  where the continuous function  $\lambda : V \rightarrow \mathbb{R}$  would be the quadruple eigenvalue of  $N$  and such that

$$\mathcal{Q}_4 \equiv (N - \lambda \mathbb{I})^2$$

would be a continuous linear map all over  $V$ . As  $\mathcal{Q}_4(m) \neq 0$ , there would be an open neighbourhood  $B \subset V$  of  $m$  such that  $\mathcal{Q}_4(B) \neq 0 \Rightarrow B \subset [(31)]^\circ \Rightarrow m \in W \cap [(31)]^\circ$ , contradicting once more the definition of the remainder  $X$ . Thus possibility 5 cannot hold either and, in summary, the initial assumption of a non-empty set  $W \cap Z$  is not valid. Hence

$$W \cap Z = \emptyset.$$

Now that we know that  $W \cap Z$  must be empty, the only way to have a non-empty interior for  $X$  would be that the open set  $W \subseteq X$  satisfied  $W \cap [(1, 111)] \neq \emptyset$ . But this is impossible, because  $\mathcal{M} = Y \cup Z \cup [(1, 111)]$  and from the previous results we know that  $W = W \cap \mathcal{M} = W \cap [(1, 111)]$ , and therefore  $W \cap [(1, 111)]$  would be an open set, so that  $W \cap [(1, 111)] = W \cap [(1, 111)]^\circ = \emptyset$  from the definition of  $X$ . This finally proves that

$$X^\circ = \emptyset,$$

that is to say, the remainder  $X$  defined from the decomposition (8) has no interior in the manifold topology. ■

Once that we have Proposition 1 at hand, and in order to achieve the main result, we must only refine the above decomposition using the character of the eigenspaces of  $N$  to ‘separate’ the sets sharing the same box in Table 1, which are packaged together in open union sets in (8). To this end, one could make use of the results given in [6] on limits of spacetimes applied to continuous paths between points within a given spacetime. This would need the explanation of the more refined algebraic classification of the Ricci tensor presented in [21] where some different invariants are used. We refer the reader to [25], [26] and [27] where this more refined classification is studied (and compared with the one used herein) by means of geometrical and topological considerations. However, the result can be also obtained while keeping the simplicity as follows.

**Theorem 1:** *Any spacetime  $\mathcal{M}$  can be decomposed in the following union of disjoint subsets*

$$\begin{aligned}\mathcal{M} = & [1, 111] \cup [1, 1(11)]^\circ \cup [(1, 1)11]^\circ \cup [(1, 1)(11)]^\circ \cup [(1, 11)1]^\circ \cup [1, (111)]^\circ \\ & \cup [(1, 111)]^\circ \cup [211]^\circ \cup [2(11)]^\circ \cup [(21)1]^\circ \cup [(211)]^\circ \\ & \cup [31]^\circ \cup [(31)]^\circ \cup [z\bar{z}11] \cup [z\bar{z}(11)]^\circ \cup X,\end{aligned}$$

with  $X^\circ = \emptyset$ , where  $X$  is the necessarily closed subset defined from the decomposition.

Thus  $\mathcal{M} - X$  is an open dense subset of  $\mathcal{M}$ . Since  $\mathcal{M}$  is connected,  $X$  is empty if and only if the Segre type is constant on  $\mathcal{M}$ .

*Proof:* Given the preliminary decomposition of Proposition 1, we only need to prove the following two statements,

- (a)  $\left([1, (111)] \cup [(1, 11)1]\right)^\circ = [1, (111)]^\circ \cup [(1, 11)1]^\circ,$
- (b)  $\left([1, 1(11)] \cup [(1, 1)11]\right)^\circ = [1, 1(11)]^\circ \cup [(1, 1)11]^\circ,$

from which the theorem follows immediately.

Consider first statement (a). Take any point  $p \in ([1, (111)] \cup [(1, 11)1])^\circ$  so that there is an open neighbourhood  $U'$  of  $p$  contained in  $([1, (111)] \cup [(1, 11)1])^\circ$  (if there is no point in  $([1, (111)] \cup [(1, 11)1])^\circ$ , then the result is trivial). Suppose first that  $p \in [(1, 11)1] \cap ([1, (111)] \cup [(1, 11)1])^\circ$ . Since  $N$  has a simple eigenvalue at  $p$ , and using result (i), we can choose an open neighbourhood  $U \subset U'$  of  $p$  where there is a continuous function  $\lambda : U \rightarrow \mathbb{R}$  representing the simple eigenvalue of  $N$  at each point in  $U$ . As  $N$  is trace-free, the MP of  $N$  in  $U \subset ([1, (111)] \cup [(1, 11)1])^\circ$  has the form  $(N - \lambda\mathbb{I})(N + \lambda\mathbb{I}/3)$ , and thus, the eigentensor for  $\lambda$  is given by  $N_\lambda = (N + \lambda\mathbb{I}/3)$ , which hence is also continuous all over  $U$ . Denoting by  $\vec{v}_\lambda$  the  $\lambda$ 's eigendirection and using  $\text{tr}N_\lambda = 4\lambda/3$ , relation (6) implies

$$\text{sign}(\vec{v}_\lambda \cdot \vec{v}_\lambda) = \text{sign}(\lambda) \text{sign}[N_\lambda(\vec{w} \cdot \vec{w})] \quad \text{in } U, \quad (9)$$

where  $\vec{w}$  is an arbitrary vector field that can be chosen continuous and non-orthogonal to  $\vec{v}_\lambda$ . At  $p \in [(1, 11)1]$  we have  $\text{sign}(\vec{v}_\lambda \cdot \vec{v}_\lambda) = 1$ , so this must also be the case in a sufficient small open neighbourhood  $V \subset U$  of  $p$  due to the continuity of the functions involved in the righthand side of expression (9). We have thus proven that for every  $p \in [(1, 11)1] \cap ([1, (111)] \cup [(1, 11)1])^\circ$ , there is an open neighbourhood  $V \subset ([1, (111)] \cup [(1, 11)1])^\circ$  such that  $V \cap [1, (111)] = \emptyset$ . Then, the following chain holds

$$\begin{aligned}V & \subseteq ([1, (111)] \cup [(1, 11)1])^\circ \cap V = (([1, (111)] \cup [(1, 11)1]) \cap V)^\circ = \\ & = ([1, (11)1] \cap V)^\circ = [(1, 11)1]^\circ \cap V\end{aligned}$$

which means that, for every  $p \in [(1, 11)1] \cap ([1, (111)] \cup [(1, 11)1])^\circ$ , necessarily  $p \in [(1, 11)1]^\circ$ . A similar reasoning serves to prove that for any point  $p \in [1, (111)] \cap$

$([1, (111)] \cup [(1, 11)1])^\circ$ , there is an open neighbourhood  $V \subset ([1, (111)] \cup [(1, 11)1])^\circ$  such that  $V \cap [(1, 11)1] = \emptyset$ . This again implies that for every  $p \in [1, (111)] \cap ([1, (111)] \cup [(1, 11)1])^\circ$ , necessarily  $p \in [1, (111)]^\circ$ . These two results together mean that for every  $p \in ([1, (111)] \cup [(1, 11)1])^\circ$ , either  $p \in [(1, 11)1]^\circ$  or  $p \in [1, (111)]^\circ$ , from where  $([1, (111)] \cup [(1, 11)1])^\circ = [1, (111)]^\circ \cup [(1, 11)1]^\circ$  follows. This proves statement (a) and corresponds to the forbidden limit between the two sets  $[(1,11)1]$  and  $[1,(111)]$  proved in [6].

Consider finally statement (b). Choose any point  $p \in ([1, 1(11)] \cup [(1, 1)11])^\circ$  so that there is an open neighbourhood  $U'$  of  $p$  contained in  $([1, 1(11)] \cup [(1, 1)11])^\circ$  (again, if there is no such point the result is immediate). As  $N$  has two simple eigenvalues at  $p$ , using result (i) we can choose an open neighbourhood  $U \subset U'$  of  $p$  where there are two continuous functions  $\lambda_1, \lambda_2 : U \rightarrow \mathbb{R}$  giving the values of the two simple eigenvalues of  $N$  all over  $U$ . The eigentensor corresponding to the double eigenvalue  $\lambda_3 = -(\lambda_1 + \lambda_2)/2$  is given by  $N_{\lambda_3} = (N + \lambda_1 \mathbb{I})(N + \lambda_2 \mathbb{I})$ , which is continuous in  $U$ . Since  $\lambda_3$  is a simple root of the MP of  $N$  in  $U$ , the trace of  $N_{\lambda_3}$  does not vanish in  $U$ , see [7]. As  $E_{\lambda_3}$  is spacelike at  $p$ , from (7) we have that  $\mathcal{N}_{\lambda_3}(F, F)(p) > 0$  for an arbitrary continuous simple bivector field  $F$ , so  $\mathcal{N}_{\lambda_3}(F, F) > 0$  in a sufficient small open neighbourhood  $V \subset U$  of  $p$ . Therefore, for every  $p \in [1, 1(11)] \cap ([1, 1(11)] \cup [(1, 1)11])^\circ$ , there is an open neighbourhood  $V \subset ([1, 1(11)] \cup [(1, 1)11])^\circ$  such that  $V \cap [(1, 1)11] = \emptyset$ . Similarly, one can prove that for every  $p \in [(1, 1)11] \cap ([1, 1(11)] \cup [(1, 1)11])^\circ$ , there is an open neighbourhood  $V \subset ([1, 1(11)] \cup [(1, 1)11])^\circ$  such that  $V \cap [1, 1(11)] = \emptyset$ . Thus, a reasoning similar to that used to prove statement (a) leads to  $([1, 1(11)] \cup [(1, 1)11])^\circ = [1, 1(11)]^\circ \cup [(1, 1)11]^\circ$ , as we wanted to show. ■

## 4 Conclusions

The theorem just proven giving the decomposition of any spacetime according to the Segre types of the Ricci tensor assures that, for each point belonging to the open dense subset  $\mathcal{M} - X$  there is an open neighbourhood in which the same Segre type holds. That is, there are open neighbourhoods of constant Segre type almost everywhere.

This was the main aim of the paper. Nevertheless, other important results can arise when the theorem is applied to (or refined for) particular cases. For example, it should be noted that, when certain Segre types are forbidden in a given spacetime, some additional stronger results can be obtained by just following the characterizations via the invariants and the ranks given in Table 1. Besides, if the differentiability requirements for the Ricci tensor are strengthened, for instance by demanding analyticity, then much stronger results hold, see below for an example. The corresponding results on limits of spacetimes [6] can also be very useful in the possible determination of  $X$ . Of course, all of this also works for the original results [1] concerning the Petrov types.

There are several known situations in which the presence of some regions of  $\mathcal{M}$  with different algebraic type of the energy-momentum tensor arise *explicitly*. Many of these situations concern spacetimes with perfect-fluid regions admitting  $G_3$  and  $G_2$  Killing

groups, or three-dimensional conformal ( $C_3$ ) or homothetic ( $H_3$ ) symmetry groups [10]-[18], where the use of non-comoving coordinates or merely the convenience of a system of coordinates adapted to the symmetries implied the imposition of the Einstein equations in such a way that, in fact, 4 different possible Segre types appear, namely  $[1, (111)]$ ,  $[(1, 11)1]$ ,  $[(1, 111)]$ , and  $[(211)]$ . Sometimes this situation is hidden under a choice of coordinates in which the solutions are actually extendible (coordinates such as those which are valid only in the region  $[1, (111)]$ ), and such that if any extension is to be performed, it must include zones with a different algebraic type of the energy-momentum tensor, see [10] for a brief discussion. Since in these cases there are only four possible Segre types, further results can be shown. As an illustrative example, here we present a result which appeared in [10]:

*If  $\mathcal{M} = [1, (111)] \cup [(1, 11)1] \cup [(1, 111)] \cup [(211)]$ , then  $[1, (111)]$  and  $[(1, 11)1]$  are both open sets, and hence  $[(1, 111)] \cup [(211)]$  is closed.*

To see this, note first from the invariants in Table 1 that  $I_1 = I_3 = 0$  everywhere in  $\mathcal{M}$ , hence  $[1, (111)] \cup [(1, 11)1]$  is open because  $I_2 > 0$  only there. Taking into account the statement (a) appearing in the proof of the main theorem, that is  $([1, (111)] \cup [(1, 11)1])^\circ = [1, (111)]^\circ \cup [(1, 11)1]^\circ$ , and since  $[1, (111)] \cap [(1, 11)1] = \emptyset$ , the result follows. In fact, from the intermediate results in the proof of (a), which correspond to the forbidden [6] limit between  $[1, (111)]$  and  $[(1, 11)1]$ , it follows that between these two sets there must always be points of  $[(1, 111)] \cup [(211)]$ , so it is a kind of a border. Furthermore, if we demand analyticity for the Ricci tensor it follows that  $[(1, 111)] \cup [(211)]$  has no interior, as otherwise we would have  $I_2 = 0$  on the whole manifold. Finally, the remainder  $X$  in the general decomposition is, in this analytical case, the set  $[(1, 111)] \cup [(211)]$  itself.

In [1] the two theorems of decompositions, the first devoted to the Petrov types (6 open subsets), and the second to the character of the symmetry orbits (3 open subsets), are combined to give a third theorem in which any spacetime is decomposed into 18 disjoint open subsets in which the character of the orbit and the Petrov type remain constants, together with a closed subset without interior. Following the same procedure, the Segre decomposition (15 open subsets) can be combined with the Petrov one, with the type of orbits, or with both of them, providing then decompositions of  $\mathcal{M}$  into  $(6 \times 15 =) 90$ ,  $(3 \times 15 =) 45$ ,  $(6 \times 3 \times 15 =) 270$  disjoint open subsets plus a closed subset without interior, *provided* that the combinations of Segre, Petrov and orbit types are consistent: in many cases, the existence of isotropy implies the degeneration of the Petrov types [9, 28] and of the Ricci tensor [9]. For instance, the action of the group  $G_3$  on  $S_2$  implies Petrov types D or O, and there are also two equal real eigenvalues in the Ricci tensor [9]. See references [29, 30, 31] and the tables at the end of the book [9] for studies on the relation between the existence of isotropy groups, the algebraic classification of the Ricci and the Weyl tensors, and related matters.

An important point to be raised is that, both in the results of [1] as well as in this paper, the basic ingredient is the continuity of the maps defined by the Weyl or the Ricci tensors. Thus, it seems feasible that similar decomposition results could be found using other tensors which can be considered as continuous endomorphisms on

arbitrarily dimensional vector spaces ‘tangent’ to an  $n$ -dimensional manifold, by just taking into account the continuity of both the eigenvalues and their corresponding eigenspaces, together with the invariants characterizing the characteristic polynomials and also the possible ranks for each algebraic case. In what follows we present the sketch of the *general* proof for any such case.

This general proof needs, first of all, an adequate table such as that used in this paper, where the invariants discriminating between different CP and MP are shown. Then, as a preliminary result, the spacetime is decomposed into the union of the interior of the sets having the same CP and the same MP, which we are going to call CPMP sets, plus the remainder  $X$ . The refinement of this preliminary decomposition can be left to the end. The first aim is then to show that  $X$  has no interior. To that end, one starts by discarding the CPMP sets which are open in general and need no further treatment. We call this set  $Y$  and trivially  $X \cap Y = \emptyset$ . After this, one constructs the set  $Z$  defined by the complementary of  $Y$  minus the CPMP set  $[(1 \dots 1)]$ . Obviously, the union  $Y \cup Z$  has positive rank and is open thanks to the rank theorem and the continuity assumed on the traceless tensor which provides the decomposition. Moreover,  $X^\circ \cap (Y \cup Z) = X^\circ \cap Z$  is open. The next step is to prove that, in fact,  $X^\circ \cap Z = \emptyset$ . To do that, one takes the groups of sets with the same form of the CP (columns), and orders them according to the (non-strictly) increasing number of vanishing invariants. This order is important and one has to proceed one by one accordingly by proving that  $X^\circ$  has vanishing intersection with all of them.

Thus, one takes the ‘first’ column (minimum number of vanishing invariants) which will be formed by  $k$  (say) CPMP sets. One can further order these  $k$  sets by means of the increasing exponents of their corresponding MP, denoting them by  $\text{CPMP}_i$  with  $i = 1 \dots k$ , so that  $\text{CPMP}_1$  is the set with the minimum values of the exponents of the MP within this column. Due to the continuity of the invariants, one can always manage to find an open neighbourhood  $U$  contained in the column, and also included in  $X^\circ$  whenever the intersection of  $X^\circ$  with the column is non-empty. Then, it is possible to define continuous functions giving the eigenvalues of our tensor inside  $U$ . From the disjointness of the decomposition,  $U$  cannot be entirely contained in any of the  $\text{CPMP}_i$  sets if it is in  $X^\circ$ , and in particular it cannot be included in the set  $\text{CPMP}_1$ . But then there should be at least one point  $p \in U$  lying in the union of the other  $(k - 1)$  sets  $\text{CPMP}_i$  with  $i = 2 \dots k$ . If  $p$  were in the set  $\text{CPMP}_k$ , the MP corresponding to the contiguous set  $\text{CPMP}_{k-1}$  would not vanish at  $p$  and by continuity it would not vanish in an open neighbourhood  $V \subset U$  of  $p$ , which would be included in both  $X^\circ$  and  $(\text{CPMP}_k)^\circ$ , contradicting the definition of the remainder. One repeats this reasoning orderly with the sets  $\text{CPMP}_{k-1}, \dots, \text{CPMP}_2$  and finally reaches the conclusion that the only possibility is that  $X^\circ$  has no intersection with this column. Once the ‘first’ column has been discarded, the same procedure is used with the next ones in order, proving that  $X^\circ \cap Z = \emptyset$ . One must finally take into account that the last column was not complete because the set with the vanishing tensor is not included in  $Z$ . But the intersection of  $X^\circ$  with this last set is necessarily open due to the previous results, and therefore the disjointness of the decomposition implies finally that this intersection is

empty, that is to say,  $X$  has no interior.

The final step is the refinement separating the interiors of the various CPMP subsets into the interiors of more specialized algebraic subsets differing only on the character of the eigenspaces. The continuity of these characters within a given CPMP set may be invoked for this purpose using analogous arguments to those shown here for the proof of the main theorem. Notice that the continuity of the character of the eigenspaces could be proven, in principle, following intrinsic characterizations analogous to those in [7], by using the corresponding eigentensors to construct projectors onto the different eigenspaces. Of course, other alternative procedures, such as those in [6], can also be used for this last step.

## Acknowledgements

The authors wish to thank G. S. Hall for a fruitful correspondence and his valuable comments and suggestions. We also thank B. Coll for a helpful discussion and G. S. Hall, M. Mars and A. Molina for providing copies of some of the references appearing in this work. R. V. acknowledges the financial support of the *Direcció General de Recerca, Generalitat de Catalunya*.

## References

- [1] Hall G S 1996 *Class. Quantum Grav.* **13** 1479
- [2] Churchill R V 1932 *Trans. Am. Math. Soc.* **34** 784
- [3] Plebański J 1964 *Acta Phys. Polon.* **26** 963
- [4] Segre C 1883 *Mem. Reale Accad. Sci. Torino Ser II* **36** 3
- [5] Hall G S and Rendall A D 1989 *Int. J. Theor. Phys.* **28** 365
- [6] Paiva F M, Rebouças M J, Hall G S and MacCallum M A H 1998 *Class. Quantum Grav.* **15** 1031
- [7] Bona C, Coll B and Morales J A 1992 *J. Math. Phys.* **33** 670
- [8] Hawking S W, and Ellis G F R 1973 *The large scale structure of space-time*, Cambridge Univ. Press, Cambridge
- [9] Kramer D, Stephani H, Herlt E, and MacCallum M A H 1980 *Exact solutions of Einstein's field equations*, Cambridge University Press, Cambridge
- [10] Senovilla J M M and Vera R 1998 *Class. Quantum Grav.* **15** 1373
- [11] McVittie G C and Wiltshire R J 1977 *Int. J. Theor. Phys.* **16** 121

- [12] Bonnor W B and Knutsen H 1992 *Int. J. Theor. Phys.* **32** 1061
- [13] Mars M and Senovilla J M M 1994 *Class. Quantum Grav.* **11** 3049
- [14] Carot J, Coley A A and Sintes A M 1995 *Gen. Rel. Grav.* **28** 311
- [15] Mars M and Senovilla J M M 1996 *Class. Quantum Grav.* **13** 2763
- [16] Mars M and Senovilla J M M 1996 *Phys. Rev. D* **54** 6166
- [17] Mars M and Wolf T 1997 *Class. Quantum Grav.* **14** 2303
- [18] Carot J and Sintes A M 1997 *Class. Quantum Grav.* **14** 1183
- [19] Sopuerta C F 1998 *J. Math. Phys.* **39** 1024
- [20] Hall G S 1984 in *Diff. Geom.*, Banach Center Publications, Vol. 12, pp. 53-73. Polish Scientific Publishers, Warsaw
- [21] Ludwig G and Scanlan G 1971 *Commun. Math. Phys.* **20** 291
- [22] Joly G C and MacCallum M A H 1990 *Class. Quantum Grav.* **7** 541
- [23] Seixas W 1991 *Class. Quantum Grav.* **8** 1577
- [24] Dieudonné J 1966 *Fundamentos de análisis moderno*, Editorial Reverté, Barcelona
- [25] Penrose R, 1972 in *Gravitation: problems, prospects*. Dedicated to the memory of A. Z. Petrov. Izdat, Naukova Dumka, Kiev
- [26] Crade R F and Hall G S 1982 *Acta Phys. Polon.* **B13** 405
- [27] Hall G S 1985 in *Proceedings of the Balatonszéplak Relativity Workshop*, Hungary.
- [28] Ehlers J and Kundt W “Exact solutions of the gravitational fields equations”, *Gravitation*, L. Witten, Ed. (Wiley, New York, 1962), p. 49
- [29] Plebański J and Stachel J 1968 *J. Math. Phys.* **9** 269
- [30] Goenner H and Stachel J 1970 *J. Math. Phys.* **11** 3358
- [31] Goenner H 1970 *Commun. Math. Phys.* **16** 34