SINGULAR Z_N CURVES, RIEMANN-HILBERT PROBLEM AND MODULAR SOLUTIONS OF THE SCHLESINGER EQUATIONS

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ABSTRACT. We are solving the classical Riemann-Hilbert problem of rank $N > 1$ on the extended complex plane punctured in $2m + 2$ points, for $N \times N$ quasi-permutation monodromy matrices with $2m(N-1)$ parameters. Our approach is based on the finite gap integration method applied to study the Riemann-Hilbert by Deift, Its, Kapaev and Zhou [1] and Kitaev and Korotkin [2, 3]. This permits us to solve the Riemann-Hilbert problem in terms of the Szegö kernel of certain Riemann surfaces branched over the given $2m+2$ points. These Riemann surfaces are constructed from a permutation representation of the symmetric group S_N to which the quasi-permutation monodromy representation has been reduced. The permutation representation of our problem generates the cyclic subgroup Z_N. For this reason the corresponding Riemann surfaces of genus $N(m-1)$ have Z_N symmetry. This fact enables us to write the matrix entries of the solution of the $N \times N$ Riemann-Hilbert problem as a product of an algebraic function and θ-function quotients. The algebraic function turns out to be related to the Szegö kernel with zero characteristics. The $2N(m-1)$ monodromy parameters are in one to one correspondence with the $2N(m-1)$ characteristics of the θ-functions. The symmetry of the problem enables us to show that if two monodromy representations are equivalent up to multiplication by N-th root of unity, then the corresponding θ-characteristics differ only at rational numbers k/N, $k = 1, \ldots, N-1$.

From the solution of the Riemann- Hilbert problem we automatically obtain a particular solution of the Schlesinger system. The τ-function of the Schlesinger system is computed explicitly in terms of θ-functions and the holomorphic projective connection of the Riemann surface. In the course of the computation we also derive Thomae-type formulae for a class of non-singular $1/N$-periods.

Finally we study in detail the solution of the rank 3 problem with four singular points $(0, t, 1, \infty)$. The corresponding Riemann surface $\mathcal{C}_{3,1}$ is of genus two branched at the above four points and admits the dihedral group D_3 of automorphisms. This implies that $\mathcal{C}_{3,1}$ is a 2-sheeted cover of two elliptic curves which are 3-isogenous. As a result, the corresponding solution of the Riemann-Hilbert problem and the Schlesinger system is given in terms of Jacobi's ϑ-function with modulus $T = T(t)$, $\operatorname{Im} T > 0$. The function $T = T(t)$ is invertible if it belongs to the Siegel upper half space modulo the subgroup $\Gamma_0(3)$ of the modular group. The inverse function $t = t(T)$ is given explicitly in terms of Jacobi's ϑ-functions, and generates a solution of a general Halphen system. The analytic counterpart of this picture is given by Goursat's higher identities for hypergeometric functions.

CONTENTS

1. Introduction 2
Acknowledgments 8
2. The $N \times N$ matrix Riemann Hilbert problem 8
3. Riemann surface of an algebraic curve 11
3.1. The curve and differentials 11
3.2. θ-function 11
3.3. Zeros of the θ-function 12
3.4. Kernel-forms 13
4. Z_N curves 15
4.1. Homologies and periods of Z_N-curves 15

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4.2. Characteristics supported on branch points 17
4.3. Szegö kernel for $\frac{1}{N}$-periods 20
5. Solution of the Riemann-Hilbert problem for the Z_N-curve 25
5.1. τ-function of the Schlesinger equations 31
6. Example: $m = 1$ 33
6.1. Decomposition of two-dimensional Jacobian to elliptic curves: $N = 3$ and $m = 1$ 34
6.2. Solution of the 3×3 matrix R-H problem with four singular points 39
6.3. Explicit solution of the Schlesinger equation 41
7. Conclusion 43
8. Appendix : Derivation of the Thomae formula 44
References 47

1. Introduction

The Riemann-Hilbert problem (R-H problem) in its classical formulation consists of deriving a linear differential equation of Fuchsian type with a given set D of singular points and a given monodromy representation

$$\mathcal{M} \,:\, \pi_1(\mathbb{CP}^1 \backslash D, \lambda_0) \to GL(N, \mathbb{C}), \quad N \geq 2, \tag{1.1}$$

of the fundamental group $\pi_1(\mathbb{CP}^1\backslash D, \lambda_0)$. An element γ of the group $\pi_1(\mathbb{CP}^1\backslash D, \lambda_0)$ is a loop contained in $\mathbb{CP}^1\backslash D$ with initial and end point λ_0, $\lambda_0 \notin D$. Not all the representation (1.1) can be realized as the monodromy representation of a Fuchsian system, [4],[5]. For $N = 3, 4$ representations (1.1) for which the R-H problem cannot be solved are given in [6] and [7] respectively. In dimension $N = 2$ the R-H problem is always solvable [8] for arbitrary number of singular points. For $N \geq 3$, every irreducible representation (1.1) can be realized as the monodromy representation of some Fuchsian system [6],[9]. In general, among the solvable cases, the solution of the matrix R-H problem cannot be computed analytically in terms of known special functions [10],[11]. Nevertheless, there are special cases when the R-H problem can be solved explicitly in terms of θ-functions [1],[2],[3]. We discuss one of these cases.

The method of solution proposed by Plemelj [12] consists of reducing the R-H problem to a homogeneous boundary value problem in the complex plane for a $N \times N$ matrix function $Y(\lambda)$. The boundary can be chosen in the form of a polygon line $\mathcal{L}$, by connecting all the singular points of the set $D := \{\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}, \lambda_{2m+2} = \infty\}$. The line $\mathcal{L}$ divides the complex plane into two domains, C_- and C_+ (see Figure. 1). Let $\gamma_1, \gamma_2, \ldots, \gamma_{2m+2}$ denote the set of generators of the fundamental group $\pi_1(\mathbb{CP}^1\backslash D, \lambda_0)$, i.e. the homotopy class γ_k corresponds to a small clock-wise loop around the point λ_k (see Figure 1). Then the matrices $\mathcal{M}(\gamma_k) = M_k \in SL(N, \mathbb{C})$, $k = 1, \ldots, 2m + 2$, form a set of generators of the monodromy group. Since the homotopy relation

$$\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2m+2} \simeq \lambda_0,$$

the generators M_k satisfy the cyclic relation

$$M_\infty M_{2m+1} \ldots M_1 = 1_N,$$

where $M_{2m+2} = M_\infty$. Let us construct the matrices G_k defined by

$$G_k = M_k M_{k-1} \ldots M_1, \quad k = 1, \ldots, 2m + 2. \tag{1.2}$$

The homogeneous Hilbert boundary value problem formulated by Plemely is the following [12]: find the $N \times N$ matrix function $Y(\lambda)$ which satisfies the following conditions

(i) $Y(\lambda)$ is analytic in $\mathbb{CP}^1\backslash\mathcal{L}$;

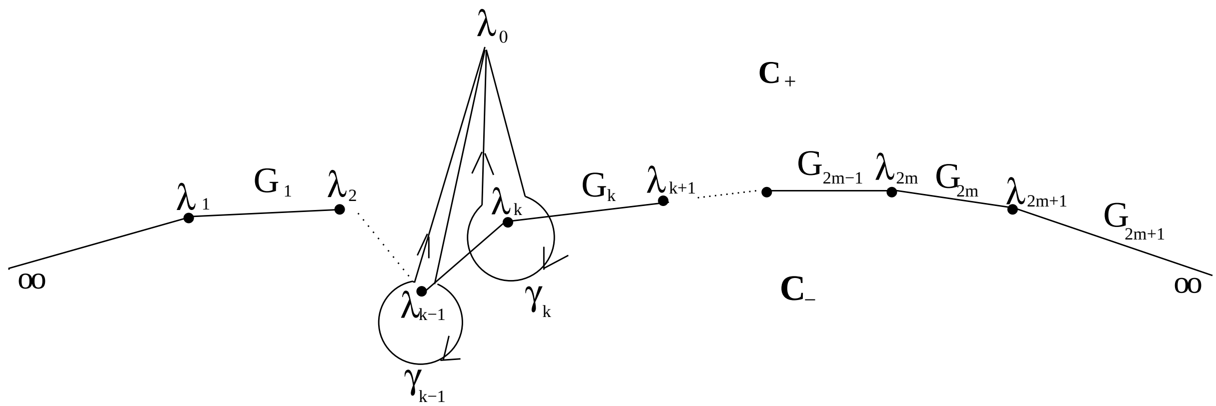

FIGURE 1. The contour $\mathcal{L}$

(ii) the L_2-limits $Y_\pm(\lambda)$ as $\lambda \to \mathcal{L}_\pm$ satisfy the jump conditions

$$Y_-(\lambda) = Y_+(\lambda)G_k, \quad \lambda \in [\lambda_k, \lambda_{k+1}], \ \ k = 0, \dots, 2m+1, \ \lambda_0 = \lambda_{2m+2} = \infty;$$

(iii) for $0 \le \epsilon < 1$,

$$Y\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right)^\epsilon \to 0 \ \text{ as } \ \lambda \to \infty \quad \text{and} \ \ Y_\pm(\lambda)(\lambda-\lambda_j)^\epsilon \to 0 \quad \text{as} \ \ \lambda \to \lambda_j,$$

over C_+ or C_- respectively;

(iv) $Y(\lambda_0) = 1_N, \quad \lambda_0 \in \mathbb{C}_+ \backslash D$.

There is always a solution of **(i)-(iv)** such that $\det Y(\lambda) \neq 0$ for $\lambda \neq D$. The analytic continuation of the solution $Y(\lambda)$ along a small loop γ_k around λ_k is determined by the matrix M_k, namely

$$Y(\gamma_k(\lambda)) = Y(\lambda)M_k, \quad \lambda \in C_+ \backslash D, \ \ k = 1, \dots, 2m+2.$$

It is possible to show that the solution $Y(\lambda)$ of the R-H problem **(i)-(iv)** satisfies a Fuchsian equation

$$\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda} = \sum_{k=1}^{2m+1} \frac{A_k}{\lambda - \lambda_k}\, Y(\lambda), \quad A_k \in SL(N, \mathbb{C}), \tag{1.3}$$

if one of the monodromy matrices is diagonalisable [5],[4]. Without this condition, Plemelj original argument does not go through.

The method of [12] was used by Deift, Its, Kapaev, and Zhou [1] to solve the 2×2 matrix R-H problem when all the matrices G_{2k} are diagonal and all the matrices G_{2k-1}, $k = 1, \dots, m+1$, are off-diagonal. The idea of the construction in [1] is to consider a hyperelliptic covering $\mathcal{C}$ over $\mathbb{CP}^1$ which is ramified in D and use the natural monodromy of the hyperelliptic curve. The application of methods of finite-gap integration [13, 14] permits to obtain a new θ-functional solution for the problem depending on $2m$ parameters. Similar results were obtained by Kitaev and Korotkin [2] by another method.

The extension of the 2×2 matrix R-H problem to higher dimensional matrices leads naturally to non-hyperelliptic curves. This fact was pointed out by Zverovich [15], who considered the $N \times N$ problem **(i)-(iv)** when all the matrices G_{2k} are diagonal, and the non-zero entries of the matrices G_{2k-1}, $k = 1, \dots, m+1$ are

$$\begin{aligned} &(G_{2k-1})_{i,i-1} \neq 0, \ \ i = 2, \dots, N, \\ &(G_{2k-1})_{N,1} \neq 0, \quad k = 1, \dots, m+1. \end{aligned} \tag{1.4}$$

The solvability of the corresponding $N\times N$ matrix R-H problem is proved by lifting it to a scalar problem on the curve

$$\mathcal{C}_{N,m} := \left\{(\lambda, y), \quad y^N = q^{N-1}(\lambda)p(\lambda)\right\}, \tag{1.5}$$

$$q(\lambda)=\prod_{j=1}^{m}(\lambda-a_{2k}), \quad p(\lambda)=\prod_{j=0}^{m}(\lambda-a_{2k+1}). \tag{1.6}$$

The curve $\mathcal{C}_{N,m}$ has singularities at the points $(\lambda_{2k},0)$, $k=1,\ldots,m$. These singularities can be easily resolved [16] to give rise to a compact Riemann surface which we still denote by $\mathcal{C}_{N,m}$. Such surface can be identified with N copies (sheets) of the complex λ-plane cut along the segments $\mathcal{L}_0=\cup_{k=1}^{m+1}[\lambda_{2k-1},\lambda_{2k}]$ and glued together according to the permutation rule $\begin{pmatrix}1 & 2 & \ldots & N-1 & N\\ 2 & 3 & \ldots & N & 1\end{pmatrix}$, that is the first sheet is pasted to the second, the second to the third and so on. The pre-image $\pi^{-1}(\lambda)$, $\lambda\in\mathbb{C}\backslash D$, of the projection $\pi:\mathcal{C}_{N,m}\to\mathbb{C}$, consists of N points $P^{(s)}=(\lambda,\rho^{s-1}y)$, where ρ is the N-th root of unity. In this paper, we solve explicitly the $N\times N$ matrix R-H problem considered by Zverovich.

The algebraic-geometrical approach to the R-H problem was developed further by Korotkin [3]. He showed that for quasi-permutation monodromy matrices (in which each row and each column have only one non-zero element), the R-H problem can be solved in terms of the Szegö kernel of a Riemann surface.

The procedure to obtain the Riemann surface from the monodromy matrices relies on the Riemann existence theorem [17, 16]. In detail the existence theorem associates a permutation representation

$$\mathcal{S}:\pi_1(\mathbb{CP}^1\backslash D,\lambda_0)\to S_N,$$

to a compact ramified cover $\mathcal{C}$ of degree N over the Riemann sphere with a set D of n prescribed branch points in such a way that the product $\mathcal{S}(\gamma_1)\mathcal{S}(\gamma_2),\ldots,\mathcal{S}(\gamma_n)=1$. The correspondence is one-to-one between isomorphism classes of covers and equivalent permutation monodromy representations (this latter equivalence relation simply reflects a relabeling of the points in the fiber of the covering over the base point λ_0). The cover $\mathcal{C}$ is connected if the only invariant subspace of the permutation representation is the N-dimensional column vector $(1,1,\ldots,1)^t$. The genus g of the surface $\mathcal{C}$ is obtained from the Riemann-Hurwitz relation

$$2(N+g-1)=\sum_{i=1}^{n}\mathrm{Tran}[\mathcal{S}(\gamma_i)], \tag{1.7}$$

where $\mathrm{Tran}[\mathcal{S}(\gamma_i)]$ is the number of transpositions in the permutation $\mathcal{S}(\gamma_i)$.

For a given monodromy representation (1.1), where the matrices $\mathcal{M}(\gamma_i)$ are quasi-permutation, the corresponding elements $\mathcal{S}(\gamma_i)$ of the symmetric group S_N are obtained by setting all the non-zero entries of $\mathcal{M}(\gamma_i)$, $i=1,\ldots,n$, equal to unity. However, the Riemann existence theorem is just an *existence* theorem, that is, it does not produce explicitly algebraic equations for the coverings. In the case under consideration, the permutation representation induced by the monodromy matrices $G_kG_{k-1}^{-1}$, with G_k being defined in (1.2), is

$$\begin{aligned}\mathcal{S}(\gamma_{2k-1})&=\begin{pmatrix}1 & 2 & \ldots & N-1 & N\\ 2 & 3 & \ldots & N & 1\end{pmatrix}, \quad k=1,\ldots,m+1,\\ \mathcal{S}(\gamma_{2k})&=\begin{pmatrix}1 & 2 & \ldots & N-1 & N\\ N & 1 & \ldots & N-2 & N-1\end{pmatrix} \quad k=1,\ldots,m+1.\end{aligned} \tag{1.8}$$

We observe that the points $P^{(s)}=(\lambda,\rho^s y)\in\mathcal{C}_{N,m}$, $s=1,\ldots,N$, $\lambda\notin D$, belonging to the pre-image $\pi^{-1}(\lambda)=(P^{(1)},P^{(2)},\ldots,P^{(N)})$, are permuted, when λ_0 moves along the path γ_k, according to the rule

$$(P^{(1)},P^{(2)},\ldots,P^{(N)})\longrightarrow\mathcal{S}(\gamma_k)(P^{(1)},P^{(2)},\ldots,P^{(N)}), \quad k=1,\ldots,2m+2.$$

Therefore $\mathcal{C}_{N,m}$ given in (1.5) is the Riemann surface associated with the permutation representation (1.8). In general the derivation of an algebraic expression for the cover from the permutation representation is a hard task.

The genus of the surface $\mathcal{C}_{N,m}$ obtained from (1.7) is equal to $g = (N-1)m$. We observe that in our case, the complex dimension of the space of quasi-permutation monodromy matrices $M_k = G_k G_{k-1}^{-1} \in SL(N, \mathbb{C})$, $k = 1, \dots, 2m+1$ is equal to $(N-1)(2m+1)$. In order to solve the R-H problem by using only the Szegö kernel as suggested in [3], the complex dimension of the space of monodromy matrices must be at most equal to $2g$. For this reason one of the monodromy matrices must be fixed as a suitable permutation or quasi-permutation matrix.

Our derivation of the solution of the R-H problem **(i)-(iv)**, incorporates both the method of [1], implemented for hyperelliptic curves and the general treatment of [3]. First, we solve the so-called canonical R-H problem, namely the problem **(i)-(iv)** when all the matrices G_{2k} are set equal to the identity and all the matrices G_{2k+1} are set equal to the quasi-permutation $\mathcal{P}_N$, where

$$\mathcal{P}_N = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & (-1)^{N-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{1.9}$$

More precisely the canonical R-H problem consists of finding a matrix valued function $X(\lambda)$ analytic in the complex plane off the segment $\mathcal{L}_0 = \cup_{k=1}^{m+1}[\lambda_{2k-1}, \lambda_{2k}]$ such that

$$\begin{aligned} X_-(\lambda) &= X_+(\lambda)\mathcal{P}_N, \quad \lambda \in \mathcal{L}_0, \\ X(\lambda_0) &= 1_N, \quad \lambda_0 \in C_+. \end{aligned} \tag{1.10}$$

The solution of the R-H problem (1.10) can be obtained in an elementary way by diagonalising the matrix $\mathcal{P}_N = U \mathrm{e}^{2\pi\imath\sigma_N} U^{-1}$, where the matrix σ_N reads

$$\sigma_N = \mathrm{Diag}\left(\frac{-N+1}{2N}, \frac{-N+3}{2N}, \dots, \frac{N-3}{2N}, \frac{N-1}{2N}\right), \tag{1.11}$$

and the matrix U is chosen so that $U_{1k} = 1,\ k = 1, \dots, N$ and $\mathrm{Det} U \neq 0$. Then it is quite immediate to verify that

$$X(\lambda) = U \left(\frac{p(\lambda)}{q(\lambda)} \frac{q(\lambda_0)}{p(\lambda_0)}\right)^{\sigma_N} U^{-1} \tag{1.12}$$

solves the R-H problem (1.10). The entries of the matrix $X(\lambda)$ can be expressed in terms of the Szegö kernel with zero characteristics, $S[0](P,Q)$, defined on $\mathcal{C}_{N,m}$. We show that

$$S[0](P,Q) = \frac{1}{N} \frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)} \sum_{k=0}^{N-1} \left(\frac{q(z(P))}{p(z(Q))} \frac{p(z(Q))}{q(z(P))}\right)^{-\frac{k}{N}+\frac{N-1}{2N}}, \quad P, Q \in \mathcal{C}_{N,m}, \tag{1.13}$$

where $z(P)$ is a local coordinate near the point P and the polynomials p and q have been defined in (1.6). Then the entries of the matrix $X(\lambda)$ in (1.12) can be written in the form

$$\begin{aligned} X_{rs}(\lambda) =& S[0](P^{(s)}, P_0^{(r)}) \frac{z(P)-z(Q)}{\sqrt{\mathrm{d}z(P)\mathrm{d}z(P_0)}} \\ =& \frac{1}{N} \sum_{k=0}^{N-1} \left(\mathrm{e}^{2\pi\imath\frac{(s-r)}{N}} \sqrt[N]{\frac{p(\lambda)}{q(\lambda)} \frac{q(\lambda_0)}{p(\lambda_0)}}\right)^{-k+\frac{N-1}{2}}, \quad \lambda_0 \notin D, \end{aligned}$$

where $P^{(s)}=(\lambda,\rho^{s-1}y)$ and $P_0^{(r)}=(\lambda_0,\rho^{r-1}y_0)$, $r,s=1,\ldots,N$, denote the points on the s-th and $r-$th sheet of $\mathcal{C}_{N,m}$ respectively. When $N=2$ and $\sqrt[4]{\dfrac{q(\lambda_0)}{p(\lambda_0)}}=1$, such a formula coincides with the canonical solution obtained in [1].

The solution $Y(\lambda)$, of the full R-H problem **(i)-(iv)**, where the constant matrices G_k, $k=1,\ldots,2m{+}1$, are parametrised by $2(N-1)m$ arbitrary complex constants, is obtained, following [3], using the Szegö kernel with non-zero characteristics. From the relation (1.13), we are able to write the global solution $Y(\lambda)=(Y_{rs}(\lambda))_{r,s,=1,\ldots,n}$ of the R-H problem **(i)-(iv)** in the form

$$Y_{rs}(\lambda)=X_{rs}(\lambda)\frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}\left(\int\limits_{P_0^{(r)}}^{P^{(s)}}\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left(\int\limits_{P_0^{(r)}}^{P^{(s)}}\mathrm{d}\boldsymbol{v};\Pi\right)}\frac{\theta(\mathbf{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)},\quad r,s=1,\ldots,N, \tag{1.14}$$

where $\mathrm{d}\boldsymbol{v}$ is the vector of normalized holomorphic differentials on $\mathcal{C}_{N,m}$, Π is the period matrix with respect to $\mathrm{d}\boldsymbol{v}$, $\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}$ is the canonical θ-function with characteristics $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ determined from the non-zero entries of the matrices G_k, $k=1,\ldots,2m+1$. The solution (1.14) exists if

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)\neq 0,$$

that is if $\Pi\boldsymbol{\delta}+\boldsymbol{\epsilon}\notin(\Theta)$, where (Θ) is the θ-divisor in the Jacobian variety of the Riemann surface. This solution coincides with the solution obtained in [1] for $N=2$. The advantage of the formula (1.14) is that permits us to evaluate *explicitly* the characteristics $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ in terms of the monodromy matrices entries thus solving the R-H problem effectively.

The Fuchsian system (1.3) is recovered from the solution (1.14) by evaluating the residue

$$A_k=A_k(\lambda_1,\ldots,\lambda_{2m+1}|M_1,\ldots,M_{2m+1})=\operatorname*{Res}_{\lambda=\lambda_k}\left[\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y^{-1}(\lambda)\right]. \tag{1.15}$$

If none of the monodromy matrices M_i, $i=1,\ldots,2m+2$, depends on the position of the singular points λ_k, $k=1,\ldots,2m+1$, then the matrices A_k satisfy the Schlesinger system [18] (see below 2.14). The Jimbo-Miwa-Ueno [19] τ-function

$$\frac{\partial}{\partial\lambda_k}\log\tau=\frac{1}{2}\operatorname*{Res}_{\lambda=\lambda_k}\ \mathrm{Tr}\left(\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y(\lambda)^{-1}\right)^2,$$

corresponding to the particular solution (1.15) of the Schlesinger system, has the form

$$\tau(\lambda_1,\ldots,\lambda_{2m+1})=\frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)}{\theta(\mathbf{0};\Pi)}\frac{\prod\limits_{\substack{k<i\\ i,k=0}}^{m}(\lambda_{2k+1}-\lambda_{2i+1})^{\frac{N^2-1}{6N}}\prod\limits_{\substack{k<i\\ k,i=1}}^{m}(\lambda_{2k}-\lambda_{2i})^{\frac{N^2-1}{6N}}}{\prod\limits_{\substack{i<j\\ i,j=1}}^{2m+1}(\lambda_i-\lambda_j)^{\frac{N^2-1}{12N}}}, \tag{1.16}$$

which, to our knowledge, is the first explicit expression of τ-function for a system of dimension $N>2$. The τ-function can be written in a different form by using the Thomae-type formula which we derive for the families of curves $\mathcal{C}_{N,m}$

$$\theta^8(\mathbf{0};\Pi)=\frac{\prod_{s=1}^{N-1}\det\mathcal{A}_s^4}{(2\pi)^{4m(N-1)}}\prod_{i<j}(\lambda_{2i}-\lambda_{2j})^{2(N-1)}\prod_{k<l}(\lambda_{2k+1}-\lambda_{2l+1})^{2(N-1)}.$$

The form (1.14) of the solution of the R-H problem enables us to show the following:

(1) if the non-singular characteristics $\boldsymbol{\delta}$, $\boldsymbol{\epsilon}$ correspond to a non-special divisor supported on the branch points, then $\boldsymbol{\delta}, \boldsymbol{\epsilon} \in (\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$ and the solution of the R-H problem corresponds to a reducible monodromy representation;

(2) when two solutions $Y(\lambda)$ and $\tilde{Y}(\lambda)$ have their corresponding characteristics equivalent modulo $(\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$, the matrix entries $Y_{rs}(\lambda)$ and $\tilde{Y}_{rs}(\lambda)$ are related by an algebraic transformation. The corresponding monodromy representations $\mathcal{M} = \{M_1, M_2, \ldots, M_{2m+1}, M_\infty\}$ and $\tilde{\mathcal{M}} = \{\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_{2m+1}, \tilde{M}_\infty\}$ are equivalent up to multiplication by N-roots of unity. That is $\tilde{M}_k = \mathrm{e}^{\frac{2\pi i}{N} j_k} M_k$, j_k integer, $\sum_{k=1}^{2m+2} j_k = 0 \bmod N$.

We remark that the result in (2) has been suggested by Dubrovin and Mazzocco [20] following their investigations of the symmetries for the Schlesinger system. These symmetries generalise the Okamoto symmetries derived in the 2×2 case [10].

Finally, we investigate in detail the case $N = 3$ and $m = 1$. The monodromy matrices read

$$M_1 = \begin{pmatrix} 0 & 0 & c_1 \\ \dfrac{c_2}{c_1} & 0 & 0 \\ 0 & \dfrac{1}{c_2} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \dfrac{c_1 d_1}{c_2} & 0 \\ 0 & 0 & c_2 d_2 \\ \dfrac{1}{c_1 d_1 d_2} & 0 & 0 \end{pmatrix}, \tag{1.17}$$

$$M_3 = \begin{pmatrix} 0 & 0 & d_1 d_2 \\ \dfrac{1}{d_1} & 0 & 0 \\ 0 & \dfrac{1}{d_2} & 0 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where c_1, c_2, d_1, d_2 are non-zero constants. Without loss of generality we choose the singular points $\lambda_1 = 0$, $\lambda_2 = t$, $\lambda_3 = 1$, $0 < \mathrm{Re}\, t < 1$. Then the solution of the R-H problem is defined in terms of the Szegö kernel of the genus two Riemann surface

$$\mathcal{C}_{3,1} : \, y^3 = \lambda(\lambda - 1)(\lambda - t)^2.$$

The period matrix of the surface has the symmetric form

$$\Pi = \begin{pmatrix} 2T & T \\ T & 2T \end{pmatrix}, \quad \mathrm{Im}\, T > 0,$$

with

$$T = \frac{\imath\sqrt{3}}{3} \frac{F\left(\frac{1}{3}, \frac{2}{3}, 1; 1 - t\right)}{F\left(\frac{1}{3}, \frac{2}{3}, 1; t\right)},$$

where $F\left(\frac{1}{3}, \frac{2}{3}, 1; 1 - t\right)$ and $F\left(\frac{1}{3}, \frac{2}{3}, 1; t\right)$ are two independent solutions of the Gauss hypergeometric equation

$$t(1 - t)F'' + (1 - 2t)F' - \frac{2}{9}F = 0.$$

The inverse function $t = t(T)$ is in general not single valued. For T belonging to Siegel half space $\mathcal{H}_1$ modulo the sub-group $\Gamma_0(3)$ of the modular group, the function $t = t(T)$ is single-valued and reads

$$t = 27\vartheta_3^4(0; 3T) \frac{(\vartheta_3^4(0; 3T) - \vartheta_3^4(0; T))^2}{(3\vartheta_3^4(0; 3T) + \vartheta_3^4(0; T))^3}. \tag{1.18}$$

Clearly, the above expression is automorphic under the action of the group $\Gamma_0(3)$. From the classical theory of the hypergeometric equation it follows that the function $t = t(T)$ satisfies the Schwartz equation (see for example [21])

$$\{t, T\} + \frac{\dot{t}^2}{2} V(t) = 0,$$

where $\dot{t} = \dfrac{dt}{dT}$, $\{\ ,\ \}$ is the Schwarzian derivative,

$$\{t, T\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2 \tag{1.19}$$

and the potential $V(t)$ is given by

$$V(t) = \frac{1-\beta^2}{t^2} + \frac{1-\gamma^2}{(t-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{t(t-1)}, \quad \alpha = \frac{1}{3},\ \beta = \gamma = 0.$$

From the function $t = t(T)$ it is possible to derive an expression for the solution of the corresponding general Halphen system equivalent to the one derived in [22]. The surface $\mathcal{C}_{3,1}$ is a two-sheeted cover of two elliptic curves that are 3-isogenous. As a result, the solution of the R-H problem and of the Schlesinger equations can be expressed explicitly in terms of Jacobi's ϑ-functions. The corresponding τ-function of the Schlesinger system reads

$$\begin{aligned}\tau(t,\delta_1,\delta_2,\epsilon_1,\epsilon_2) &= \frac{\theta\left[\begin{smallmatrix}\delta_1 & \delta_2\\ \epsilon_1 & \epsilon_2\end{smallmatrix}\right](\mathbf{0};\Pi)}{\theta(\mathbf{0};\Pi)}\frac{1}{(t(t-1))^{\frac{2}{9}}} = \frac{\mathrm{e}^{2\pi\imath[T(\delta_1^2+\delta_1\delta_2+\delta_2^2)+\epsilon_1\delta_1+\epsilon_2\delta_2]}}{(t(t-1))^{\frac{2}{9}}}\\ &\times \frac{\sum_{k=2}^{3}\vartheta_k(\epsilon_1+\epsilon_2+3T(\delta_1+\delta_2);\,6T)\vartheta_k(\epsilon_1-\epsilon_2+T(\delta_1-\delta_2);\,2T)}{\vartheta_3(0;\,6T)\vartheta_3(0;\,2T)+\vartheta_2(0;\,6T)\vartheta_2(0;\,2T)},\end{aligned} \tag{1.20}$$

where ϑ_i, $i = 2, 3$ are the Jacobi's ϑ-functions and

$$\epsilon_i = \frac{1}{2\pi\imath}\log c_i, \quad \delta_i = \frac{1}{2\pi\imath}\log d_i, \quad i = 1, 2.$$

This paper is organized as follows. In the first section we give some general backgrounds about the theory of R-H problems and we describe the R-H problem we are going to solve. In the Section 2 we give some backgrounds about classical algebraic geometry of Riemann surfaces and kernel forms. We describe in detail the curve $\mathcal{C}_{N,m}$ in the Section 3, namely its homology basis, the characteristics supported on branch points, the Szegö kernel for $1/N$ characteristics and the projective connection. This section contains mainly new material. In the Section 4 we solve the R-H problem for quasi-permutation monodromy matrices and we study the symmetry properties of the solution which are inherited from the symmetries of the curve. We derive the τ-function for the Schlesinger system and Thomae-type formula for the Z_n curve in the Section 5. We describe extensively an example for a 3×3 matrix R-H problem with four singular points in the sixth Section and we derive the solution of the corresponding 3×3 Schlesinger system. Interesting relations with the modular surface $\mathcal{H}_1/\Gamma_0(3)$ are derived. We draw our conclusion in last section.

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2. The $N \times N$ matrix Riemann Hilbert problem

The method of [12] to solve the R-H problem consists of reducing it to the so-called homogeneous Hilbert boundary value problem of the theory of singular equations [25]. The reduction is carried out in the following way. Let us assume that the set of points $\lambda_1, \ldots, \lambda_{2m+1}$ satisfy the relation

$$\operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_3 < \operatorname{Re}\lambda_3 < \cdots < \operatorname{Re}\lambda_m < \operatorname{Re}\lambda_{2m+1}.$$

Let $\mathcal{L}$ be the oriented polygonal line which connects this set of points and infinity

$$\mathcal{L} = [\infty, \lambda_1] \cup [\lambda_1, \lambda_2] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2m}, \lambda_{2m+1}] \cup [\lambda_{2m+1}, \infty].$$

We denote by C_+ and C_- the positive and negative parts of the plane $\mathbb{C}$ with respect to $\mathcal{L}$ (see Figure 1).

Let us consider the set of $2(N-1)m$ non-zero complex constants $c_1, \ldots, c_{(N-1)m}$ and $d_1, \ldots, d_{(N-1)m}$ and define the $N \times N$ quasi-permutation matrices $G_k \in SL(N, \mathbb{C})$ as

$$G_{2k-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & (-1)^{N-1} c_k \\ \frac{c_{k+m}}{c_k} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \frac{c_{k+2m}}{c_{k+m}} & 0 & \ldots & 0 & 0 \\ \ldots & & \ldots & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \frac{c_{k+(N-2)m}}{c_{k+(N-3)m}} & 0 & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{c_{k+(N-2)m}} & 0 \end{pmatrix} \tag{2.1}$$

for $k = 1, \ldots, m$ and $G_{2m+1} = \mathcal{P}_N$, where $\mathcal{P}_N$ has been defined in (1.9); the diagonal matrix G_{2k} reads

$$G_{2k} = \mathrm{dig}\left(d_k, d_{k+m}, \ldots, d_{k+(N-2)m}, \prod_{j=0}^{N-2} \frac{1}{d_{k+jm}} \right), \tag{2.2}$$

for $k = 1, \ldots, m$ and $G_0 = G_{2m+1} = 1_N$. We define the $N \times N$ matrix function $Y(\lambda)$ as the solution of the following R-H problem:

$$Y(\lambda) \text{ is analytic in } \mathbb{CP}^1 \backslash \mathcal{L}, \tag{2.3}$$

$$\text{The } L_2-\text{limits } Y_\pm(\lambda) \text{ as } \lambda \to \mathcal{L}_\pm \text{ satisfy the jump conditions:}$$

$$Y_-(\lambda) = Y_+(\lambda) G_k, \quad \lambda \in [\lambda_k, \lambda_{k+1}],\ k = 0, \ldots, 2m+1,\ \lambda_0 = \lambda_{2m+2} = \infty \tag{2.4}$$

$$Y(\lambda_0) = 1_N, \quad \lambda_0 \in C_+ \backslash D. \tag{2.5}$$

Assuming the existence of the solution of the R-H problem (2.3)-(2.5), one can find that the monodromy matrices are obtained from (1.2) by

$$Y(\gamma_k(\lambda)) = Y(\lambda) M_k,$$

where

$$M_k = G_k \left(G_{k-1}\right)^{-1}, \quad k = 1, \ldots, 2m+2, \tag{2.6}$$

$$Y(\lambda)_{\frac{1}{\lambda} \to \frac{1}{\lambda} e^{2\pi i}} = Y(\lambda) M_\infty,$$

where

$$M_\infty = \mathcal{P}_N^{-1}. \tag{2.7}$$

Remark 2.1. *The monodromy representation described by the matrices (2.6) is irreducible if*

$$c_{k+sm} \neq \xi_k^{s+1}, \quad d_{k+sm} \neq \zeta_k, \quad s = 0, \ldots, N-2,$$

where ξ_k and ζ_k, $k = 1, \ldots, m$, are any N-th root of unity and $\xi_{m+1} = \zeta_{m+1} = 1$. Indeed on the contrary, the matrices G_k read

$$G_{2k} = \zeta_k 1_N, \quad G_{2k-1} = \xi_k \mathcal{P}_N, \quad k = 1, \ldots, m.$$

The corresponding reducible monodromy representation is given by the matrices

$$M_{2k} = G_{2k}(G_{2k-1})^{-1} = \frac{\zeta_k}{\xi_k}\mathcal{P}_N^{-1}, \quad k = 1, \dots, m+1,$$
$$M_{2k-1} = G_{2k-1}(G_{2k-2})^{-1} = \frac{\xi_k}{\zeta_{k-1}}\mathcal{P}_N, \quad k = 1, \dots, m+1. \tag{2.8}$$

The matrices M_k can be written in the form

$$M_k = U_k^{-1} e^{2\pi\imath\sigma_N} U_k, \quad k = 1, \dots, 2m+1, \;\; U_k \in GL(N, \mathbb{C}), \tag{2.9}$$

where the matrix σ_N reads

$$\sigma_N = \mathrm{Diag}\left(\frac{-N+1}{2N}, \frac{-N+3}{2N}, \dots, \frac{N-3}{2N}, \frac{N-1}{2N}\right). \tag{2.10}$$

The function $Y(\lambda)$ has regular singularities of the following form near the points λ_k

$$Y(\lambda) = \hat{Y}_k(\lambda)(\lambda - \lambda_k)^{\sigma_N} U_k^{\pm}, \;\; \lambda \in C_{\pm},$$

where the matrices $\hat{Y}_k(\lambda)$ are holomorphic and invertible at $\lambda = \lambda_k$, $U_k^+ = U_k$ and $U_k^- = U_k G_{k-1}$, $k = 1, \dots, 2m+1$.

It follows from the above expansion that $\frac{dY(\lambda)}{d\lambda} Y^{-1}(\lambda)$ is meromorphic in $\mathbb{CP}^1$ with simple poles at $\lambda_1, \lambda_2, \dots \lambda_{2m+1}$ and ∞. Therefore $Y(\lambda)$ satisfies the Fuchsian equation

$$\frac{dY(\lambda)}{d\lambda} = \sum_{k=1}^{2m+1} \frac{A_k}{\lambda - \lambda_k} Y(\lambda), \tag{2.11}$$

where

$$A_k = A_k(\lambda_1, \dots, \lambda_{2m+1} | M_1, \dots, M_{2m+1}) = \operatorname*{Res}_{\lambda=\lambda_k}\left[\frac{dY(\lambda)}{d\lambda} Y^{-1}(\lambda)\right]$$
$$= \hat{Y}_k(\lambda_k)\sigma_N \hat{Y}_k^{-1}(\lambda_k), \;\; k = 1, \dots, 2m+1. \tag{2.12}$$

If none of the monodromy matrices depend on the position of the singular points λ_k, $k = 1, \dots, 2m+1$, the function $Y(\lambda; \lambda_1, \dots, \lambda_{2m+1})$ in addition to (2.11) satisfies the following equations

$$\frac{\partial}{\partial \lambda_k} Y(\lambda) = \left(\frac{A_k}{\lambda_0 - \lambda_k} - \frac{A_k}{\lambda - \lambda_k}\right) Y(\lambda), \quad k = 1, \dots, 2m+1. \tag{2.13}$$

Compatibility conditions of (2.11) and (2.13) are described by the system of Schlesinger equations [18]

$$\frac{\partial}{\partial \lambda_j} A_k = \frac{[A_k, A_j]}{\lambda_k - \lambda_j} - \frac{[A_k, A_j]}{\lambda_0 - \lambda_j}, \quad j \neq k,$$
$$\frac{\partial}{\partial \lambda_k} A_k = -\sum_{\substack{j \neq k \\ j=1}}^{2m+1} \left(\frac{[A_k, A_j]}{\lambda_k - \lambda_j} - \frac{[A_k, A_j]}{\lambda_0 - \lambda_j}\right). \tag{2.14}$$

Thus the solution of the R-H problem (2.3)-(2.5) leads immediately to the particular solution (2.12) of the Schlesinger system (2.14).

From the solution of the Schlesinger equation (2.12) one can define the corresponding holomorphic τ-function given by the formula [19]

$$\frac{\partial}{\partial \lambda_k} \log \tau = \frac{1}{2} \operatorname*{Res}_{\lambda=\lambda_k} \mathrm{Tr}\left(\frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1}\right)^2. \tag{2.15}$$

The set of zeros of the τ-function in the space of singularities of the R-H problem is called the Malgrange divisor (θ) [26]. It plays a crucial role in the discussion of the solvability of the R-H problem with the given monodromy data.

3. Riemann surface of an algebraic curve

In order to solve the R-H problem (2.4)-(2.5), we first need to introduce some basic objects on Riemann surfaces.

3.1. **The curve and differentials.** Let $\mathcal{C}$ be the Riemann surface of the algebraic equation

$$y^N + p_1(\lambda)y^{N-1} + \ldots + p_N(\lambda) = 0,$$

where $p_1, \ldots, p_N$ are polynomials in λ. In a neighbourhood U_R of the point $R = (\lambda_0, y_0) \in \mathcal{C}$, a local coordinate $z(P)$, $P = (\lambda, y) \in U_R$, is the function defined by

$$z(P) = \begin{cases} \lambda - \lambda_0 & \text{if} \quad R \quad \text{is an ordinary point,} \\ \sqrt[l]{\lambda - \lambda_0} & \text{if} \quad R \quad \text{is a finite branch point of order } l, \\ \frac{1}{\lambda} & \text{if} \quad R \quad \text{is an ordinary point at infinity,} \\ \frac{1}{\sqrt[m]{\lambda}} & \text{if} \quad R \quad \text{is a branch point at infinity of order} \quad m. \end{cases} \tag{3.1}$$

Introduce the canonical homology basis in $H_1(\mathcal{C}, \mathbb{Z})$ of α and β-cycles, $(\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g)$. Denote by $\mathrm{d}\boldsymbol{v}(P) = (\mathrm{d}v_1(P), \ldots, \mathrm{d}v_1(P))$, the associated basis of normalized holomorphic differentials,

$$\oint_{\alpha_i} \mathrm{d}v_k(P) = \delta_{ik}, \quad i, k = 1 \ldots, g. \tag{3.2}$$

The matrix of beta-periods

$$\Pi = \left(\oint_{\beta_i} \mathrm{d}v_k(P) \right)_{i,k=1,\ldots,g} \tag{3.3}$$

belongs to the Siegel half space, $\mathcal{H}_g = \{\Pi | \Pi^t = \Pi, \operatorname{Im} \Pi > 0\}$. The Jacobian variety of the curve $\mathcal{C}$ is denoted by $\operatorname{Jac}(\mathcal{C}) = \mathbb{C}^g/(1_g \oplus \Pi)$.

We also mention the variation formulas which describe the dependence of the period matrix Π on the variation of the moduli. These formula can be already found in the hyperelliptic case in Thomae [27], while for general surfaces the formulae are due to Rauch [28] (see also Fay [29]):

$$\frac{\partial}{\partial \lambda_k} \Pi_{ij} = 2\pi\imath \operatorname*{Res}_{\lambda=\lambda_k} \left\{ \frac{1}{(\mathrm{d}z(P))^2} \sum_{s=1}^{N} \mathrm{d}v_i(P^{(s)}) \mathrm{d}v_j(P^{(s)}) \right\}, \tag{3.4}$$

where $i, j = 1, \ldots, 2m,\ \ k = 1, \ldots, 2m+1$ and $P^{(s)}$ is a point on the sheet s of $\mathcal{C}$.

3.2. **θ-function.** Any point $\mathbf{e} \in \mathbb{C}^g$ can be written uniquely as $\mathbf{e} = (\boldsymbol{\epsilon}, \boldsymbol{\delta})\binom{1_g}{\Pi}$, where $\boldsymbol{\epsilon}, \boldsymbol{\delta} \in \mathbb{R}^g$ are the characteristics of $\mathbf{e}$. We use the notation $[\mathbf{e}] = [{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}]$.

If $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ are half integer, then we say that the corresponding characteristics $[\mathbf{e}]$ are half-integer. The half-integer characteristics is odd or even, whenever $4\langle \boldsymbol{\delta}, \boldsymbol{\epsilon} \rangle$ is equal to 1 or 0 modulo 2. The brackets $\langle\ ,\ \rangle$, denotes the standard Euclidean scalar product.

The Riemann θ-function with characteristics $\left[{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}\right]$ is given on $\mathcal{H}_g \times \operatorname{Jac}(\mathcal{C})$ as the Fourier series

$$\theta \left[{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}\right] (\boldsymbol{z}; \Pi) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp(\pi\imath \langle \Pi \boldsymbol{n} + \Pi \boldsymbol{\delta}, \boldsymbol{n} + \boldsymbol{\delta} \rangle + 2\pi\imath \langle \boldsymbol{z} + \boldsymbol{\epsilon}, \boldsymbol{n} + \boldsymbol{\delta} \rangle). \tag{3.5}$$

The θ-function is an entire function in the variable $\boldsymbol{z}$ and a modular function of weight $\frac{1}{2}$ in the variable τ with

- **periodicity properties**:

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z}+\boldsymbol{e}_k;\Pi) = \mathrm{e}^{2\pi\imath\delta_k}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi), \tag{3.6}$$

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z}+\boldsymbol{e}_k\Pi;\Pi) = \mathrm{e}^{-2\pi\imath\epsilon_k}\mathrm{e}^{-2\pi\imath z_k}\mathrm{e}^{-\pi\imath\Pi_{kk}}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi), \tag{3.7}$$

$$\theta\begin{bmatrix}\boldsymbol{\delta}+\boldsymbol{n}'\\ \boldsymbol{\epsilon}+\boldsymbol{n}''\end{bmatrix}(\boldsymbol{z};\Pi) = \mathrm{e}^{2\pi\imath\langle\boldsymbol{\epsilon},\boldsymbol{n}'\rangle}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi), \tag{3.8}$$

where $\boldsymbol{e}_k = (0,\ldots,\overset{k\downarrow}{1},\ldots,0)$ is the standard basis in $\mathbb{C}^g$, $\boldsymbol{n}'$ and $\boldsymbol{n}''$ integer vectors;
- **modular property**: if the homology basis changes to

$$\begin{pmatrix}\tilde{\boldsymbol{\alpha}}^t\\ \tilde{\boldsymbol{\beta}}^t\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}^t\\ \boldsymbol{\beta}^t\end{pmatrix},$$

where $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$, the transformation law for θ-function is

$$\theta\begin{bmatrix}\tilde{\boldsymbol{\delta}}\\ \tilde{\boldsymbol{\epsilon}}\end{bmatrix}(\tilde{\boldsymbol{z}};\tilde{\Pi}) = k\sqrt{\det(b\,\Pi+a)}\exp\left(\sum_{i<j} z_i z_j \frac{\partial \log\det(b\Pi+a)}{\partial \Pi_{ij}}\right)\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi) \tag{3.9}$$

with some constant k independent of $\boldsymbol{z}$ and Π,

$$\tilde{\Pi} = (d\Pi + c)(b\Pi + a)^{-1} \tag{3.10}$$

and

$$\begin{pmatrix}\tilde{\boldsymbol{\delta}}^t\\ \tilde{\boldsymbol{\epsilon}}^t\end{pmatrix} = \begin{pmatrix} a & -b\\ -c & d\end{pmatrix}\begin{pmatrix}\boldsymbol{\delta}^t\\ \boldsymbol{\epsilon}^t\end{pmatrix} + \frac{1}{2}\mathrm{Diag}\begin{pmatrix} ba^t\\ dc^t\end{pmatrix}$$

Diag denotes the column vector of the diagonal entries to $\begin{pmatrix} ba^t\\ dc^t\end{pmatrix}$.

When $\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}$ is equal to zero we write $\theta\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}(\boldsymbol{z};\Pi) = \theta(\boldsymbol{z};\Pi)$. The function $\theta(\boldsymbol{z};\Pi)$ is even and clearly satisfies the relation

$$\frac{\partial}{\partial z_i}\theta(\boldsymbol{z};\Pi)\Big|_{\boldsymbol{z}=0} = 0, \quad i = 1,\ldots,g. \tag{3.11}$$

The θ-function with arbitrary characteristics satisfies the heat equation

$$\frac{\partial^2}{\partial z_k \partial z_l}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi) = 2\imath\pi(1+2\delta_{k,l})\frac{\partial}{\partial \Pi_{kl}}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi), \quad k,l = 1,\ldots,g. \tag{3.12}$$

3.3. **Zeros of the θ-function.** [**Riemann vanishing theorem**] Let $\boldsymbol{e} \in \mathrm{Jac}(\mathcal{C})$ be an arbitrary vector. Then the multi-valued function

$$\theta\left(\int_{Q_0}^{P}\mathrm{d}\boldsymbol{v} - \boldsymbol{e};\Pi\right),$$

of P has on $\mathcal{C}$ exactly g zeros $Q_1, Q_2, \ldots, Q_g$ provided it does not vanish identically. There is a one-to-one correspondence between $\boldsymbol{e} \in \mathrm{Jac}(\mathcal{C})$ and the divisor $\sum_{i=1}^g Q_i$

$$\boldsymbol{e} = \sum_{i=1}^{g}\int_{Q_0}^{Q_i}\mathrm{d}\boldsymbol{v} - \boldsymbol{K}_{Q_0},$$

where $\boldsymbol{K}_{Q_0}$ is the vector of Riemann constants

$$(\boldsymbol{K}_{Q_0})_j = \frac{1+\Pi_{jj}}{2} - \sum_{i=1, i\neq j}^{g}\oint_{\alpha_j}\mathrm{d}v_i(P)\int_{Q_0}^{P}\mathrm{d}v_j. \tag{3.13}$$

Remark 3.1. *The vector of Riemann constants depends on the homology basis $(\alpha_1,\dots,\alpha_g;\beta_1,\dots,\beta_g)\in H_1(\mathcal{C},\mathbb{Z})$ and the base point $Q_0\in\mathcal{C}$.*

For a point $P\in\mathcal{C}$, we define the Abel map

$$\mathfrak{A}:\mathcal{C}\longrightarrow \mathrm{Jac}(\mathcal{C})$$

by setting

$$\mathfrak{A}(P)=\int_{P_0}^{Q}\mathrm{d}\boldsymbol{v} \tag{3.14}$$

for some base point $P_0\in\mathcal{C}$. For a divisor $\mathcal{D}$ of degree n the Abel map reads

$$\mathfrak{A}(\mathcal{D})=\int_{nP_0}^{\mathcal{D}}\mathrm{d}\boldsymbol{v}.$$

For a divisor of degree zero the Abel map does not depend on the base point P_0. There exists a non-positive divisor Δ of degree $g-1$ such that

$$\mathfrak{A}(\Delta-(g-1)Q_0)=\boldsymbol{K}_{Q_0}, \tag{3.15}$$

where $\boldsymbol{K}_{Q_0}$ has been defined in (3.13). The divisor Δ is called the Riemann divisor and satisfies the condition

$$2\Delta=\mathcal{K}_{\mathcal{C}}, \tag{3.16}$$

where $\mathcal{K}_{\mathcal{C}}$ is the canonical class (that is the class of divisors of Abelian differentials).

Definition 3.1. *The characteristic $\left[{\boldsymbol{\delta}\atop\boldsymbol{\epsilon}}\right]$ of a point $\boldsymbol{e}=\boldsymbol{\epsilon}+\boldsymbol{\delta}\Pi$ is called singular if*

$$\theta(\boldsymbol{e};\Pi)=0.$$

The odd characteristics $\left[{\hat{\boldsymbol{\delta}}\atop\hat{\boldsymbol{\epsilon}}}\right]$ of a point $\boldsymbol{\gamma}=\hat{\boldsymbol{\epsilon}}+\hat{\boldsymbol{\delta}}\Pi$ is non-singular if among the derivatives

$$\left.\frac{\partial}{\partial z_j}\theta[\boldsymbol{\gamma}](\boldsymbol{z};\Pi)\right|_{\boldsymbol{z}=\boldsymbol{0}}$$

there is at least one non-vanishing.

3.4. **Kernel-forms.** The Schottky-Klein prime form $E(P,Q)$, $P,Q\in\mathcal{C}$ is a skew-symmetric $(-\frac{1}{2},-\frac{1}{2})$-form on $\mathcal{C}\times\mathcal{C}$ [30]

$$E(P,Q)=\frac{\theta[\gamma]\left(\int\limits_{Q}^{P}\mathrm{d}\boldsymbol{v};\Pi\right)}{h(P)h(Q)}, \tag{3.17}$$

where $[\gamma]$ is a non-singular odd half-integer characteristics and

$$h^2(P)=\sum_{j=1}^{g}\frac{\partial}{\partial z_j}\theta[\boldsymbol{\gamma}](\boldsymbol{0};\Pi)\mathrm{d}v_j(P).$$

The prime form does not depend on the point $\boldsymbol{\gamma}$. The automorphic factors of the prime form along all cycles α_k are trivial; the automorphic factor along each β_k cycle in the Q variable equals $\exp\{-\pi\imath\Pi_{kk}-2\pi\imath\int_P^Q\mathrm{d}v_k\}$. If the points P and Q are placed in the vicinity of the point R with local coordinate z, $z(R)=0$, then the prime form has the following local behaviour as $P\to Q$

$$E(P,Q)=\frac{z(P)-z(Q)}{\sqrt{\mathrm{d}z(P)}\sqrt{\mathrm{d}z(Q)}}\left(1+O(1)\right). \tag{3.18}$$

The prime form $E(P,Q)$ is the generating form of the Bergmann and Szegö kernels. Let $P=(\lambda,y)$ and $Q=(\mu,w)$. Then the Bergmann kernel $\mathrm{d}\omega(P,Q)$ is defined as a symmetric 2-differential,

$$\mathrm{d}\omega(P,Q)=\mathrm{d}_\lambda\mathrm{d}_\mu\log E(P,Q). \tag{3.19}$$

All the α-periods of $\mathrm{d}\omega(P,Q)$ with respect to any of its two variables vanish. The period of the Bergmann kernel with respect to the variable P or Q, along the β_k cycle, is equal to $2\pi\imath\mathrm{d}v_k(Q)$ or $2\pi\imath\mathrm{d}v_k(P)$ respectively. The Bergmann kernel has a double pole along the diagonal with the following local behaviour

$$\mathrm{d}\omega(P,Q)=\left(\frac{1}{(z(P)-z(Q))^2}+H(z(P),z(Q))+\text{higher order terms}\right)\mathrm{d}z(P)\mathrm{d}z(Q), \tag{3.20}$$

where $H(z(P),z(Q))\mathrm{d}z(P)\mathrm{d}z(Q)$ is the non-singular part of $\mathrm{d}\omega(P,Q)$ in each coordinte chart. The restriction of H on the diagonal gives the projective connection (see for example [31])

$$R(z(P))=6H(z(P),z(P)) \tag{3.21}$$

which non-trivially depends on the chosen system of local coordinates. Namely the projective connection transforms as follows with respect to a change of local coordinates $z\to f(z)$

$$R(z)\to R(f(z))[f'(z)]^2+\{f(z),z\}$$

where $\{\,,\,\}$ is the Schwarzian derivative.

The Szegö kernel $S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q)$ is defined for all non-singular characteristics $\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]$ as the $(1/2,1/2)$-form on $\mathcal{C}\times\mathcal{C}$ [30]

$$S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q)=\frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]\left(\int_Q^P\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\boldsymbol{0};\Pi)E(P,Q)}. \tag{3.22}$$

The local behaviour of the Szegö kernel when $P\to Q$ is

$$S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q)=\frac{\sqrt{\mathrm{d}z(P)}\sqrt{\mathrm{d}z(Q)}}{z(P)-z(Q)}\left[1+T(z(P))(z(P)-z(Q))+O((z(P)-z(Q)^2)\right], \tag{3.23}$$

where $T(z(P))$ is regular in each coordinate chart. The Szegö kernel transforms when the variable P goes around α_k and β_k-cycles as follows

$$S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P+\alpha_k,\,Q)=\mathrm{e}^{2\pi\imath\delta_k}S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q), \tag{3.24}$$

$$S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P+\beta_k,\,Q)=\mathrm{e}^{-2\pi\imath\epsilon_k}S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q),\quad k=1,\dots,g. \tag{3.25}$$

The Riemann divisor Δ is the divisor class of the Szegö kernel with zero characteristics $\left[\begin{smallmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{smallmatrix}\right]$ ([30], p. 7).

Another important relation [30], Cor. 2.12, connects the Szegö and Bergmann kernels

$$S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q)S\left[\begin{smallmatrix}-\boldsymbol{\delta}\\ -\boldsymbol{\epsilon}\end{smallmatrix}\right](P,Q)=\mathrm{d}\omega(P,Q)+\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\boldsymbol{0};\Pi)\mathrm{d}v_k(P)\mathrm{d}v_l(Q). \tag{3.26}$$

Finally we point the following equality, [30], Cor. 2.19,

$$\det\left(\left(S\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](P_j,Q_k)\right)_{j,k=1,\dots,n}\right)=\frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]\left(\sum_{j=1}^{n}\int_{Q_j}^{P_j}\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\boldsymbol{0};\Pi)}\frac{\prod_{1\le j<k\le n}E(P_j,P_k)E(Q_k,Q_j)}{\prod_{j,k=1}^{n}E(P_j,Q_k)}, \tag{3.27}$$

for any two sets of points $P_1,\dots,P_n$, and $Q_1,\dots,Q_n$, $n\ge g$, and non-singular characteristics $\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]$.

4. Z_N CURVES

In order to solve the R-H problem (2.3)-(2.5) explicitly we need to study in detail the Riemann surface $\mathcal{C}_{N,m}$ of the curve

$$y^N = p(\lambda)q(\lambda)^{N-1}, \tag{4.1}$$

where $p(\lambda)$ and $q(\lambda)$ have been defined in (1.6). The curve (4.1) has singularities at the points $(\lambda_{2k}, 0)$, $k = 1, \ldots, m$. These singularities can be easily resolved [16] to give rise to a compact Riemann surface which we denote by $\mathcal{C}_{N,m}$. The genus g of the curve (4.1) can be computed from (1.7) and is equal to $N(m-1)$.

The branch points of the curve are $\lambda_1, \ldots, \lambda_{2m+1}$ and infinity. The projection $\pi : (\lambda, y) \to \lambda$, defines $\mathcal{C}_{N,m}$ as a $N-$sheeted covering of the complex plane $\mathbb{CP}^1$. Therefore the pre-image of a non-Weierstrass point $\lambda \in \mathbb{CP}^1$ consists of N points. The N-cyclic automorphism J of $\mathcal{C}_{N,m}$ is given by the action $J : (\lambda, y) \to (\lambda, \rho y)$, where ρ is the N-primitive root of unity, namely $\rho = \mathrm{e}^{\frac{2\pi\imath}{N}}$. In a neighbourhood U_R of the point $R = (\lambda_0, y_0) \in \mathcal{C}_{N,m}$, a local coordinate $z(P)$, $P = (\lambda, y) \in U_R$, is the function defined by

$$z(P) = \begin{cases} \lambda - \lambda_0, & \text{if} \quad R \quad \text{is an ordinary point,} \\ \sqrt[N]{\lambda - \lambda_0}, & \text{if} \quad R = (\lambda_k, 0), \ k = 1, \ldots, 2m+1, \\ \frac{1}{\sqrt[N]{\lambda}}, & \text{if} \quad R = (\infty, \infty). \end{cases} \tag{4.2}$$

4.1. **Homologies and periods of Z_N-curves.**

The canonical homology basis,

$$(\alpha_1, \ldots, \alpha_{(N-1)m}; \beta_1, \ldots, \beta_{(N-1)m}) \in H(\mathcal{C}, \mathbb{Z})$$

of $\mathcal{C}_{N,m}$ is shown in the Figure2. Namely the cycles α_{j+km}, $j = 1, \ldots, m$ lie on the $k+1$ sheet,

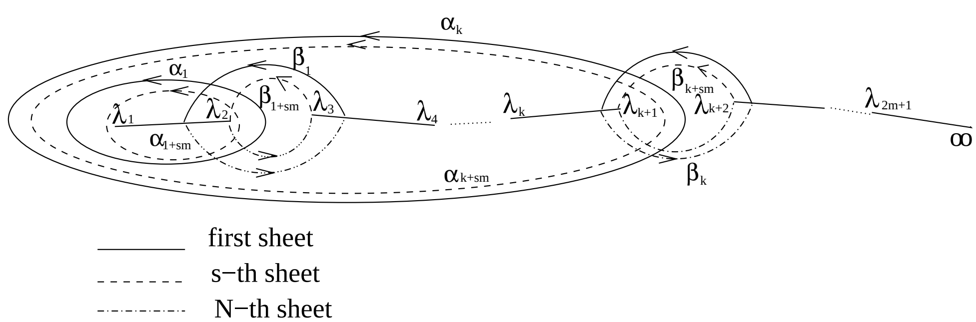

FIGURE 2. The homology basis.

$k = 0, \ldots, N-2$. The cycles β_{j+km}, $j = 1, \ldots, m$, $k = 0, \ldots, N-2$, emerges on the $(k+1)$th sheet on the cut $(\lambda_{2j-1}, \lambda_{2j})$, pass anti-clockwise to the Nth sheet through the cut $(\lambda_{2j+1}, \lambda_{2j+2})$ and return to the initial point through the Nth sheet.

Remark 4.1. *We remark that on Figure 3, at $N > 3$, the β-cycles placed from the second to the $(N-2)$th sheet should intersect the cuts only on the branch points. If we drop this requirement we need to draw a more complicated but equivalent homology basis.*

The action of the automorphism J on the basis of cycles is given by

$$J\alpha_{i+sm} = \alpha_{i+(s+1)m}, \ i = 1, \ldots, m, \ s = 0, \ldots, N-3, \tag{4.3}$$

$$J\alpha_{i+(N-2)m} = -\sum_{s=0}^{N-2} \alpha_{i+sm}, \quad i = 1, \ldots, m, \tag{4.4}$$

$$J\beta_{i+sm} = \beta_{i+(s+1)m} - \beta_i, \ \ s = 0, \dots, N-3, \quad J\beta_{i+(N-2)m} = -\beta_i, \quad i = 1, \dots, m. \tag{4.5}$$

The action of the automorphism J can be represented by an element of the symplectic group $\mathrm{Sp}(2(N-1)m;\mathbb{Z})$ in the following way. Let us introduce the $(N-1)m \times (N-1)m$ matrix

$$C = \underbrace{\begin{pmatrix} 0_m & 1_m & 0_m & \dots & 0_m & 0_m \\ 0_m & 0_m & 1_m & \dots & 0_m & 0_m \\ \dots & & \dots & \dots & & \dots \\ 0_m & 0_m & 0_m & \dots & 0_m & 1_m \\ -1_m & -1_m & -1_m & \dots & -1_m & -1_m \end{pmatrix}}_{N-1}, \tag{4.6}$$

where 0_m is the $m \times m$ zero matrix and 1_m is the $m \times m$ identity matrix. Then the element of the symplectic group induced by the automorphism J is given by

$$J \longleftrightarrow \begin{pmatrix} C & 0 \\ 0 & (C^t)^{-1} \end{pmatrix} \in \mathrm{Sp}(2(N-1)m;\mathbb{Z}).$$

The basis of canonical holomorphic differentials reads

$$\mathrm{d}u_{j+sm}(P) = \frac{\lambda^{j-1} q(\lambda)^s}{y^{s+1}} \mathrm{d}\lambda, \quad j = 1, \dots, m, \ \ s = 0, \dots, N-2. \tag{4.7}$$

The $(N-1)m \times (N-1)m$ matrices $\mathcal{A}$ of α-periods and $\mathcal{B}$ of β-periods are expressible in terms of $m \times m$-matrices

$$(\mathcal{A}_{s+1})_{kj} = \oint_{\alpha_j} \mathrm{d}u_{k+ms}, \quad (\mathcal{B}_{s+1})_{kj} = \oint_{\beta_j} \mathrm{d}u_{k+ms}, \quad j,k = 1, \dots, m, \ s = 0, \dots, N-2, \tag{4.8}$$

in the following way. Let us introduce the $(N-1)m \times (N-1)m$ dimensional matrices

$$\mathcal{R}_A = \left(\frac{\rho^{-i(k-1)} - \rho^{-ik}}{1 - \rho^{-i}} \right)_{i,k=1,\dots,N-1} \otimes 1_m, \tag{4.9}$$

$$\mathcal{R}_B = \left(\frac{\rho^{-i(k-1)} - \rho^{-i(N-1)}}{1 - \rho^{-(N-1)i}} \right)_{i,k=1,\dots,N-1} \otimes 1_m. \tag{4.10}$$

Then

$$\mathcal{A} = \left(\oint_{\alpha_j} \mathrm{d}u_k \right)_{k,j=1,\dots,(N-1)m} = \mathrm{Diag}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{N-1}) \mathcal{R}_{\mathcal{A}}, \tag{4.11}$$

$$\mathcal{B} = \left(\oint_{\beta_j} \mathrm{d}u_k \right)_{k,j=1,\dots,(N-1)m} = \mathrm{Diag}(\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{N-1}) \mathcal{R}_{\mathcal{B}}, \tag{4.12}$$

where

$$\mathrm{Diag}(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{N-1}), \quad \mathrm{Diag}(\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{N-1})$$

are the block diagonal $(N-1)m \times (N-1)m$ dimensional matrices having as entries the matrices A_s and B_s, $s = 1, \dots, N-1$, respectively.

The basis of normalized holomorphic differentials

$$\mathrm{d}\boldsymbol{v} = (\mathrm{d}v_1, \dots, \mathrm{d}v_{(N-1)m}), \quad \oint_{\alpha_j} \mathrm{d}v_k = \delta_{jk},$$

is written as

$$\mathrm{d}v_j = \sum_{k=1}^{g} (\mathcal{A}^{-1})_{jk} \mathrm{d}u_k, \quad j = 1, \ldots, (N-1)m.$$

The action of the automorphism J on the normalized holomorphic differentials is given by [32], p. 288,

$$J(\mathrm{d}\boldsymbol{v}) = \mathrm{d}\boldsymbol{v}\, C^{-1}$$

with C defined in (4.6).

The period matrix Π,

$$\Pi_{j,k} = \oint_{\beta_k} \mathrm{d}v_j, \quad j,k = 1, \ldots, (N-1)m$$

is given by

$$\Pi = \mathcal{R}_{\mathcal{A}}^{-1} \mathrm{Diag}(\mathcal{A}_1^{-1}\mathcal{B}_1, \mathcal{A}_2^{-1}\mathcal{B}_2, \ldots, \mathcal{A}_{N-1}^{-1}\mathcal{B}_{N-1}) \mathcal{R}_{\mathcal{B}} \tag{4.13}$$

with $\mathcal{R}_A$ and $\mathcal{R}_B$ defined in (4.9) and (4.10) respectively.

Example 4.2. *In the case $m = 1$ and $N = 3$ we have*

$$\mathcal{R}_A = \begin{pmatrix} 1 & \rho^2 \\ 1 & \rho \end{pmatrix}, \quad \mathcal{R}_B = \begin{pmatrix} 1 & -\rho \\ 1 & -\rho^2 \end{pmatrix}$$

so that the period matrix Π defined in (4.13) reads

$$\Pi = \frac{1}{\rho - 1} \begin{pmatrix} -\frac{\mathcal{B}_1}{\mathcal{A}_1} + \rho\frac{\mathcal{B}_2}{\mathcal{A}_2} & \rho\frac{\mathcal{B}_1}{\mathcal{A}_1} - \frac{\mathcal{B}_2}{\mathcal{A}_2} \\ \rho^2\frac{\mathcal{B}_1}{\mathcal{A}_1} - \rho^2\frac{\mathcal{B}_2}{\mathcal{A}_2} & -\frac{\mathcal{B}_1}{\mathcal{A}_1} + \rho\frac{\mathcal{B}_2}{\mathcal{A}_2} \end{pmatrix}.$$

Since the period matrix must be symmetric, it follows that

$$\rho\frac{\mathcal{B}_1}{\mathcal{A}_1} - \frac{\mathcal{B}_2}{\mathcal{A}_2} = \rho^2\frac{\mathcal{B}_1}{\mathcal{A}_1} - \rho^2\frac{\mathcal{B}_2}{\mathcal{A}_2}$$

which gives the identity

$$\frac{\mathcal{B}_1}{\mathcal{A}_1} = -\rho\frac{\mathcal{B}_2}{\mathcal{A}_2}.$$

The above relation can be also derived from the Riemann bilinear relations. Defining

$$T = \frac{1}{1-\rho}\frac{\mathcal{B}_1}{\mathcal{A}_1},$$

the period matrix simplifies to the form

$$\Pi = \begin{pmatrix} 2T & T \\ T & 2T \end{pmatrix}.$$

4.2. **Characteristics supported on branch points.** In this section we are going to compute the integrals of the form

$$\int_{\infty}^{\lambda_k} \mathrm{d}v_j, \quad j,k = 1, \ldots, (N-1)m,$$

in terms of the period matrix Π.

Lemma 4.3. *The following relations are satisfied for $k = 1, \ldots, m$, $s = 0, \ldots, N-2$,*

$$\int_{\lambda_{2k}}^{\lambda_{2k-1}} \mathrm{d}v_{k+sm} = \frac{N-1-s}{N}, \tag{4.14}$$

$$\int_{\lambda_{2k+2}}^{\lambda_{2k+1}} \mathrm{d}v_{k+sm} = -\frac{N-1-s}{N}, \tag{4.15}$$

$$\int_{\lambda_{2k+2}}^{\lambda_{2k+1}} \mathrm{d}v_{j+sm} = 0, \quad j \neq k, k+1, \; j = 1, \ldots, m \tag{4.16}$$

and

$$\int_{\lambda_{2j}+1}^{\lambda_{2j}} \mathrm{d}v_{k+sm} = \frac{N-1}{N}\Pi_{k+sm,j} - \frac{1}{N}\sum_{r=1}^{N-2}\Pi_{k+sm,j+rm}, \tag{4.17}$$

for $k, j = 1, \ldots, m, \; s = 0, \ldots, N-2.$

Proof. To prove (4.14) we observe that for $r, s = 0, \ldots, N-2$,

$$\oint_{\alpha_{j+rm}} \mathrm{d}v_{k+sm} = 0 = \sum_{l=1}^{j}\int_{\lambda_{2l}}^{\lambda_{2l-1}} (J^{(r)}(\mathrm{d}v_{k+sm}) - J^{r+1}(\mathrm{d}v_{k+sm})), \quad j < k.$$

Since

$$\sum_{r=0}^{N-1} J^{(r)}(\mathrm{d}\boldsymbol{v}) = 0, \tag{4.18}$$

the above two equations imply that

$$\int_{\lambda_{2j-1}}^{\lambda_{2j}} J^{(r)}(\mathrm{d}v_{k+sm}) = 0, \quad j < k, \quad r, s = 0, \ldots, N-2. \tag{4.19}$$

Therefore for $k = 1, \ldots, m$ and $s = 0, \ldots, N-2$

$$\begin{aligned} \oint_{\alpha_{k+sm}} \mathrm{d}v_{k+sm} &= 1 = \int_{\lambda_{2k-1}}^{\lambda_{2k}} \left(J^{(s)}(\mathrm{d}v_{k+sm}) - J^{(s+1)}(\mathrm{d}v_{k+sm})\right), \\ \oint_{\alpha_{k+rm}} \mathrm{d}v_{k+sm} &= 0 = \int_{\lambda_{2k-1}}^{\lambda_{2k}} \left(J^{(r)}(\mathrm{d}v_{k+sm}) - J^{(r+1)}(\mathrm{d}v_{k+sm})\right), \quad r \neq s. \end{aligned} \tag{4.20}$$

Combining (4.18) and (4.20) we can write the system

$$\begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 1 & 1 & 1 & \ldots & 1 & 2 \end{pmatrix} \begin{pmatrix} \int_{\lambda_{2k-1}}^{\lambda_{2k}} \mathrm{d}v_{k+sm} \\ \int_{\lambda_{2k-1}}^{\lambda_{2k}} J(\mathrm{d}v_{k+sm}) \\ \ldots \\ \int_{\lambda_{2k-1}}^{\lambda_{2k}} J^{(N-3)}(\mathrm{d}v_{k+sm}) \\ \int_{\lambda_{2k-1}}^{\lambda_{2k}} J^{(N-2)}(\mathrm{d}v_{k+sm}) \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 1 \\ \ldots \\ 0 \end{pmatrix} \overset{s+1}{\leftarrow} \tag{4.21}$$

for $\quad s = 0, \ldots, N-2 \quad$ and $\quad k = 1, \ldots, m,$

which leads to (4.14). The relation (4.15) follows from the combination of (4.19), (4.21) and the fact that

$$\oint_{\alpha_{k+1+rm}} \mathrm{d}v_{k+sm} = 0 \;\text{ for }\; r, s = 0, \ldots, N-2.$$

The relation (4.16) follows from (4.14), (4.15) and the fact that $\mathrm{d}\boldsymbol{v}$ are normalized differentials.

Finally to prove (4.17) we observe that

$$\int_{\beta_{j+rm}} \mathrm{d}v_{k+sm} = \Pi_{k+sm,\, j+rm} = \int_{\lambda_{2j+1}}^{\lambda_{2j}} J^{(r)}(\mathrm{d}v_{k+sm}) - J^{(N-1)}(\mathrm{d}v_{k+sm}), \quad r = 0, \ldots, N-2.$$

Writing the above equation in matrix form and using (4.18) we obtain

$$\begin{pmatrix} 2 & 1 & 1 & \dots & 1 & 1\\ 1 & 2 & 1 & \dots & 1 & 1\\ \dots & \dots & & \dots & & \\ 1 & 1 & 1 & \dots & 2 & 1\\ 1 & 1 & 1 & \dots & 1 & 2 \end{pmatrix} \begin{pmatrix} \int_{\lambda_{2j+1}}^{\lambda_{2j}} \mathrm{d}v_{k+sm}\\ \int_{\lambda_{2j+1}}^{\lambda_{2j}} J(\mathrm{d}v_{k+sm})\\ \dots\\ \int_{\lambda_{2j+1}}^{\lambda_{2j}} J^{(N-3)}(\mathrm{d}v_{k+sm})\\ \int_{\lambda_{2j+1}}^{\lambda_{2j}} J^{(N-2)}(\mathrm{d}v_{k+sm}) \end{pmatrix} = \begin{pmatrix} \Pi_{k+sm,\,j}\\ \Pi_{k+sm,\,j+m}\\ \dots\\ \Pi_{k+sm,\,j+(N-3)m}\\ \Pi_{k+sm,\,j+(N-2)m} \end{pmatrix} \tag{4.22}$$
$$\text{for} \quad s=0,\dots,N-2 \quad \text{and} \quad k=1,\dots,m,$$

which is equivalent to (4.17). □

We observe that the quantities in (4.15) and (4.17) satisfy

$$\frac{N-1}{N}\Pi_{k+sm,j} = -\frac{1}{N}\Pi_{k+sm,j} \ \text{ modulo lattice},$$

$$-\frac{N-1-s}{N} = \frac{s+1}{N} \ \text{ modulo lattice}.$$

From the relations (4.14)-(4.17) and the above observation we are able to write the characteristics $[\boldsymbol{\mathfrak{U}}_k]$ of the vectors

$$\boldsymbol{\mathfrak{U}}_k = \int_\infty^{\lambda_k} \mathrm{d}\boldsymbol{v}$$

in the form

$$[\boldsymbol{\mathfrak{U}}_{2m+1}] \ = \left[\underbrace{\begin{matrix} 0 & \dots & 0 & \overset{m\downarrow}{0}\\ 0 & \dots & 0 & \frac{1}{N} \end{matrix}}_{m} \dots \underbrace{\begin{matrix} 0 & \dots & 0 & \overset{sm\downarrow}{0}\\ 0 & \dots & 0 & \frac{s}{N} \end{matrix}}_{m} \dots \underbrace{\begin{matrix} 0 & \dots & 0 & \overset{(N-1)m\downarrow}{0}\\ 0 & \dots & 0 & \frac{N-1}{N} \end{matrix}}_{m} \right],$$

which immediately follows from (4.15). To pass from $[\boldsymbol{\mathfrak{U}}_{2m+1}]$ to $[\boldsymbol{\mathfrak{U}}_{2m}]$, and in general from $[\boldsymbol{\mathfrak{U}}_{2k+1}]$ to $[\boldsymbol{\mathfrak{U}}_{2k}]$ we use (4.16), while for passing from $[\boldsymbol{\mathfrak{U}}_{2k}]$ to $[\boldsymbol{\mathfrak{U}}_{2k-1}]$ we use (4.14) and (4.15), thus obtaining

$$[\boldsymbol{\mathfrak{U}}_{2m}] = \left[\underbrace{\begin{matrix} 0 & \dots & 0 & \overset{m\downarrow}{-\frac{1}{N}}\\ 0 & \dots & 0 & \frac{1}{N} \end{matrix}}_{m} \dots \underbrace{\begin{matrix} 0 & \dots & 0 & \overset{sm\downarrow}{-\frac{1}{N}}\\ 0 & \dots & 0 & \frac{s}{N} \end{matrix}}_{m} \dots \underbrace{\begin{matrix} 0 & \dots & 0 & \overset{(N-1)m\downarrow}{-\frac{1}{N}}\\ 0 & \dots & 0 & \frac{N-1}{N} \end{matrix}}_{m} \right],$$

$$\vdots$$

$$[\boldsymbol{\mathfrak{U}}_{2k+1}] = \left[\underbrace{\begin{matrix} 0 & \dots & \overset{k\downarrow}{0} & -\frac{1}{N} & \dots & -\frac{1}{N}\\ 0 & \dots & \frac{1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \dots \underbrace{\begin{matrix} 0 & \dots & \overset{k+(s-1)m\downarrow}{0} & -\frac{1}{N} & \dots & -\frac{1}{N}\\ 0 & \dots & \frac{s}{N} & 0 & \dots & 0 \end{matrix}}_{m} \dots \right.$$
$$\left. \dots \underbrace{\begin{matrix} 0 & \dots & \overset{k+(N-2)m\downarrow}{0} & -\frac{1}{N} & \dots & -\frac{1}{N}\\ 0 & \dots & \frac{N-1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \right],$$

$$[\mathcal{U}_{2k}] = \left[\underbrace{\begin{matrix} 0 & \dots & \overset{k\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & \dots & \frac{1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \; \underbrace{\begin{matrix} 0 & \dots & \overset{k+(s-1)m\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & \dots & \frac{s}{N} & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \right.$$

$$\left. \dots \; \underbrace{\begin{matrix} 0 & \dots & \overset{k+(N-2)m\downarrow}{-\frac{1}{N}} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & \dots & \frac{N-1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \right],$$

$$\vdots$$

$$[\mathcal{U}_2] = \left[\underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ \frac{1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \; \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ \frac{s}{N} & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \; \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ \frac{N-1}{N} & 0 & \dots & 0 \end{matrix}}_{m} \right],$$

$$[\mathcal{U}_1] = \left[\underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \; \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & 0 & \dots & 0 \end{matrix}}_{m} \; \dots \; \underbrace{\begin{matrix} -\frac{1}{N} & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 0 & 0 & \dots & 0 \end{matrix}}_{m} \right].$$

These formulas will be useful for the construction of $1/N$ non-singular characteristics.

4.3. **Szegö kernel for $\frac{1}{N}$-periods.** In this section we construct the Szegö kernel for $1/N$ characteristics. For the purpose we first need to determine the vector of Riemann constants of the curve $\mathcal{C}_{N,m}$. In the following we denote by $P_k = (\lambda_k, 0)$, $k = 1, \dots, 2m+1$, the branch points of the curve $\mathcal{C}_{N,m}$.

Lemma 4.4. *The vector of Riemann constants computed in the homology basis described in Figure 2 and with base point ∞ equals*

$$\boldsymbol{K}_\infty = (N-1) \sum_{k=1}^{m} \int_{\infty}^{P_{2k}} \mathrm{d}\boldsymbol{v}. \tag{4.23}$$

The proof of the above relation is obtained by direct calculations from the definition (3.13) following the lines of the proof of the Lemma 4.3.

Lemma 4.5. *The Riemann divisor Δ of the curve $\mathcal{C}_{N,m}$ in the homology basis described in Figure 2 is equivalent to*

$$\Delta = (N-1) \sum_{k=1}^{m} P_{2k} - \infty. \tag{4.24}$$

Proof. The above relation follows immediately from (4.23). □

We are going to verify, for the later convenience, that $2\Delta = \mathcal{K}_{\mathcal{C}}$. The canonical divisor $\mathcal{K}_{\mathcal{C}}$ is the divisor class of any Abelian differential on $\mathcal{C}_{N,m}$. Choosing $\mathrm{d}\lambda$ as representative differential, we have according to (4.2) that

$$\mathcal{K}_{\mathcal{C}} = (N-1) \sum_{i=1}^{2m+1} P_i - (N+1)\,\infty. \tag{4.25}$$

Because of the existence of the meromorphic function

$$\frac{y}{\prod_{k=0}^{m} (\lambda - \lambda_{2k+1})}$$

the comparison holds

$$(N-1)\sum_{k=1}^{m} P_{2m} + (N-1)\infty = (N-1)\sum_{k=0}^{m} P_{2m+1},$$

as divisor class. Therefore, using the above relation we obtain

$$2\Delta = 2(N-1)\sum_{k=0}^{m} P_{2k} - 2\,\infty = (N-1)\sum_{k=1}^{2m+1} P_k - (N+1)\,\infty \equiv \mathcal{K}_{\mathcal{C}}.$$

According to the results of Section 4.2, the formula (3.14) when Q_0 is a branch point, put into correspondence divisors $\mathcal{D}$ consisting of branch points with $1/N$-periods.

Following Diez[33], we describe a family of non-special divisors on $\mathcal{C}_{N,m}$ supported on the branch points. For $m \le l \le 2m+1$, let $s_1, \dots, s_l$ be positive integers such that

$$\sum_{i=1}^{l} s_i = (N-1)m, \quad s_i \le N-1, \tag{4.26}$$

i.e. when $l = m$ all $s_i = N-1$. For each l let us define the divisor class $\mathcal{D}_l$ supported on the branch points

$$\mathcal{D}_l = s_1 P_{i_1} + \ldots + s_l P_{i_l}, \quad \mathbb{I}_l = \{i_1, \dots, i_l\} \in \{1, \dots, 2m+1\}. \tag{4.27}$$

In particular, the divisor class $\mathcal{D}_m$ contains $\begin{pmatrix} 2m+1 \\ m \end{pmatrix}$ divisors

$$\mathcal{D}_m = (N-1)P_{i_1} + \ldots + (N-1)P_{i_{m-1}} + (N-1)P_{i_m}. \tag{4.28}$$

Among the divisors with $m+1$ branch points we consider the divisor class $\mathcal{D}_{m+1,1}$ which contains $\frac{1}{2}(m+1)m \begin{pmatrix} 2m+1 \\ m+1 \end{pmatrix}$ divisors

$$\mathcal{D}_{m+1,1} = (N-1)P_{i_1} + \ldots + (N-1)P_{i_{m-1}} + (N-2)P_{i_m} + P_{i_{m+1}}. \tag{4.29}$$

while the divisor class $\mathcal{D}_{m+1,k}$ is given by

$$\mathcal{D}_{m+1,k} = (N-j_1)P_{i_1} + \ldots + (N-j_{m-1})P_{i_{m-1}} + (N-j_m)P_{i_m} + kP_{i_{m+1}},$$

where

$$\sum_{l=1}^{m} j_l - k = m, \quad k = 1, \dots, N-2.$$

It is out of the scope of the present manuscript to classify all the non-singular divisors of the form (4.27). However we can single out two families of non-special divisors.

Lemma 4.6. *The divisors $\mathcal{D}_m$ defined in (4.28) are non-special and the divisors $\mathcal{D}_{m+1,1}$ defined in (4.29) are non-special at $N > 3$. At $N = 3$ the divisors*

$$\begin{aligned} &\mathcal{D}_{m+1,1} = 2P_{i_1} + \ldots + 2P_{i_{m-1}} + P_{i_m} + P_{i_{m+1}}, \\ &i_m \in \{1, 3, 5, \dots, 2m+1\}, \quad i_{m+1} \in \{2, 4, 6, \dots, 2m\}, \end{aligned} \tag{4.30}$$

are non-special.

Proof. Assume the opposite: suppose that the divisor $\mathcal{D}_m$ or $\mathcal{D}_{m+1,1}$ are special, this means that there exists a non-constant meromorphic function $f(\lambda, y)$ whose divisor of poles is $\mathcal{D}_m$ or $\mathcal{D}_{m+1,1}$. Then the function

$$\phi(\lambda, y) = f(\lambda, y) \prod_{i_j \in \mathbb{I}_l} (\lambda - \lambda_{i_j}), \quad l = m, m+1,$$

has poles only at infinity. It follows from the Weierstrass gap theorem, that the ring of meromorphic functions with poles at infinity is generated in the case of the curve $y^N = p(\lambda)q^{N-1}(\lambda)$ by powers of λ and functions $y^i/q(\lambda)^{i-1}$, $i = 0, \dots, N-1$. Therefore the function $\phi(\lambda, y)$ can be written in the form

$$\phi(\lambda, y) = \sum_{i=0}^{N-1} R_i(\lambda) \frac{y^i}{q^{i-1}(\lambda)}, \tag{4.31}$$

where $R_i(\lambda)$ are polynomials in λ and $q(\lambda) = \prod_{k=1}^{m}(\lambda - \lambda_k)$.[1]

We remark that $\mathrm{ord}_\infty \left(R_i(\lambda) \frac{y^i}{q^{i-1}} \right) \neq \mathrm{ord}_\infty \left(R_j(\lambda) \frac{y^j}{q^{j-1}} \right)$ for $i \neq j$ because otherwise

$$N \mathrm{ord}_\lambda R_i(\lambda) + i \;\mathrm{deg} y - Nim = N \mathrm{ord}_\lambda R_j(\lambda) + j \;\mathrm{deg} y - Njm, \tag{4.32}$$

and N and $\deg y$ would not be relatively prime. This observation implies that

$$\mathrm{ord}_\infty (f(\lambda, y) \prod_{i_l \in \mathbb{I}_l} (\lambda - \lambda_{i_l})) = \mathrm{ord}_\infty \left(R_j(\lambda) \frac{y^j}{q^{j-1}(\lambda)} \right) \tag{4.33}$$

for some $0 \leq j \leq N-1$. Moreover

$$\mathrm{ord}_\infty (f(\lambda, y) \prod_{i_j \in \mathbb{I}_l} (\lambda - \lambda_{i_j})) = -N|\mathbb{I}_l| + k_l, \quad l = m, m+1,$$

where k_l, $l = m, m+1$ is the order at infinity of $f(\lambda, y)$ and the number of elements $|\mathbb{I}_m| = m$, $|\mathbb{I}_{m+1}| = m+1$. From the equation of the curve we get $\deg y = mN + 1$. Therefore the equality (4.33) can be written as

$$N|\mathbb{I}_l| - k_l = N(r_j + m) + j$$

so that

$$j = N(|\mathbb{I}_l| - r_j - m) - k_l \geq 0.$$

When $l = m$ that is $|\mathbb{I}_l| = m$ it follow that $r_j = 0$, $j = 0$, $k_m = 0$ and

$$f(\lambda, y) = \frac{1}{\prod_{i_n \in \mathbb{I}_m} (\lambda - \lambda_{i_n})}$$

and contradicts the assumption that $f(\lambda, y)$ has divisor $\mathcal{D}_m$.

When $l = m+1$, that is $|\mathbb{I}_l| = m+1$ two possibility occurs: (i) $r_j = 0,\ j = N - k_{m+1},\ 0 \leq k_{m+1} < N$ and (ii) $r_j = 1,\ k_l = 0,\ j = 0$. This latter case can be easily excluded while for the former one we have

$$f(\lambda, y) = \frac{1}{\prod_{i_j \in \mathbb{I}_{m+1}} (\lambda - \lambda_{i_j})} \frac{y^{N-k_{m+1}}}{q^{N-k_{m+1}-1}(\lambda)}, \quad \mathrm{ord}_\infty(f(\lambda, y)) = k_{m+1},$$

which has divisor

$$\mathrm{Div} f(\lambda, y) = -N \sum_{i_n \in \mathbb{I}_{m+1}} P_{i_n} + (N - k_{m+1}) \sum_{j=1}^{m+1} P_{2j+1} + k_{m+1} \sum_{j=1}^{m} P_{2j}.$$

Namely the divisors of poles of $f(\lambda, y)$ is

$$\mathrm{Div}_{\mathrm{poles}} f(\lambda, y) = (N - k_{m+1}) \sum_{i_n \in \mathbb{I}_{m+1}, i_n \text{ even}} P_{i_n} + k_{m+1} \sum_{i_n \in \mathbb{I}_{m+1}, i_n \text{ odd}} P_{i_n}$$

[1] In this point our proof differs from that given in [33] which is working for Galois covers of the form $y^N = \prod_{i=1}^{mN}(\lambda - \lambda_i)$ where the ansatz for the function (4.31) can be written as $\sum R_i y^i$

and for $N > 3$, differs from $\mathcal{D}_{m+1,1}$. This contradicts the assumption unless f is constant. For $N = 3$ the divisor of poles of $f(\lambda, y)$ coincides with $\mathcal{D}_{m+1,1}$ in the following two cases:

$$\mathcal{D}_{m+1,1} = 2\sum_{k=1}^{m-1} P_{i_k} + P_{i_m} + P_{i_{m+1}},$$

with

$$i_m,\, i_{m+1} \in \{2,4,6\dots,2m\}, \quad i_k \in \{1,3,5,\dots,2m+1\},\ k=1,\dots,m-1$$

or

$$i_m,\, i_{m+1} \in \{1,3,5\dots,2m+1\}, \quad i_k \in \{2,4,6,\dots,2m\},\ k=1,\dots,m-1.$$

We conclude that the divisors (4.30) where i_m and i_{m+1} have different parity are non-special. □

Remark 4.7. *The importance of the divisor classes $\mathcal{D}_m$ and $\mathcal{D}_{m+1,1}$ is due to the fact that one can construct meromorphic functions with zeros and poles in prescribed branch points. Indeed let $\mathcal{D}_m$ and $\mathcal{D}_{m+1,1}$ be the divisors defined in (4.28) and (4.29). Then the function*

$$f(P) = C\left(\frac{\theta\left(\int_{\infty}^{P}\mathrm{d}\boldsymbol{v} - \int_{g\infty}^{\mathcal{D}_m}\mathrm{d}\boldsymbol{v} + \boldsymbol{K}_\infty;\Pi\right)}{\theta\left(\int_{\infty}^{P}\mathrm{d}\boldsymbol{v} - \int_{g\infty}^{\mathcal{D}_{m+1,1}}\mathrm{d}\boldsymbol{v} + \boldsymbol{K}_\infty;\Pi\right)}\right)^N, \tag{4.34}$$

with C a constant, has the only zero of N-th order at the point P_{i_m} and the only pole of N-th order at the point $P_{i_{m+1}}$. When the normalising constant C is chosen in an appropriate way, the function $f(P)$ can be identified with the coordinate λ of the curve.

Now we associate to the non-special divisors $\mathcal{D}_m$ the Szegö kernel corresponding to such divisors. The Szegö kernel for the complete class of divisors (4.27) will be considered in a separate publication.

We define the divisor class $\mathfrak{D}$ as

$$\mathfrak{D} = P + J(P) + J^{(2)}(P) + \cdots + J^{(N-1)}(P), \tag{4.35}$$

which is independent from the point $P \in \mathcal{C}_{N,m}$. The following relations hold

$$\mathfrak{D} = NP_i, \quad i=1,\dots,2m+1, \quad \mathfrak{D} = NP_\infty.$$

Let us associate to the divisor $\mathcal{D}_m$ the divisor of degree $m(N-1)-1$

$$\tilde{\mathcal{D}}_m = \mathcal{D}_m + (N-1)P_\infty - \mathfrak{D}, \tag{4.36}$$

and the corresponding $1/N$-period $[\tilde{\mathcal{D}}_m]$

$$[\tilde{\mathcal{D}}_m] = \mathfrak{A}\left(\mathcal{D}_m + (N-1)P_\infty - \mathfrak{D} - \Delta\right), \tag{4.37}$$

where Δ is the Riemann divisor. We observe that when the base point is at infinity then

$$[\tilde{\mathcal{D}}_m] = \sum_{i_k \in I_m} \int_{P_\infty}^{P_{i_k}} \mathrm{d}\boldsymbol{v} - \boldsymbol{K}_\infty,$$

therefore, the characteristics $[\tilde{\mathcal{D}}_m]$ is non-singular because of Lemma 4.6. Let us define the function

$$\psi_k(P,Q) = \frac{z(P)-\lambda_k}{z(Q)-\lambda_k}, \quad k=1,\dots,2m+1. \tag{4.38}$$

and agree to omit the arguments (P,Q) if no ambiguities appear.

Theorem 4.8. *The Szegö kernel associated to the characteristics* $[\tilde{\mathcal{D}}_m]$ *reads*

$$S[\tilde{\mathcal{D}}_m](P,Q) = \frac{1}{N}\frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)}\sum_{s=0}^{N-1}\left(\frac{\prod\limits_{i_k\in\mathbb{I}_m}\psi_{i_k}(P,Q)}{\prod\limits_{j_k\in\mathbb{J}_{m+1}}\psi_{j_k}(P,Q)}\right)^{-\frac{s}{N}+\frac{N-1}{2N}} \tag{4.39}$$

where $\mathbb{I}_m=\{i_1,\ldots,i_m\}\subset\{1,2,\ldots,2m+1\}$ *and* $\mathbb{J}_{m+1}=\{1,2,\ldots,2m+1\}\backslash\{i_1,\ldots,i_m\}$*. In particular, the Szegö kernel with zero characteristics reads*

$$S[0](P,Q) = \frac{1}{N}\frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)}\sum_{s=0}^{N-1}\left(\frac{q(z(P))}{p(z(P))}\frac{p(z(Q))}{q(z(Q))}\right)^{-\frac{s}{N}+\frac{N-1}{2N}}. \tag{4.40}$$

Proof. The Szegö kernel $S[\tilde{\mathcal{D}}_m](P,Q)$ is the unique, up to a constant, $(\frac{1}{2},\frac{1}{2})$-form on $\mathcal{C}_{N,m}\times\mathcal{C}_{N,m}$ with the only pole along the diagonal $P=Q$ and which has divisor $\mathcal{K}_{\mathcal{C}}-\tilde{\mathcal{D}}_m$ in the variable P and $\tilde{\mathcal{D}}_m$ in the variable Q (see e.g. Narasimhan [34]). Here $\mathcal{K}_{\mathcal{C}}$ is the canonical divisor and $\tilde{\mathcal{D}}_m$ has been defined in (4.36). Therefore we just need to verify that the right hand sides of the expressions (4.39) and (3.22) have the same divisor. It is enough to show this by setting $Q=P_{j_1}$. Regarding the formula (3.22) we have

$$\mathrm{Div}\left(\frac{\theta[\tilde{\mathcal{D}}_m]\left(\int\limits_{P_{j_1}}^{P}\mathrm{d}\boldsymbol{v}\right)}{E(P,P_{j_1})}\right) = (N-1)\sum_{k=2}^{m+1}P_{j_k}-P_{j_1} = \mathcal{K}_{\mathcal{C}_{N,m}}-\widetilde{\mathcal{D}}_m,$$

Next putting $Q=P_{j_1}$ into the expression (4.39) we obtain

$$\mathrm{div}\left(\frac{1}{N}\frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)}\sum_{s=0}^{N-1}\left(\frac{\prod\limits_{k=1}^{m}\psi_{i_k}(P,Q)}{\prod\limits_{k=1}^{m+1}\psi_{j_k}(P,Q)}\right)^{-\frac{s}{N}+\frac{N-1}{2N}}\right)\Bigg|_{Q=P_{j_1}} = \frac{1}{\sqrt{N}}\left(\frac{\prod\limits_{k=1}^{m}(\lambda_{j_1}-\lambda_{i_k})}{\prod_{k=2}^{m+1}(\lambda_{j_1}-\lambda_{j_k})}\right)^{\frac{N-1}{2N}}\times$$

$$\times\,\mathrm{div}\left(\frac{\sqrt{\mathrm{d}z(P)}}{z(P)-\lambda_{j_1}}\left(\frac{\prod\limits_{k=1}^{m+1}(z(P)-\lambda_{j_k})}{\prod\limits_{k=1}^{m}(z(P)-\lambda_{i_k})}\right)^{\frac{N-1}{2N}}\right) = (N-1)\sum_{k=2}^{m+1}P_{j_k}-P_{j_1} = \mathcal{K}_{\mathcal{C}_{N,m}}-\widetilde{\mathcal{D}}_m,$$

which shows that the two expressions (4.39) and (3.22) have the same divisor class in P. In the same way one can check the divisor class in Q. Therefore the expressions (3.22) and (4.39) of the Szegö kernel differ at most from a constant. This constant is equal to one because when $P\to Q$ the expression (4.39) has the following expansion

$$S[\tilde{\mathcal{D}}_m](P,Q) = \frac{\sqrt{\mathrm{d}z(P)}\sqrt{\mathrm{d}z(Q)}}{z(P)-z(Q)}\left[1+O((z(P)-z(Q))\right],$$

which coincides with the leading coefficient of the expansion (3.23). ☐

Example 4.9. *In particular for* $N=3$*, the above families of Szegö kernels read*

$$S\{2P_{i_1}+\ldots+2P_{i_m}\}(P,Q) = \frac{1}{3}\left(\sqrt[3]{\frac{\psi_{i_1}\cdots\psi_{i_m}}{\psi_{j_1}\cdots\psi_{j_{m+1}}}}+1+\sqrt[3]{\frac{\psi_{j_1}\cdots\psi_{j_{m+1}}}{\psi_{i_1}\cdots\psi_{i_m}}}\right)\frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)},$$

for $N = 4$,

$$S\{3P_{i_1} + \ldots + 3P_{i_m}\}(P,Q) = \frac{1}{4}\frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)}$$
$$\left(\sqrt[8]{\frac{\psi_{i_1}^3\cdots\psi_{i_m}^3}{\psi_{j_1}^3\cdots\psi_{j_{m+1}}^3}} + \sqrt[8]{\frac{\psi_{i_1}\cdots\psi_{i_m}}{\psi_{j_1}\cdots\psi_{j_{m+1}}}} + \sqrt[8]{\frac{\psi_{j_1}\cdots\psi_{j_{m+1}}}{\psi_{i_1}\cdots\psi_{i_m}}} + \sqrt[8]{\frac{\psi_{j_1}^3\cdots\psi_{j_{m+1}}^3}{\psi_{i_1}^3\cdots\psi_{i_m}^3}}\right).$$

The following corollary can be checked in a straightforward manner.

Corollary 4.10. *The expansion of the Szegö kernel with zero characteristics as $P \to Q$ reads*

$$S[0](P,Q) = \frac{\sqrt{\mathrm{d}z(P)\mathrm{d}z(Q)}}{z(P)-z(Q)}$$
$$\times\left\{1 + \left(\frac{1}{6}\{z(P),P\} + \frac{N^2-1}{24N^2}\left[\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{p(z(P))}{q(z(P))}\right]^2\right)(z(P)-z(Q))^2 + \ldots\right\}, \tag{4.41}$$

where $\{z(P),P\}$ is the Schwarzian derivative (1.19).

5. Solution of the Riemann-Hilbert problem for the Z_N-curve

Now we are ready to solve the canonical R-H problem (1.10), that is we determine a $N \times N$ matrix valued function $X(\lambda)$ that satisfies

$$\begin{aligned} &X_-(\lambda) = X_+(\lambda)\mathcal{P}_N, \quad \lambda \in \cup_{k=0}^m(\lambda_{2k+1},\lambda_{2k+2}),\\ &X(\lambda_0) = 1_N, \quad \lambda_0 \in C_+. \end{aligned} \tag{5.1}$$

The quasi-permutation monodromy matrix $\mathcal{P}_N$ can be diagonalised to the form

$$\mathcal{P}_N = U\mathrm{e}^{2\pi\imath\sigma_N}U^{-1},$$

where the diagonal matrix σ_N is defined in (2.10) and the matrix U can be chosen in the form $U_{1k} = 1$, $k = 1,\ldots,N$, $\mathrm{Det}U \neq 0$. In this way the canonical R-H problem (5.1) is reduced to the form

$$\begin{aligned} &(U^{-1}XU)_- = (U^{-1}XU)_+\mathrm{e}^{2\pi\imath\sigma_N} \quad \lambda \in \cup_{k=0}^m(\lambda_{2k+1},\lambda_{2k+2}),\\ &U^{-1}X(\lambda_0)U = 1_N, \quad \lambda_0 \in C_+. \end{aligned}$$

It is easy to verify that the diagonal matrix

$$U^{-1}X(\lambda)U = \left(\frac{p(\lambda)}{q(\lambda)}\frac{q(\lambda_0)}{p(\lambda_0)}\right)^{\sigma_N},$$

where the polynomials $p(\lambda)$ and $q(\lambda)$ has been defined in (1.6), solves the above R-H problem. Indeed choosing the function

$$\left(\frac{p(\lambda)}{q(\lambda)}\right)^{\frac{N-1}{2N}} \to \lambda^{\frac{N-1}{2N}}, \quad \lambda \to \imath\infty,\ \arg\lambda = \frac{\pi}{2}$$

it follows that

$$\left(\frac{p(\lambda)}{q(\lambda)}\right)_-^{-k+\frac{N-1}{2N}} = \mathrm{e}^{2\pi\imath(-k+\frac{N-1}{2N})}\left(\frac{p(\lambda)}{q(\lambda)}\right)_+^{-k+\frac{N-1}{2N}}, \quad k = 0,\ldots,N-1.$$

Furthermore

$$\det(U^{-1}X(\lambda)U) = \det\left(\left(\frac{p(\lambda)}{q(\lambda)}\frac{q(\lambda_0)}{p(\lambda_0)}\right)^{\sigma_N}\right) = 1, \quad \lambda \in \mathbb{C}\cup\infty$$

and clearly $U^{-1}X(\lambda_0)U = 1_N$. Therefore the matrix function

$$X(\lambda) = U\left(\frac{p(\lambda)}{q(\lambda)}\frac{q(\lambda_0)}{p(\lambda_0)}\right)^{\sigma_N} U^{-1}$$

solves the canonical R-H problem (5.1). The entries of the matrix $X(\lambda)$ can be expressed in terms of the Szegö kernel with zero characteristics, $S[0](P,Q)$, defined on $\mathcal{C}_{N,m}$ and derived in (4.40). Indeed it turns out that the entries $X_{rs}(\lambda)$ of $X(\lambda)$ are also equal to

$$\begin{aligned} X_{rs}(\lambda) =& S[0](P^{(s)}, P_0^{(r)})\frac{z(P)-z(Q)}{\sqrt{\mathrm{d}z(P)\mathrm{d}z(P_0)}} \\ =& \frac{1}{N}\sum_{k=0}^{N-1}\left(\mathrm{e}^{2\pi\imath\frac{(s-r)}{N}}\sqrt[N]{\frac{p(\lambda)}{q(\lambda)}\frac{q(\lambda_0)}{p(\lambda_0)}}\right)^{-k+\frac{N-1}{2}}, \quad \lambda_0\notin D,\ r,s,=1,\ldots,N, \end{aligned}$$

where $P^{(s)} = (\lambda, \rho^{s-1}y)$ and $P_0^{(r)} = (\lambda_0, \rho^{r-1}y_0)$, $r,s = 1,\ldots,N$, denote the points on the s-th and $r-$th sheet of $\mathcal{C}_{N,m}$ respectively. When $N = 2$ and $\sqrt[4]{\frac{q(\lambda_0)}{p(\lambda_0)}} = 1$, such formula coincides with the canonical solution obtained in [1].

We also observe that the matrix $X(\lambda)$ satisfies the differential equation

$$\frac{\mathrm{d}X(\lambda)}{\mathrm{d}\lambda} = \sum_{k=1}^{2m+1}\frac{A_k}{\lambda-\lambda_k}X(\lambda)$$

where the matrices A_k are given by

$$A_k = (-1)^{k-1}U\sigma_N U^{-1}, \quad k = 1,\ldots,2m+1, \tag{5.2}$$

with σ_N defined (2.10). Therefore the canonical R-H problem gives a constant solution of the Schlesinger system (2.14).

We are now ready to derive the solution of the R-H problem (2.4)-(2.5) for arbitrary non-zero values of the constants c_k and d_k, $k = 1,\ldots(N-1)m$.

Theorem 5.1 (Main Theorem)**.** *Let us define the characteristics* $\boldsymbol{\epsilon},\boldsymbol{\delta}\in\mathbb{C}^{(N-1)m}$ *as*

$$\epsilon_{k+sm} = \frac{1}{2\pi\imath}\log\frac{c_{k+sm}}{c_{k+1+sm}}, \quad s = 0,\ldots,N-2,\ \ k = 1,\ldots,m-1, \tag{5.3}$$

$$\epsilon_{sm} = \frac{1}{2\pi\imath}\log c_{sm}, \quad s = 1,\ldots,N-1,$$

$$\delta_k = \frac{1}{2\pi\imath}\log d_k \ \ k = 1,\ldots,(N-1)m. \tag{5.4}$$

Let us suppose $\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi)\neq 0$. *Then the matrix valued function* $Y(\lambda) = (Y_{rs}(\lambda))_{r,s,=1,\ldots,N}$

$$Y_{rs}(\lambda) = X_{rs}(\lambda)\frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]\left(\int_{P_0^{(r)}}^{P^{(s)}}\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left(\int_{P_0^{(r)}}^{P^{(s)}}\mathrm{d}\boldsymbol{v};\Pi\right)}\frac{\theta(\mathbf{0};\Pi)}{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi)}, \quad r,s = 1,\ldots,N \tag{5.5}$$

solves the R-H problem (2.3)-(2.5) and $\det Y(\lambda)\neq 0$ *for* $\lambda\neq\lambda_k$, $k = 1,\ldots,2m+2$.

Proof. First of all we show that matrix (5.5) is holomorphic outside the singular set $\lambda \neq \lambda_1, \dots, \lambda_{2m+1}, \infty$. Indeed combining (3.22) and (4.40), the entries of the matrix (5.5) can be written in the form given by [3]

$$Y_{rs}(\lambda) = S\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(P_0^{(r)}, P^{(s)})\frac{z(P^{(s)}) - z(P_0^{(r)})}{\sqrt{\mathrm{d}z(P^{(s)})\mathrm{d}z(P_0^{(r)})}}, \quad r, s = 1, \dots, N. \tag{5.6}$$

From the properties of the Szegö kernel, the matrix (5.6) is clearly holomorphic for $\lambda \neq \lambda_1, \dots, \lambda_{2m+1}, \infty$. Furthermore, using the formula (5.6) for the entries of $Y(\lambda)$ and applying the relation (3.27) and (4.18) we conclude that

$$\det Y(\lambda) = \left(\frac{z(P^{(1)}) - z(P_0^{(1)})}{\sqrt{\mathrm{d}z(P^{(1)})\mathrm{d}z(P_0^{(1)})}}\right)^N \frac{\prod\limits_{1\leq r<s\leq N} E(P_0^{(r)}, P_0^{(s)})E(P^{(r)}, P^{(s)})}{\prod\limits_{r,s=1}^{N} E(P_0^{(r)}, P^{(s)})} \neq 0, \tag{5.7}$$

for $P \neq (\lambda_k, 0)$ or (∞, ∞). In the above formula we have used the relation $z(P^{(r)}) = z(P^{(1)})$ and $z(P_0^{(r)}) = z(P_0^{(1)})$, $r = 1, \dots, N$. Evidently we have that

$$\det Y(\lambda_0) = 1_N.$$

In order to prove that (5.5) does indeed satisfy the R-H problem (2.4)-(2.5) the following considerations are needed. The action of the automorphism J on $\mathrm{d}v_j$ is given by the relation

$$J(\mathrm{d}v_j(\lambda, y)) = \sum_{k=1}^{m}\sum_{r=1}^{N}\gamma_{j,k}^r \lambda^{m-k}\frac{(q(\lambda))^{r-1}}{\rho^r y^r}\mathrm{d}\lambda, \tag{5.8}$$

where γ_{jk}^r are the normalisation constants of the holomorphic differentials and ρ is the Nth root of unity. Let us consider the Abelian integral

$$\boldsymbol{v}(P) = \int_\infty^P \mathrm{d}\boldsymbol{v}. \tag{5.9}$$

The action of the automorphism J on $\boldsymbol{v}(P)$ is naturally given by

$$J(\boldsymbol{v}(P)) = \int_\infty^{J(P)} \mathrm{d}\boldsymbol{v} = \int_\infty^P J(\mathrm{d}\boldsymbol{v}). \tag{5.10}$$

When $P^{(s)} = (\lambda, \rho^{s-1}y)$ is on the s-th sheet, we denote by $J^{(s-1)}(\boldsymbol{v}(\lambda))$ the natural restriction of the integral $\boldsymbol{v}(P^{(s)})$ on $C_+ \cup C_-$:

$$J^{(s-1)}(\boldsymbol{v}(\lambda)) := \int_\infty^{(\lambda,y)} J^{(s-1)}(\mathrm{d}\boldsymbol{v}) = \sum_{k=1}^{m}\sum_{r=1}^{N}\gamma_{j,k}^r\int_\infty^{(\lambda,y)}\xi^{m-k}\frac{(q(\xi))^{r-1}}{\rho^{r(s-1)}w^r}\mathrm{d}\lambda, \quad (\xi, w) \in \mathcal{C}_{N,m}.$$

The integral itself is taken on the first sheet of $\mathcal{C}_{N,m}$ and the integration path lies in C_+ and C_- for $\lambda \in C_+$ or $\lambda \in C_-$ respectively. The integral in (5.5) is defined as

$$\int\limits_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v} := J^{(s-1)}(\boldsymbol{v}(\lambda)) - J^{(r-1)}(\boldsymbol{v}(\lambda_0)).$$

If $\lambda \in \mathcal{L}$ the integrals $\boldsymbol{v}_\pm(\lambda)$, are shown on Figure 3, namely the integration path of $\boldsymbol{v}_\pm(\lambda)$, lies in $C_\pm$ respectively. From the properties of the homology basis (4.3)-(4.5) the following relations can be easily

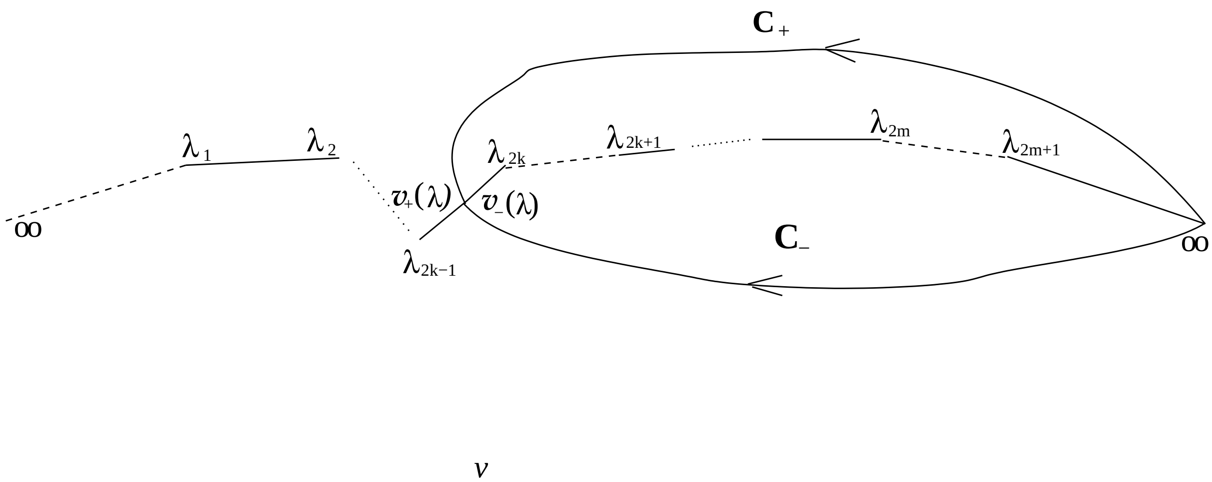

FIGURE 3. The different paths for $\boldsymbol{v}_-(\lambda)$ and $\boldsymbol{v}_+(\lambda)$

derived:

$$\left[J^{(s-1)}\boldsymbol{v}_-(\lambda) - J^{(s)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2k-1},\lambda_{2k}]} = \sum_{j=k}^{m}\left(\oint\limits_{\beta_{j+(s-1)m}} \mathrm{d}\boldsymbol{v} - \oint\limits_{\beta_{j+sm}} \mathrm{d}\boldsymbol{v}\right) \tag{5.11}$$

for $s = 1, \ldots, N-2$ and $k = 1, \ldots, m$,

$$\left[J^{(N-2)}\boldsymbol{v}_-(\lambda) - J^{(N-1)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2k-1},\lambda_{2k}]} = \sum_{j=k}^{m}\oint\limits_{\beta_{j+(N-2)m}} \mathrm{d}\boldsymbol{v}, \tag{5.12}$$

$$\left[J^{(N-1)}\boldsymbol{v}_-(\lambda) - \boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2k-1},\lambda_{2k}]} = -\sum_{j=k}^{m}\oint_{\beta_j} \mathrm{d}\boldsymbol{v} \tag{5.13}$$

for $\quad k = 1, \ldots, m$ and

$$\left[J^{(s-1)}\boldsymbol{v}_-(\lambda) - J^{(s)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2m+1},\infty]} = 0, \quad s = 1, \ldots, N. \tag{5.14}$$

In the same way we obtain

$$\left[J^{(s)}\boldsymbol{v}_-(\lambda) - J^{(s)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2k},\lambda_{2k+1}]} = \oint_{\alpha_{k+sm}} \mathrm{d}\boldsymbol{v}, \;\; s = 0, \ldots, N-2, \tag{5.15}$$

$$\left[J^{(N-1)}\boldsymbol{v}_-(\lambda) - J^{(N-1)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\lambda_{2k},\lambda_{2k+1}]} = -\sum_{s=0}^{N-2}\oint_{\alpha_{k+sm}} \mathrm{d}\boldsymbol{v}, \tag{5.16}$$

for $\quad k = 1, \ldots, m$ and

$$\left[J^{(s)}\boldsymbol{v}_-(\lambda) - J^{(s)}\boldsymbol{v}_+(\lambda)\right]\Big|_{[\infty,\lambda_1]} = 0, \quad s = 0, \ldots, N-1. \tag{5.17}$$

Now let us suppose that $\lambda \in [\lambda_{2k-1}, \lambda_{2k}]$. Then for $s = 1, \dots, N-2$ and $r = 1, \dots, N$ we have

$$
\begin{aligned}
(Y_-(\lambda))_{rs} &= (X_-(\lambda))_{rs} \frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v}_-;\Pi\right)}{\theta\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v}_-;\Pi\right)} \frac{\theta(\mathbf{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)} \\
&= (X_+(\lambda)\mathcal{P}_N)_{rs} \frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}\left(\int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+ + \int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v}_- - \int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+;\Pi\right)}{\theta\left(\int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+ \int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v}_- - \int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+;\Pi\right)} \frac{\theta(\mathbf{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)} \\
&= (X_+(\lambda)\mathcal{P}_N)_{rs} \frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}\left(\int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+;\Pi\right)}{\theta\left(\int_{P_0^{(r)}}^{P^{(s+1)}} \mathrm{d}\boldsymbol{v}_+;\Pi\right)} \frac{\theta(\mathbf{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)} \times \begin{cases} \dfrac{\mathrm{e}^{\sum_{j=k}^m \epsilon_{j+sm}}}{\mathrm{e}^{\sum_{j=k}^m \epsilon_{j+(s-1)m}}}, & s = 1, \dots, N-2 \\ \mathrm{e}^{-\sum_{j=k}^m \epsilon_{j+(N-2)m}}, & s = N-1 \\ \mathrm{e}^{\sum_{j=k}^m \epsilon_j}, & s = N, \end{cases} \qquad (5.18)
\end{aligned}
$$

where the quantities $(Y_-(\lambda))_{rs}$, $(X_-(\lambda))_{rs}$, and $(X_-(\lambda)\mathcal{P}_N)_{rs}$ denote the r, s entry of the matrix $Y_-(\lambda)$, $X_-(\lambda)$, and $X_-(\lambda)\mathcal{P}_N$ respectively, and in the last identity we have used the relation (5.11)-(5.13) and the periodicity property (3.7) of the θ-function. From (5.18) it is immediate to verify that the constants $\epsilon_1, \dots \epsilon_{(N-1)m}$ and $c_1, \dots, c_{(N-1)m}$ are related by (5.3) if and only if the matrix $Y(\lambda)$ satisfies

$$Y_-(\lambda) = Y_+(\lambda)G_{2k-1}, \quad \lambda \in [\lambda_{2k-1}, \lambda_{2k}],\ k = 1, \dots, m+1,\ \lambda_{2m+2} = \infty,$$

where the matrix G_{2k-1} has been define in (1.2). Repeating the same procedure for $\lambda \in [\lambda_{2k}, \lambda_{2k+1}]$ and using (5.15)-(5.17) and the periodicity properties (3.6) of the θ-function, we derive (5.4) if and only if the matrix valued function $Y(\lambda)$ satisfies

$$Y_-(\lambda) = Y_+(\lambda)G_{2k}, \quad \lambda \in [\lambda_{2k}, \lambda_{2k+1}],\ k = 0, \dots, m,\ \lambda_0 = \infty$$

where G_{2k} has been define in (2.2). We conclude that the matrix (5.5) satisfies the R-H problem (2.4)-(2.5). ☐

The form of the solution (5.5) and the Z_N symmetry of the curve $\mathcal{C}_{N,m}$ enable us to prove the following.

Proposition 5.2. *Let $\boldsymbol{\delta}_N$, $\boldsymbol{\epsilon}_N \in (\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$ be the characteristics associated to the non-singular divisor $\mathcal{D}_l$ supported on the branch points, that is*

$$\boldsymbol{\epsilon}_N + \boldsymbol{\delta}_N \Pi = \sum_{i_l} s_{i_l} \int_\infty^{P_{i_l}} \mathrm{d}\boldsymbol{v} - \boldsymbol{K}_\infty, \quad \sum_{i_l} s_{i_l} = (N-1)m, \qquad (5.19)$$

where $i_l \in \{1, 2, \dots, 2m+1\}$ and $\boldsymbol{K}_\infty$ is the vector of Riemann constants (4.23). Then the matrix $Y(\lambda)$ with entries

$$Y_{rs}(\lambda) = X_{rs}(\lambda) \frac{\theta\begin{bmatrix}\boldsymbol{\delta}_N\\ \boldsymbol{\epsilon}_N\end{bmatrix}\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v};\Pi\right)} \frac{\theta(\mathbf{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}_N\\ \boldsymbol{\epsilon}_N\end{bmatrix}(\mathbf{0};\Pi)}, \quad r, s = 1, \dots, N, \qquad (5.20)$$

solves the R-H problem (2.4)-(2.5) with reducible monodromy representation (2.8).

Proof. Using the notations of Section 4.2 we write (5.19) in the form

$$\boldsymbol{\epsilon}_N + \boldsymbol{\delta}_N \Pi = \sum_{i_l} s_{i_l}[\mathfrak{U}_{i_l}] - (N-1)\sum_{l=1}^{m}[\mathfrak{U}_{2l}], \tag{5.21}$$

where $[\mathfrak{U}_k]$ are the characteristics defined in Sect.4.2. Then we associate to the characteristics $[\mathfrak{U}_k]$, the constants $c_j,\ d_j,\ j=1,\dots,(N-1)m$ according to the rule

$$[\mathfrak{U}_1] \longleftrightarrow c_j = 1, \quad d_j = \mathrm{e}^{-\frac{2\pi\imath}{N}},\ j=1,\dots(N-1)m;$$

for $k=2,\dots,m$

$$\begin{aligned}[\mathfrak{U}_{2k-1}] \longleftrightarrow c_{k-1+sm} &= \mathrm{e}^{\frac{2\pi\imath(s+1)}{N}},\ \ c_{j+sm}=1,\ j\neq k-1,\ j=1,\dots,m,\\ d_{j+sm} &= \mathrm{e}^{-\frac{2\pi\imath}{N}},\ k\le j\le m,\ \ d_{j+sm}=1,\ 1\le j<k,\ \ s=0,\dots,N-2;\end{aligned}$$

for $k=1,\dots,m$

$$\begin{aligned}[\mathfrak{U}_{2k}] \leftrightarrow c_{k+sm} &= \mathrm{e}^{\frac{2\pi\imath(s+1)}{N}},\ \ c_{j+sm}=1,\ j\neq k,\ j=1,\dots,m,\\ d_{j+sm} &= \mathrm{e}^{-\frac{2\pi\imath}{N}},\ k\le j\le m,\ \ d_{j+sm}=1,\ 1\le j<k,\ \ s=0,\dots,N-2.\end{aligned}$$

Combining the above relations it is possible to verify that the monodromy representation associated to the non-singular characteristics (5.21) is

$$\left\{ M_k = \frac{\prod_{i_{l_k}} \xi_{i_{l_k}}^{s_{i_l}}}{\prod\limits_{n_k \text{even}} \xi_{n_k}} \mathcal{P}^{(-1)^{k-1}},\ k=1,\dots,2m+1,\ M_\infty = \mathcal{P}_N^{-1},\quad \prod_{k=1}^{2m+1} \frac{\prod_{i_{l_k}} \xi_{i_{l_k}}^{s_{i_l}}}{\prod\limits_{n_k \text{even}} \xi_{n_k}} = 1,\right.$$
$$\left. \xi_{i_{l_k}}, \xi_{n_k} \in \{1, \mathrm{e}^{\frac{2\pi\imath}{N}}, \dots, \mathrm{e}^{\frac{2\pi\imath(N-1)}{N}}\},\quad \sum_{i_l} s_{i_l} = (N-1)m \right\}.$$

According to remark 2.1, the above monodromy representation is reducible. □

Remark 5.3. *The solution (2.12) of the Schlesinger equations obtained from the solution (5.20) of the R-H problem, with non-singular $1/N$ characteristics supported on the branch points, is constant. This statement can be directly verified for some class of $1/N$ characteristics, using the explicit expression of the corresponding Szegö kernel (4.39). For general non-singular $1/N$ characteristics the statement follows from a result of Dubrovin and Mazzocco* [20]*, which in our case says that the matrix entries of the solution of the Schlesinger system for non-singular $1/N$ characteristics are rational functions of the matrix entries of the solution of the Schlesinger system with zero characteristics. Since the solution of the Schlesinger systems with zero characteristics is constant and given in (5.2), it follows that the solution of the Schlesinger system derived from (5.20) is constant as well.*

In the following we consider a couple of divisors whose difference is a non-singular divisor supported on the branch points. If $\left[{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}\right]$ is a non-singular characteristics corresponding to a divisor in general position and if $\boldsymbol{\epsilon}_N, \boldsymbol{\delta}_N \in (\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$ is a characteristics corresponding to the non-special divisors $\mathcal{D}_l$ defined in (4.27), then $\left[{\boldsymbol{\delta}+\boldsymbol{\delta}_N \atop \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_N}\right]$ is a non-singular characteristics. Indeed the characteristics $\left[{\boldsymbol{\delta}+\boldsymbol{\delta}_N \atop \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_N}\right]$ correspond to a divisor of degree $2g$. But all divisors $\mathcal{D}$ of degree $\deg\mathcal{D} > 2g-2$ are non-special [32].

Theorem 5.4. *Let $\left[{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}\right]$ and $\left[{\boldsymbol{\delta}+\boldsymbol{\delta}_N \atop \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_N}\right]$ be as above. Then the entries $Y_{rs}(\lambda)$ and $\widetilde{Y}_{rs}(\lambda)$ of the solutions $Y(\lambda)$ and $\widetilde{Y}(\lambda)$ of the R-H problem (2.3)-(2.5) with characteristics $\left[{\boldsymbol{\delta} \atop \boldsymbol{\epsilon}}\right]$ and $\left[{\boldsymbol{\delta}+\boldsymbol{\delta}_N \atop \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_N}\right]$, are equivalent up to an algebraic transformation. The monodromy representation $\mathcal{M} = \{M_1, M_2, \dots, M_{2m+1}, M_\infty\}$ and*

$\widetilde{\mathcal{M}} = \{\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_{2m+1}, \widetilde{M}_\infty\}$ *associated to the solutions* $Y(\lambda)$ *and* $\widetilde{Y}(\lambda)$ *respectively, are equivalent up to multiplication by* N*th roots of unity. That is* $\widetilde{M}_k = e^{\frac{2\pi\imath}{N}j_k}M_k$, j_k *integer,* $\sum_{k=1}^{2m+1} j_k = 0 \, mod N$.

Proof. If $\boldsymbol{\delta}_N$, $\boldsymbol{\epsilon}_N \in (\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$ then, by (3.6), (3.7), the ratio

$$\mathcal{F}(P^{(s)}, P_0^{(r)}) := \left(\frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}+\boldsymbol{\delta}_N\\ \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_N\end{smallmatrix}\right]\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]\left(\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v};\Pi\right)} \right)^N \tag{5.22}$$

is a single-valued function on $\mathcal{C}_{N,m}$ in both the arguments $P^{(s)}$ and $P_0^{(r)}$. Hence $\mathcal{F}(P^{(s)}, P_0^{(r)})$ is a meromorphic function. This means that $\mathcal{F}(P^{(s)}, P_0^{(r)})$ is a rational expression in $\lambda, y, \lambda_0, y_0$, therefore $\sqrt[N]{\mathcal{F}(P^{(s)}, P_0^{(r)})}$ is algebraic. Hence, from (5.5) and (5.22)

$$\widetilde{Y}_{rs}(\lambda) = \sqrt[N]{\mathcal{F}(P^{(s)}, P_0^{(r)})}Y_{rs}(\lambda), \quad r, s = 1, \ldots, N,$$

which is the first statement of the theorem. The equivalence of the corresponding monodromy representation $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ up to multiplication by Nth roots of unity, follows from the proof of proposition 5.2. □

Example 5.5. *We consider the case* $N = 3$ *and* $m = 1$ *when the Riemann surface* $\mathcal{C}_{3,1} : \{(\lambda, y), \quad y^3 = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)\}$ *is of genus* 2. *Let* $\boldsymbol{\epsilon}$ *and* $\boldsymbol{\delta}$ *be a non-singular characteristics. We consider the non-singular characteristic* $\boldsymbol{\epsilon}_3, \boldsymbol{\delta}_3$ *supported on the branch points given by*

$$\boldsymbol{\epsilon}_3 + \boldsymbol{\delta}_3\Pi = 2[\boldsymbol{\mathfrak{U}}_3] - 2[\boldsymbol{\mathfrak{U}}_2] = \left(-\frac{4}{3}, \frac{2}{3}\right)\Pi,$$

as follows from the relations derived in Section (4.2). If $\{M_1, M_2, M_3, M_\infty\}$ *are the monodromy matrices associated to the characteristics* $\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]$, *the monodromy matrices associated to the characteristics* $\left[\begin{smallmatrix}\boldsymbol{\delta}+\boldsymbol{\delta}_3\\ \boldsymbol{\epsilon}+\boldsymbol{\epsilon}_3\end{smallmatrix}\right]$ *are*

$$\{M_1, e^{\frac{4\pi\imath}{3}}M_2, e^{\frac{2\pi\imath}{3}}M_3, M_\infty\}.$$

5.1. **τ-function of the Schlesinger equations.** From the solution of the R-H problem one can derive the τ function for the Schlesinger system defined by (2.15).

Theorem 5.6. *The* τ*-function for the Schlesinger system reads*

$$\tau(\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}) = \frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi)}{\theta(\mathbf{0};\Pi)} \frac{\prod_{\substack{k<i\\ i,k=0}}^{m}(\lambda_{2k+1} - \lambda_{2i+1})^{\frac{N^2-1}{6N}} \prod_{\substack{k<i\\ k,i=1}}^{m}(\lambda_{2k} - \lambda_{2i})^{\frac{N^2-1}{6N}}}{\prod_{\substack{i<j\\ i,j=1}}^{2m+1}(\lambda_i - \lambda_j)^{\frac{N^2-1}{12N}}}. \tag{5.23}$$

Proof. We define the τ-function by the formula (2.15), where $Y(\lambda)$ is the solution (5.5) of the R-H problem It follows from the definition that the τ-function does not depend on the normalisation point λ_0. In order to obtain the explicit expression of the residue in the r.h.s. of (5.27), we use the relation obtained in [3], namely

$$\frac{1}{2}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\mathrm{Tr}\left(\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y(\lambda)^{-1}\right)^2\right\} = \frac{\partial}{\partial\lambda_k}\log\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi) - \operatorname*{Res}_{P=(\lambda_k,0)}\left\{\sum_{\substack{r<s\\ r,s=1}}^{N}\frac{\mathrm{d}\omega(P^{(r)}, P^{(s)})}{(\mathrm{d}z(P))^2}\right\}, \tag{5.24}$$

where $\mathrm{d}\omega(P,Q)$ is the Bergmann kernel and $P^{(s)}$, is on the s-th sheet of $\mathcal{C}_{N,m}$, $s=1,\dots,N$. Since

$$\operatorname*{Res}_{P=(\lambda_k,0)}\left\{\sum_{\substack{r<s\\ r,s=1}}^{N}\frac{\mathrm{d}\omega(P^{(r)},P^{(s)})}{(\mathrm{d}z(P))^2}\right\}=-\frac{1}{2}\operatorname*{Res}_{P=(\lambda_k,0)}\left\{\sum_{s=1}^{N}\frac{\mathrm{d}\omega(P^{(s)},P^{(s)})}{(\mathrm{d}z(P))^2}\right\},$$

we can write (5.24) in the form

$$\frac{1}{2}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\mathrm{Tr}\left(\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y(\lambda)^{-1}\right)^2\right\}=\frac{\partial}{\partial\lambda_k}\log\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)+\frac{1}{2}\operatorname*{Res}_{P=(\lambda_k,0)}\left\{\sum_{s=1}^{N}\frac{\mathrm{d}\omega(P^{(s)},P^{(s)})}{(\mathrm{d}z(P))^2}\right\}. \tag{5.25}$$

From the identity (3.26) and the expansion (4.41) we express (5.25) in the form

$$\frac{1}{2}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\mathrm{Tr}\left(\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y(\lambda)^{-1}\right)^2\right\}=\frac{\partial}{\partial\lambda_k}\log\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)+\frac{N^2-1}{24N}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right]\right\}^2-$$
$$-\frac{1}{2}\sum_{i,j=1}^{g}\frac{\partial^2}{\partial z_i\partial z_j}\log\theta(\mathbf{0};\Pi)\left(\operatorname*{Res}_{P=(\lambda_k,0)}\sum_{s=1}^{N}\left\{\mathrm{d}v_i(P^{(s)})\mathrm{d}v_j(P^{(s)})\right\}\right).$$

Using the relation (3.4), the heat equation (3.12) and the property (3.11), the above formula can be reduced to the form

$$\begin{aligned}\frac{1}{2}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\mathrm{Tr}\left(\frac{\mathrm{d}Y(\lambda)}{\mathrm{d}\lambda}Y(\lambda)^{-1}\right)^2\right\}=&\frac{\partial}{\partial\lambda_k}\log\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)+\frac{N^2-1}{24N}\operatorname*{Res}_{\lambda=\lambda_k}\left\{\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\frac{p(\lambda)}{q(\lambda)}\right]\right\}^2-\\ &-\frac{\partial}{\partial\lambda_k}\log\theta(\mathbf{0};\Pi).\end{aligned} \tag{5.25}$$

From the above formula, we can easily obtain the τ function defined in (5.23). □

We remark that the formula for the τ-function obtained in [3] for the case of a general N-sheeted Riemann surface reads $\tau(\lambda_1,\dots,\lambda_{2m+1})=F(\lambda_1,\dots,\lambda_{2m+1})\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)$, where the function F does not depend on the characteristics $\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}$ and depends only on the projective connection of the Riemann surface. In the formula (5.23) the term derived from the projective connection is explicitly evaluated.

The set of zeros of the τ-function in the space of singularities of the R-H problem, that is the set

$$\{(\lambda_1,\dots\lambda_{2m+1}),\ \ \lambda_i\neq\lambda_j,\ \ \ i,j=1,\dots,2m+1,\ s.t.\ \tau(\lambda_1,\dots,\lambda_{2m+1})=0\}$$

is called the Malgrange divisor (θ) [26]. From the expression (5.23) it follows that the τ-function vanishes when

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0};\Pi)=0,$$

that is when $\Pi\boldsymbol{\delta}+\boldsymbol{\epsilon}\notin(\Theta)$. Therefore the Malgrange divisor (θ) coincides with the (Θ)-divisor [3].

The expression for the τ function can be written in a different form substituting the Thomae formula for the θ-constant.

Theorem 5.7. *The Thomae-type formula for $\theta(\mathbf{0};\Pi)$ reads*

$$\theta^8(\mathbf{0};\Pi)=\frac{\prod_{s=1}^{N-1}\det\mathcal{A}_s^4}{(2\pi)^{4(N-1)m}}\prod_{i<j}(\lambda_{2i}-\lambda_{2j})^{2(N-1)}\prod_{k<l}(\lambda_{2k+1}-\lambda_{2l+1})^{2(N-1)}, \tag{5.26}$$

where the matrices $\mathcal{A}_s$, $s=1,\dots,N-1$, are defined in (4.8).

The proof of the theorem is shown in the Appendix.

Remark 5.8. *We remark that the Thomae formulae for Z_N curve $y^N = \prod_{k=1}^{mN}(\lambda-\lambda_k)$, $\lambda_i \neq \lambda_j$ was discovered by Bershadsky and Radul* [35] *and rigorously proved by Nakayashiki* [36]*. The formula (5.26) is written for singular Z_N curves and it does not follow from the results in* [35, 36].

Combining (5.23) and (5.26) we have

$$\tau(\lambda_1,\lambda_2,\ldots,\lambda_{2m+1}) = \xi(2\pi)^{(N-1)\frac{m}{2}} \frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi)}{(\prod_{s=1}^{N-1}\det\mathcal{A}_s)^{\frac{1}{2}}} \prod_{\substack{i<j\\ i,j=1}}^{2m+1}(\lambda_i-\lambda_j)^{-\frac{N^2-1}{12N}} \times$$
$$\times \prod_{\substack{k<i\\ i,k=0}}^{m}(\lambda_{2k+1}-\lambda_{2i+1})^{-\frac{(N-1)(N-2)}{12}} \prod_{\substack{k<i\\ k,i=1}}^{m}(\lambda_{2k}-\lambda_{2i})^{-\frac{(N-1)(N-2)}{12}}, \tag{5.27}$$

where $\xi^8=1$. For $N=2$ (5.27) coincides with the expression derived in [2].

6. Example: $m=1$

Consider the class of curves (1.5) for $m=1$ and set without loosing generality $\lambda_1=0, \lambda_2=t$ and $\lambda_3=1$, $0<\mathrm{Re}\, t<1$

$$\mathcal{C}_{N,1}: \quad y^N = \lambda(\lambda-1)(\lambda-t)^{N-1}. \tag{6.1}$$

The curve (6.1) is a non-ramified covering over the hyperelliptic curve

$$\mathcal{C}_{\mathrm{hyperel}}: \quad w^2 = \xi^{2N} + 2(1-2t)\xi^N + 1. \tag{6.2}$$

The coordinates of the cover $\psi: \mathcal{C}_{N,1} \to \mathcal{C}_{\mathrm{hyperel}}$ are

$$\xi = \frac{y}{\lambda-t}, \quad w = \frac{\lambda^2-2\lambda t+t}{\lambda-t}. \tag{6.3}$$

The canonical holomorphic differentials of both curves,

$$\mathrm{d}u_k(\lambda,y) = \frac{(\lambda-t)^{k-1}}{y^k}\mathrm{d}\lambda, \quad \text{and} \quad \mathrm{d}U_k(\xi,w) = \xi^{k-1}\frac{\mathrm{d}\xi}{w}, \quad k=1,\ldots,N-1 \tag{6.4}$$

are linked under the action of ψ as

$$\mathrm{d}u_k(\lambda,y) = N\mathrm{d}U_{N-k}(\xi,w). \tag{6.5}$$

The curve (6.2) admits two automorphisms $f_\pm$ of order two different from the hyperelliptic involution $\mathcal{I}$:

$$f_+(\xi,w) = \left(\frac{1}{\xi},\frac{w}{\xi^N}\right), \quad f_-(\xi,w) = (f_+\circ\mathcal{I})(\xi,w) = \left(\frac{1}{\xi},-\frac{w}{\xi^N}\right).$$

We observe that the automorphism group of the surfaces (6.2) is

$$\{Id,\mathcal{I},f_+,J\}, \quad J(\xi,w) = (\mathrm{e}^{\frac{2\pi\imath}{N}},w).$$

The quotient of the automorphism group by $\{Id,\mathcal{I}\}$ is isomorphic to the dihedral group D_N of symmetries of the N-sided regular polygon. For $N=3$ this result was pointed out in [37].

For N odd each of the maps f_+ and f_- fixes exactly two points of $\mathcal{C}_{\mathrm{hyperel}}$. The automorphism f_+ fixes the two points were $\xi=1$ while f_- fixes the two points were $\xi=-1$. According to Riemann-Hurwitz formula the quotient surfaces:

$$\mathcal{C}_\pm = \mathcal{C}_{\mathrm{hyperel}}/\{Id,f_\pm\}$$

have genus equal to $\frac{N-1}{2}$.

For N even, the map f_+ fixes the four points were $\xi=\pm1$ while f_- has no fixed points. Therefore the quotient surfaces $\mathcal{C}_\pm = \mathcal{C}_{\mathrm{hyperel}}/\{Id,f_\pm\}$ have genus equal to $\frac{N}{2}-1$ and $\frac{N}{2}$ respectively.

The Jacobian varieties $\mathrm{Jac}(\mathcal{C}_{N,1})$ and $\mathrm{Jac}(\mathcal{C}_+)\times\mathrm{Jac}(\mathcal{C}_-)$ are complex tori of dimension $N-1$. Following [38] it is possible to show that these tori are isomorphic. From the factorisation of the Jacobian variety $\mathrm{Jac}(\mathcal{C}_{\mathrm{hyperel}})$, the θ-functions defined on the surface of genus $N-1$ can be expressed in terms of θ-functions defined on two surfaces of genus $\frac{N-1}{2}$ for N odd and of genus $N/2-1$ and $N/2$ for N even. The procedure for obtaining the period matrix of the two quotient surfaces and the factorisation of the θ-function is illustrated, for automorphisms of order two, in [39] and [30].

We are going to study in detail the case $N=3$ when θ-functions decomposes as product of Jacobi's ϑ-functions.

Remark 6.1. *We remark that the curves of the form (4.1) are hyperelliptic only in the case $m=1$.*

6.1. **Decomposition of two-dimensional Jacobian to elliptic curves: $N=3$ and $m=1$.** We restrict ourselves to the curve

$$\mathcal{C}_{3,1}: \quad y^3=\lambda(\lambda-1)(\lambda-t)^2. \tag{6.6}$$

Its holomorphic differentials are

$$\mathrm{d}u_1(\lambda,y)=\frac{\mathrm{d}\lambda}{y},\quad \mathrm{d}u_2(\lambda,y)=\frac{(\lambda-t)\mathrm{d}\lambda}{y^2}.$$

The matrices of α and β-periods in the homology basis given on the Figure 3, are

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}_1 & \rho^2\mathcal{A}_1\\ \mathcal{A}_2 & \rho\mathcal{A}_2\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}\mathcal{B}_1 & -\rho\mathcal{B}_1\\ \mathcal{B}_2 & -\rho^2\mathcal{B}_2\end{pmatrix},\quad \rho=\mathrm{e}^{\frac{2\pi\imath}{3}}, \tag{6.7}$$

where $\oint_{\alpha_1}\mathrm{d}u_i=\mathcal{A}_i$, $\oint_{\beta_1}\mathrm{d}u_i=\mathcal{B}_i$, $i=1,2$. The relation

$$\frac{\mathcal{A}_1}{\mathcal{A}_2}=-\rho^2\frac{\mathcal{B}_1}{\mathcal{B}_2}, \tag{6.8}$$

follows from the bilinear Riemann relations. Writing explicitly the integrals $\mathcal{A}_i$ and $\mathcal{B}_i$ it is immediate to verify the identities (see below (6.24)-(6.25))

$$\mathcal{A}_1=-\rho^2\mathcal{A}_2,\quad \mathcal{B}_1=\mathcal{B}_2. \tag{6.9}$$

From (6.7) and (6.9) we obtained the normalized holomorphic differentials

$$\mathrm{d}v_1=\frac{1}{\mathcal{A}_1(1-\rho)}(\mathrm{d}u_1+\mathrm{d}u_2),\quad \mathrm{d}v_2=\frac{1}{\mathcal{A}_1(1-\rho^2)}(\rho\mathrm{d}u_1+\mathrm{d}u_2). \tag{6.10}$$

From (6.7) and (6.8), the Riemann Π-matrix has the form

$$\Pi=\begin{pmatrix}2T & T\\ T & 2T\end{pmatrix},\quad \mathrm{Im}\,T>0, \tag{6.11}$$

where

$$T=\frac{1}{1-\rho}\frac{\mathcal{B}_1}{\mathcal{A}_1}. \tag{6.12}$$

An element of $\pi_1(\mathbb{C}\backslash\{0,1,\infty\},t)$, where t is the base point, induces an automorphism of $H_1(\mathcal{C}_{3,1},\mathbb{Z})$ which preserves the intersection matrix of $(\alpha_1,\alpha_2;\beta_1,\beta_2)\in H_1(\mathcal{C}_{3,1},\mathbb{Z})$, so it belongs to $\mathrm{Sp}(4,\mathbb{Z})$ [40]. The induced action on the Π matrix is the following

$$\Pi=\begin{pmatrix}2T & T\\ T & 2T\end{pmatrix}\longrightarrow\widetilde{\Pi}=\begin{pmatrix}2\widetilde{T} & \widetilde{T}\\ \widetilde{T} & 2\widetilde{T}\end{pmatrix}, \tag{6.13}$$

where

$$\widetilde{T}=T\,\mathrm{mod}\,\Gamma_0(3),$$

with $\Gamma_0(3)$ the subset of the modular group defined by the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, $c = 0 \bmod 3$. Therefore T belongs to Siegel half space $\mathcal{H}_1$ modulo the group $\Gamma_0(3)$ [41].

The curve $\mathcal{C}_{3,1}$ covers the hyperelliptic curve of genus two

$$\mathcal{C}_{\text{hyperel}}: \quad w^2 = \xi^6 + 2(1-2t)\xi^3 + 1. \tag{6.14}$$

The moduli of curves of genus two form a 3-dimensional variety that was first described by Igusa [42],[43]. Bolza [37], Igusa [44] and Lange [45] have classified the curves of genus two with automorphism and in particular the curves with involutions. The moduli of such a curves describe a 2-dimensional variety of the moduli space. The generic point of this space is described by curves with reduced group of automorphism $\mathbb{Z}/2\mathbb{Z}$. The automorphism group of the curve (6.14) is generated by $\{Id, \mathcal{I}, J, f_+\}$ where now $J(\xi,w) = (e^{\frac{2\pi\imath}{3}}\xi, w)$ and

$$f_+(\xi,w) = \left(\frac{1}{\xi}, \frac{w}{\xi^3}\right).$$

The reduced group of automorphism is isomorphic to the dihedral group D_3 and such curves describes a one-dimensional variety in the moduli space of curves of genus two. The quotient surfaces $\mathcal{C}_\pm = \mathcal{C}_{\text{hyperel}}/\{Id, f_\pm\}$ with $f_- = f_+ \circ \mathcal{I}$, are elliptic surfaces.

We are going to construct the covering maps $\phi_\pm := h_\pm \circ \psi$ [30],[39],[45],[46]:

$$\mathcal{C}_{N,1} \xrightarrow{\psi} \mathcal{C}_{\text{hyperel}} \xrightarrow{h_\pm} \mathcal{C}_\pm$$

where ψ is defined in (6.3). Let a_1, a_2, b_1, b_2 be the canonical homology basis defined on $\mathcal{C}_{\text{hyperel}}$ so that $f_+(a_1) = a_2$ and $f_+(b_1) = b_2$. Then $f_-(a_1) = -a_2$ and $f_-(b_1) = -b_2$.

It is easy to verify that $a_1 = \psi(\alpha_1)$, $a_2 = -\psi(\alpha_1) - \psi(\alpha_2)$ and $b_1 = \psi(\beta_1) - \psi(\beta_2)$, $b_2 = -\psi(\beta_2)$, where $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is the canonical homology basis defined on $\mathcal{C}_{3,1}$. We fix $h_\pm(a_1) = \alpha_\pm$ and $h_\pm(b_1) = \beta_\pm$ where $\{\alpha_\pm, \beta_\pm\}$ is the canonical homology basis on $\mathcal{C}_\pm$ respectively. It follows that $h_\pm(a_2) = \pm\alpha_\pm$ and $h_\pm(b_2) = \pm\beta_\pm$ so that

$$\phi_+(\alpha_1) = \alpha_+,\ \phi_+(\alpha_2) = -2\alpha_+,\ \ \phi_+(\beta_1) = 0,\ \ \phi_+(\beta_2) = -\beta_+, \tag{6.15}$$

$$\phi_-(\alpha_1) = \alpha_-,\ \phi_+(\alpha_2) = 0,\ \ \phi_+(\beta_1) = 0,\ \ \phi_+(\beta_2) = \beta_-. \tag{6.16}$$

The action of $f_\pm$ on the holomorphic differentials $\mathrm{d}U_k$, $k = 1, 2$, of the curve $\mathcal{C}_{\text{hyperel}}$ is given by

$$f_\pm(\mathrm{d}U_1(\xi,w)) = \mp\mathrm{d}U_2(\xi,w), \quad f_\pm(\mathrm{d}U_2(\xi,w)) = \mp\mathrm{d}U_1(\xi,w).$$

Therefore the differentials $\mathrm{d}U_1 \mp \mathrm{d}U_2$ are invariant under the action of $f_\pm$. It follows that the differentials $\mathrm{d}U_1 \mp \mathrm{d}U_2 = \dfrac{1}{3}(\mathrm{d}u_2 \mp \mathrm{d}u_1)$ project to the holomorphic differentials of the curve $\mathcal{C}_\pm$. From the above considerations and from (6.10) we conclude that

$$\mathrm{d}v_+ = \mathrm{d}v_1 - 2\mathrm{d}v_2, \quad \mathrm{d}v_- = \mathrm{d}v_1, \tag{6.17}$$

where $\mathrm{d}v_\pm$ are the holomorphic differentials of the curve $\mathcal{C}_\pm$. From the relations (6.15) and (6.16) we deduce that $\mathrm{d}v_\pm$ are the normalized holomorphic differentials of $\mathcal{C}_\pm$ with periods

$$\oint_{\beta_-} \mathrm{d}v_- = T, \quad \oint_{\beta_+} \mathrm{d}v_+ = 3T.$$

Therefore the elliptic curves $\mathcal{C}_\pm$ are 3-isogenous and isomorphic to the two tori $\mathbb{C}/(1, 3T)$ and $\mathbb{C}/(1, T)$ respectively. Let us write the equations of the two elliptic curves $\mathcal{C}_\pm$ in the Legendre form:

$$\mathcal{C}_\pm: \quad z_\pm^2 = \eta(1-\eta)(1-k_\pm^2\eta).$$

The Jacobi's moduli $k_\pm$ are related by a third order transformation and are parametrised as

$$k_-^2 = \frac{1}{16p}(p+1)^3(3-p), \quad k_+^2 = \frac{1}{16p^3}(p+1)(3-p)^3, \tag{6.18}$$

where the parameter p can be expressed in terms of ϑ-constants (see e.g. [47])

$$p = \frac{3\vartheta_3^2(0;3T)}{\vartheta_3^2(0;T)}. \tag{6.19}$$

The holomorphic differentials $\mathrm{d}v_\pm$ reads

$$\begin{gathered}\mathrm{d}v_\pm = \frac{1}{4K_\pm}\frac{\mathrm{d}\eta}{z_\pm} \\ K_+ = \frac{\pi}{2}\vartheta_3^2(0;3T) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1,k_+^2\right), \quad K_- = \frac{\pi}{2}\vartheta_3^2(0;T) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1,k_-^2\right),\end{gathered} \tag{6.20}$$

where $F(\alpha,\beta,\gamma;Z)$ is the standard hypergeometric function. From the relation (6.17) we construct the coordinates of the covers $\phi_\pm : \mathcal{C}_{3,1} \to \mathcal{C}_\pm$

$$\begin{aligned}\eta &= \frac{p^2(y-\lambda+t)^2 + 3(y+\lambda-t)^2}{p^2k_+^2(y-\lambda+t)^2 + 3k_-^2(y+\lambda-t)^2}\\ z_\pm &= \frac{\mathcal{A}_1(1\pm\rho)}{4K_\pm}\frac{y^2}{y\mp(\lambda-t)}\frac{d\eta}{d\lambda}.\end{aligned}$$

Constructing the covering maps $\phi_\pm = h_\pm \circ \psi$ as in [37], [14],[48], by mapping the branch points

$$\xi_0 = \sqrt[3]{2t-1+\sqrt{t^2-t}},\ \rho\xi_0, \quad \rho^2\xi_0,\ \frac{1}{\xi_0},\ \frac{\rho}{\xi_0},\ \frac{\rho^2}{\xi_0},$$

of the hyperelliptic curve $\mathcal{C}_{hperell}$ to $(0,0),\ (\infty,\infty),\ (1,0) \in \mathcal{C}_\pm$ according to the rule

$$((\xi_0)^{\pm 1},0) \to (0,0), \quad ((\rho\xi_0)^{\pm 1},0) \to (\infty,\infty), \quad ((\rho^2\xi_0)^{\pm 1},0) \to (1,0), \tag{6.21}$$

we derive the algebraic dependence of the parameter p on t, that is

$$t = \frac{p^2(p^2-9)^2}{(p^2+3)^3}, \tag{6.22}$$

and we deduce that

$$\mathcal{A}_1 = (1-\rho^2)\frac{p^2+3}{p^{\frac{3}{2}}}K_+ = \frac{\pi}{2}(1-\rho^2)\frac{p^2+3}{p^{\frac{3}{2}}}F\left(\frac{1}{2},\frac{1}{2},1;k_+^2\right). \tag{6.23}$$

Namely the holomorphic integrals $\mathcal{A}_1$ and $\mathcal{B}_1 = T(1-\rho)\mathcal{A}_1$ defined on the trigonal curve $\mathcal{C}_{3,1}$ are reducible to elliptic integrals.

The equality (6.23) can be interpreted as the hypergeometric equalities of Goursat [49] employed by Harnad and McKay[22] to describe the solution of the Halphen system in terms of automorphic functions for groups commensurable with the modular group. We are pointing here the link between higher order Goursat hypergeometric identities and the reduction of Abelian integrals to lower genera. By the definition of $\mathcal{A}_1$ we have

$$\begin{aligned}\mathcal{A}_1 &= \int_{\alpha_1}\mathrm{d}u_1 = (1-\rho^2)\int_0^t \frac{\mathrm{d}\xi}{\sqrt[3]{\xi(1-\xi)(t-\xi)^2}} = (1-\rho^2)\int_0^1 \frac{\mathrm{d}\xi}{\sqrt[3]{\xi(1-\xi)^2(1-t\xi)}} = \\ &= \frac{2\pi}{\sqrt{3}}(1-\rho^2)F\left(\frac{1}{3},\frac{2}{3},1;t\right) = \frac{2\pi}{\sqrt{3}}(1-\rho^2)F\left(\frac{2}{3},\frac{1}{3},1;t\right) = \\ &= -\rho^2(1-\rho)\int_t^0 \frac{(\xi-t)\mathrm{d}\xi}{\sqrt[3]{(\xi(\xi-1)(\xi-t)^2)^2}} = -\rho^2\int_{\alpha_1}\mathrm{d}u_2 = -\rho^2\mathcal{A}_2\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}\mathcal{B}_1 = \int_{\beta_1} \mathrm{d}u_1 = (1-\rho)\mathrm{e}^{-\frac{\pi\imath}{3}} \int_1^t \frac{\mathrm{d}\xi}{\sqrt[3]{\xi(1-\xi)(\xi-t)^2}} = \\ = (1-\rho)\rho \int_1^{\frac{1}{t}} \frac{\mathrm{d}\xi}{\sqrt[3]{\xi(\xi-1)^2(1-\xi t)}} = \frac{2\pi}{\sqrt{3}}(1-\rho)\rho F\left(\frac{1}{3},\frac{2}{3},1;1-t\right) = \mathcal{B}_2.\end{aligned} \tag{6.25}$$

For reducing (6.24) to an elliptic integral we use the identity

$$F\left(\frac{1}{3},\frac{2}{3},1;t\right) = \frac{\sqrt{3}}{4}\frac{p^2+3}{p^{\frac{3}{2}}} F\left(\frac{1}{2},\frac{1}{2},1;k_+^2\right), \tag{6.26}$$

where p and k_+^2 have been defined in (6.19) and (6.18) respectively. The equality (6.26) follows from the superposition of the following transformations:

- [49], (126), p. 140,

$$F\left(\frac{1}{3},\frac{2}{3},1;t\right) = \left(\frac{8t^2-36t+27}{27}\right)^{-\frac{1}{6}} F\left(\frac{1}{12},\frac{7}{12},1;\frac{64t^3(t-1)}{(8t^2-36t+27)^2}\right),$$

- [49] (118), p. 138

$$\begin{aligned}F\left(\frac{1}{4},\frac{3}{4},1;\frac{4k_+^2}{(1+k_+^2)^2}\right) = \left(\frac{2k_+^4-5k_+^2+2}{2(1+k_+^2)^2}\right)^{-\frac{1}{6}} \times \\ \times F\left(\frac{1}{12},\frac{7}{12},1;-27\left(\frac{(k_+^2-1)k_+^2}{(k_+^2+1)(k_+^2-2)(2k_+^2-1)}\right)^2\right),\end{aligned}$$

- [49] (35), p. 119

$$F\left(\frac{1}{2},\frac{1}{2},1;k_+^2\right) = \frac{1}{(1+k_+^2)^{\frac{1}{2}}} F\left(\frac{1}{4},\frac{3}{4},1;\frac{4k_+^2}{(1+k_+^2)^2}\right).$$

Substituting (6.26) in (6.24) we obtain (6.23).

Remark 6.2. *Combining (6.24) and (6.25) we deduce that the modulus T defined in (6.12) reads*

$$T = \frac{1}{(\rho^2-\rho)} \frac{F\left(\frac{1}{3},\frac{2}{3},1;1-t\right)}{F\left(\frac{1}{3},\frac{2}{3},1;t\right)}, \quad \rho = e^{\frac{2\pi\imath}{3}}, \tag{6.27}$$

where $F\left(\frac{1}{3},\frac{2}{3},1;1-t\right)$ and $F\left(\frac{1}{3},\frac{2}{3},1;t\right)$ are two independent solutions of the Gauss hypergeometric equation

$$t(1-t)F'' + (1-2t)F' - \frac{2}{9}F = 0.$$

For T belonging to Siegel half space $\mathcal{H}_1$ modulo the group $\Gamma_0(3)$ the above expression is invertible and the inverse function is given in (6.22) and reads

$$t = 27\vartheta_3^4(0;3T)\frac{(\vartheta_3^4(0;3T)-\vartheta_3^4(0;T))^2}{(3\vartheta_3^4(0;3T)+\vartheta_3^4(0;T))^3}. \tag{6.28}$$

One can show, by comparing q-expansions, that

$$\frac{1}{t(T/2)} = f(T),$$

where $f(T)$ is the automorphic function of $\Gamma_0(3)$ found in [22] *Table on p. 12,*

$$f(T) = 1 + \frac{1}{27}\frac{\eta(T)}{\eta(3T)},$$

and η is the Dedekind η-function.

In [23] *it is shown that the automorphic functions of the group $\Gamma_0(3)$ can be expressed in terms of the Rosenhain moduli*

$$\kappa = \frac{\theta^2\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & 0\end{bmatrix}(\mathbf{0};\Pi)}{\theta^2(\mathbf{0};\Pi)}, \quad \nu = \frac{\theta^2\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}(\mathbf{0};\Pi)}{\theta^2\begin{bmatrix}0 & 0\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}(\mathbf{0};\Pi)}, \quad \mu = \kappa\nu.$$

The function

$$\chi(T) = \frac{\nu^2(1-\nu)^2}{(1-\nu+\nu^2)^3} = \kappa^2(1-\kappa) = \mu(1-\mu)^2 = \frac{16}{27}t(1-t)$$

as a function of T is an automorphic function of the group $\Gamma_0(3)$. From the Thomae formula (5.26) and from (6.23) we derive the identities

$$\theta(\mathbf{0};\Pi) = F\left(\frac{1}{3},\frac{2}{3},1;t\right) = \frac{3\vartheta_3^4(0;3T)+\vartheta_3^4(0;T)}{4\vartheta_3(0;3T)\vartheta_3(0;T)}.$$

Using the above expression and the decomposition of genus two θ-function (see below) in terms of Jacobi's ϑ-functions it is possible to express the function $t(T)$ in (6.28) in terms of the Rosenhain moduli. The function $t(T)$ in (6.28) gives a solution of the Schwarzian equation [50, 51]

$$\{t,T\} + \frac{\dot{t}^2}{2}V(t) = 0, \quad \{t,T\} := \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2$$

where $\{\ ,\ \}$ is the Schwarzian derivative (1.19), $\dot{t} = \dfrac{dt}{dT}$ and the potential $V(t)$ is given by (see for example [50]*)*

$$V(t) = \frac{1-\beta^2}{t^2} + \frac{1-\gamma^2}{(t-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{t(t-1)}, \quad \alpha = \frac{1}{3},\ \beta=\gamma=0.$$

It follows that the functions

$$\omega_1 = -\frac{1}{2}\frac{d}{dT}\ln\frac{\dot{t}}{t(t-1)}, \quad \omega_2 = -\frac{1}{2}\frac{d}{dT}\ln\frac{\dot{t}}{t-1}, \quad \omega_3 = -\frac{1}{2}\frac{d}{dT}\ln\frac{\dot{t}}{t},$$

solve the general Halphen system

$$\begin{aligned}
\dot{\omega_1} &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + R,\\
\dot{\omega_2} &= \omega_1\omega_3 - \omega_2(\omega_1+\omega_3) + R,\\
\dot{\omega_3} &= \omega_1\omega_2 - \omega_1(\omega_1+\omega_2) + R,
\end{aligned}$$

where

$$R = \alpha^2(\omega_1-\omega_2)(\omega_3-\omega_1) + \beta^2(\omega_2-\omega_3)(\omega_1-\omega_2) + \gamma^2(\omega_3-\omega_1)(\omega_2-\omega_3).$$

When $R=0$ the above system coincides with the classical Halphen system. The solution of the classical and general Halphen system has been investigated by many authors [52, 22, 53, 54, 55]. *The expression (6.28) gives a formula for the solution of the general Halphen system with parameters $\alpha = \frac{1}{3}$, $\beta=\gamma=0$ equivalent to the one derived in* [22].

In the following we derive the decomposition of the genus two θ-functions in terms of Jacobi's ϑ-functions.

Lemma 6.3. *The θ-function of the curve $\mathcal{C}_{3,1}$ is decomposed in terms of Jacobi's ϑ-functions of the curves $\mathcal{C}^{\pm}$ as*

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(z_1,z_2;\Pi)=e^{\pi\imath\langle\boldsymbol{\delta},\Pi\boldsymbol{\delta}\rangle+2\pi\imath\langle\boldsymbol{z}+\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle}\left[\vartheta_3(e_1;6T)\vartheta_3(e_2;2T)+\vartheta_2(e_1;6T)\vartheta_2(e_2;2T)\right] \tag{6.29}$$

where

$$e_1=z_1+z_2+\epsilon_1+\epsilon_2+3T(\delta_1+\delta_2),\quad e_2=z_1-z_2+\epsilon_1-\epsilon_2+T(\delta_1-\delta_2).$$

Proof. By definition of θ-function we obtain

$$\begin{aligned}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(z_1,z_2;\Pi)=\mathrm{e}^{\pi\imath\langle\boldsymbol{\delta},\Pi\boldsymbol{\delta}\rangle+2\pi\imath\langle\boldsymbol{z}+\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle}\sum_{n_1,n_2\in\mathbb{Z}}\exp\left[\pi\imath\left(2T(n_1^2+n_2^2+n_1n_2)+\right.\right.\\ \left.\left.+3T(\delta_1+\delta_2)(n_1+n_2)+T(\delta_1-\delta_2)(n_1-n_2)+2(z_1+\epsilon_1)n_1+2(z_2+\epsilon_1)n_2\right)\right].\end{aligned}$$

Substituting in the above $m_1=n_1+n_2$ and $m_2=n_1-n_2$ where $m_i=2k_i+r,\ i=1,2,\ r=0,1$ we obtain

$$\begin{aligned}\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(z_1,z_2;\Pi)=\mathrm{e}^{\pi\imath\langle\boldsymbol{\delta},\Pi\boldsymbol{\delta}\rangle+2\pi\imath\langle\boldsymbol{z}+\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle}\sum_{r=0,1}\sum_{k_1,k_2\in\mathbb{Z}}\exp\Big[\pi\imath\Big(6T(k_1+\frac{r}{2})^2+2T(k_2+\frac{r}{2})^2+\\ +6T(\delta_1+\delta_2)(k_1+\frac{r}{2})+2T(\delta_1-\delta_2)(k_2+\frac{r}{2})+2(k_1+\frac{r}{2})(z_1+\epsilon_1+z_2+\epsilon_2)+\\ +2(k_2+\frac{r}{2})(z_1-z_2+\epsilon_1-\epsilon_2)\Big)\Big]=\\ =\mathrm{e}^{\pi\imath\langle\boldsymbol{\delta},\Pi\boldsymbol{\delta}\rangle+2\pi\imath\langle\boldsymbol{z}+\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle}\sum_{k=2}^{3}\vartheta_k(z_1+z_2+\epsilon_1+\epsilon_2+3T(\delta_1+\delta_2);\,6T)\times\\ \times\vartheta_k(z_1-z_2+\epsilon_1-\epsilon_2+T(\delta_1-\delta_2);\,2T),\end{aligned}$$

which is equivalent to (6.29). □

6.2. **Solution of the 3×3 matrix R-H problem with four singular points.** Let us consider the R-H problem with the four singular points $\lambda_1=0$, $\lambda_1=t$, $\lambda_3=1$, $\lambda_4=\infty$ and with monodromy matrices

$$\begin{aligned}M_0=\begin{pmatrix}0&0&c_1\\ \dfrac{c_2}{c_1}&0&0\\ 0&\frac{1}{c_2}&0\end{pmatrix},\quad M_t=\begin{pmatrix}0&\dfrac{c_1d_1}{c_2}&0\\ 0&0&c_2d_2\\ \dfrac{1}{c_1d_1d_2}&0&0\end{pmatrix},\\ M_1=\begin{pmatrix}0&0&d_1d_2\\ \dfrac{1}{d_1}&0&0\\ 0&\dfrac{1}{d_2}&0\end{pmatrix},\quad M_\infty=\begin{pmatrix}0&1&0\\ 0&0&1\\ 1&0&0\end{pmatrix},\end{aligned} \tag{6.30}$$

where c_1,c_2,d_1,d_2 are non-zero constants. The solution of this R-H problem is given in (5.5) and read

$$\begin{aligned}Y_{rs}(\lambda)=&\frac{1}{3}\left(\rho^{(s-r)}\sqrt[3]{\frac{\lambda(\lambda-1)}{\lambda-t}}\sqrt[3]{\frac{\lambda_0-t}{\lambda_0(\lambda_0-1)}}+1+\rho^{(r-s)}\sqrt[3]{\frac{\lambda-t}{\lambda(\lambda-1)}}\sqrt[3]{\frac{\lambda_0(\lambda_0-1)}{\lambda_0-t}}\right)\times\\ &\times\frac{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}\left(\int_{P_0^r}^{P^s}\mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left(\int_{P_0^r}^{P^s}\mathrm{d}\boldsymbol{v};\Pi\right)}\frac{\theta(\boldsymbol{0};\Pi)}{\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{0};\Pi)},\end{aligned} \tag{6.31}$$

with $\mathrm{d}\boldsymbol{v}$ and Π defined in (6.10) and (6.11) respectively and

$$\delta_i = \frac{1}{2\pi\imath}\log d_i, \quad \epsilon_i = \frac{1}{2\pi\imath}\log c_i, \quad i=1,2.$$

Using the reduction formula (6.29) it is possible to write the above solution in terms of Jacobi's ϑ-functions

$$\begin{aligned}
Y_{rs}(\lambda) = {} & \frac{1}{3}\left(\rho^{(s-r)}\sqrt[3]{\frac{\lambda(\lambda-1)}{\lambda-t}}\sqrt[3]{\frac{\lambda_0-t}{\lambda_0(\lambda_0-1)}} + 1 + \rho^{(r-s)}\sqrt[3]{\frac{\lambda-t}{\lambda(\lambda-1)}}\sqrt[3]{\frac{\lambda_0(\lambda_0-1)}{\lambda_0-t}}\right)\times \\
& \times \frac{\mathrm{e}^{2\pi\imath\langle \boldsymbol{z},\boldsymbol{\delta}\rangle}\vartheta_3(0;6T)\vartheta_3(0;2T)+\vartheta_2(0;6T)\vartheta_2(0;2T)}{\sum_{k=2}^3(\vartheta_k(\frac{1}{2\pi\imath}\log\frac{c_1}{c_2^2}-\frac{3T}{2\pi\imath}\log d_2;\ 6T)\vartheta_k(\frac{1}{2\pi\imath}\log c_1+\frac{T}{2\pi\imath}\log d_1^2d_2;\ 2T))}\times \\
& \times\left[\sum_{k=2}^{3}\vartheta_k\left(\int\limits_{\phi_+(P_0^r)}^{\phi_+(P^s)}\mathrm{d}v_+ + \frac{1}{2\pi\imath}\log\frac{c_1}{c_2^2}-\frac{3T}{2\pi\imath}\log d_2;\ 6T\right)\times\right. \\
& \left.\times\,\vartheta_k\left(\int\limits_{\phi_-(P_0^r)}^{\phi_-(P^s)}\mathrm{d}v_- + \frac{1}{2\pi\imath}\log c_1+\frac{T}{2\pi\imath}\log d_1^2d_2;\ 2T\right)\right]\times \\
& \times\left[\sum_{k=2}^{3}\vartheta_k\left(\int\limits_{\phi_+(P_0^r)}^{\phi_+(P^s)}\mathrm{d}v_+;\ 6T\right)\vartheta_k\left(\int\limits_{\phi_-(P_0^r)}^{\phi_-(P^s)}\mathrm{d}v_-;\ 2T\right)\right]^{-1},
\end{aligned} \tag{6.32}$$

where $\mathrm{d}v_\pm$ have been defined in (6.17), the covering maps $\phi_\pm$ have been described in the previous section and

$$z_1 = \int\limits_{\phi_-(P_0^r)}^{\phi_-(P^s)}\mathrm{d}v, \quad z_2 = \frac{1}{2}\int\limits_{\phi_-(P_0^r)}^{\phi_-(P^s)}\mathrm{d}v_- - \frac{1}{2}\int\limits_{\phi_+(P_0^r)}^{\phi_+(P^s)}\mathrm{d}v_+.$$

The expression (6.32) has been obtained after performing a modular transformation of the θ-function under the action of the following symplectic transformation

$$\begin{pmatrix} C^2 & 0_2 \\ 0_2 & C^t\end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & -1\end{pmatrix}, \quad C^3 = 1,$$

induced by the automorphism J^2. The period matrix $(1_2,\Pi)$ is invariant under the action of the above symplectic transformations and the θ-function changes according to the formula

$$\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\boldsymbol{z};\Pi) = \theta\begin{bmatrix}\boldsymbol{\delta}(C^2)^t\\ \boldsymbol{\epsilon}C\end{bmatrix}(\boldsymbol{z}C;\Pi). \tag{6.33}$$

Let us make a particular choice of the point $P_0=(\lambda_0,y_0)$ so that it lies on the first sheet and

$$\frac{y_0}{\lambda_0-t} = \xi_0,$$

where $\xi_0 = \sqrt[3]{2t-1+\sqrt{t^2-t}}$. Then, according to (6.21),

$$\phi_\pm(P_0^{(1)}) = (0,0), \quad \phi_\pm(P_0^{(2)}) = (\infty,\infty), \quad \phi_\pm(P_0^{(3)}) = (1,0), \tag{6.34}$$

where $\phi_\pm$ are the covering maps.

It can be easily checked that

$$\begin{aligned}\phi_\pm((0,0)) &= \left(\eta_0^{(0,0)}, \pm\frac{1\pm\rho}{K_\pm} z_0^{(0,0)}\right), \quad \phi_\pm((1,0)) = \left(\eta_0^{1,0}, \mp\frac{1\pm\rho}{K_\pm} z_0^{(1,0)}\right),\\ \phi_\pm((t,0)) &= \left(\eta_0^{(t,0)}, \frac{1\pm\rho}{K_\pm} z_0^{(t,0)}\right), \quad \phi_\pm((\infty,\infty) = \left(\eta_0^{(\infty,\infty)}, -\frac{1\pm\rho}{K_\pm} z_0^{(\infty,\infty)}\right),\end{aligned} \tag{6.35}$$

where

$$\begin{aligned}\eta_0^{(0,0)} &= \frac{p^2+3}{k_+^2p^2+3k_-^2}, \quad z_0^{(0,0)} = 4\mathcal{A}_1\frac{(k_+^2-k_-^2)p^2}{(p^2k_+^2+3k_-^2)^2},\\ \eta_0^{(1,0)} &= \frac{(t-1)^2(p^2+3)}{k_+^2p^2(t-1)^2+3k_-^2(t+1)^2}, \quad z_0^{(1,0)} = 4\mathcal{A}_1\frac{p^2k_+^2(t-1)^2-(t+1)(p^2t-3)k_-^2}{(k_+^2p^2(t-1)^2+3k_-^2(t+1)^2)^2},\\ \eta_0^{(t,0)} &= \frac{(p^2+3)\sqrt[3]{t^2(1-t)^2}}{12k_+^2t^2}, \quad z_0^{(t,0)} = \mathcal{A}_1\frac{(p^2+3)(t-1)}{9tk_-^2},\\ \eta_0^{(\infty,\infty)} &= \frac{p^2+3}{k_+^2p^2+3k_-^2}, \quad z_0^{(\infty,\infty)} = \frac{4\mathcal{A}_1}{(p^2k_+^2+3k_-^2)^2}.\end{aligned}$$

6.3. **Explicit solution of the Schlesinger equation.** In this section we derive the explicit solution of the Schlesinger equations. The solution of the R-H problem has the following representation in the vicinity of the branch points for $\lambda \in C_+$,

$$Y(\lambda) = \Lambda_\infty\left(1+O\left(\frac{1}{\lambda}\right)\right)\lambda^{-\sigma_3}\begin{pmatrix}1&\rho&\rho^2\\1&1&1\\1&\rho^2&\rho\end{pmatrix} \quad \lambda\to\infty, \tag{6.36}$$

$$Y(\lambda) = \Lambda_0\left(1+O(\lambda)\right)t^{-\sigma_3}\lambda^{\sigma_3}\begin{pmatrix}1&\rho^2&\rho\\1&1&1\\1&\rho&\rho^2\end{pmatrix}C_0, \quad \lambda\to 0, \tag{6.37}$$

$$Y(\lambda) = \Lambda_1\left(1+O(\lambda-1)\right)(1-t)^{-\sigma_3}(\lambda-1)^{\sigma_3}\begin{pmatrix}1&\rho^2&\rho\\1&1&1\\1&\rho&\rho^2\end{pmatrix}C_1, \quad \lambda\to 1, \tag{6.38}$$

$$Y(\lambda) = \Lambda_t\left(1+O(\lambda-t)\right)(t(t-1))^{-\sigma_3}(\lambda-t)^{\sigma_3}\begin{pmatrix}1&\rho&\rho^2\\1&1&1\\1&\rho^2&\rho\end{pmatrix}C_0C_1, \quad \lambda\to t, \tag{6.39}$$

where

$$C_0 = \begin{pmatrix}1&0&0\\0&\mathrm{e}^{2\pi\imath(\epsilon_1-\epsilon_2)}&0\\0&0&\mathrm{e}^{2\pi\imath\epsilon_1}\end{pmatrix}, \quad C_1 = \begin{pmatrix}1&0&0\\0&\mathrm{e}^{2\pi\imath\delta_1}&0\\0&0&\mathrm{e}^{2\pi\imath(\delta_1+\delta_2)}\end{pmatrix}.$$

We remark that the expansion of the integral $\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v}$ when $P^{(s)}$ approaches a branch point is obtained following the rule

$$\int_{P_0^{(r)}}^{P^{(s)}} \mathrm{d}\boldsymbol{v} = \int_{P_0^{(r)}}^{\infty} \mathrm{d}\boldsymbol{v} + \int_{\infty}^{P^{(1)}} J^{(s-1)}(\mathrm{d}\boldsymbol{v}) = \int_{P_0^{(r)}}^{\infty} \mathrm{d}\boldsymbol{v} + \int_{\infty}^{P^{(1)}} \mathrm{d}\boldsymbol{v} - \left(\int_{\infty}^{P^{(1)}} \mathrm{d}\boldsymbol{v} - \int_{\infty}^{P^{(1)}} J^{(s-1)}(\mathrm{d}\boldsymbol{v})\right).$$

When $P^{(1)}$ approaches a branch point, the last term in the above expression coincides with some period.

The matrices Λ_∞, Λ_0, Λ_1 and Λ_t are defined in the following way. Let us introduce the operators

$$\mathcal{L}_1 = \frac{3}{(1-\rho)\mathcal{A}_1}\left(\frac{\partial}{\partial z_1} - \rho\frac{\partial}{\partial z_2}\right), \quad \mathcal{L}_2 = \frac{3}{(1-\rho)\mathcal{A}_1}\left(\frac{\partial}{\partial z_1} - \rho^2\frac{\partial}{\partial z_2}\right), \tag{6.40}$$

and the matrices

$$\Lambda_a^{\pm}(P) = \frac{1}{3}\frac{y_0}{q_0}\begin{pmatrix} \Psi_1(P) & (\frac{q_0}{y_0} \pm \mathcal{L}_1)\Psi_1(P) & \left(\frac{q_0^2}{y_0^2} \pm \frac{q_0}{y_0}\mathcal{L}_1 + \frac{1}{2}(\mathcal{L}_1^2 \pm \mathcal{L}_2)\right)\Psi_1(P) \\ \rho\Psi_2(P) & (\frac{q_0}{y_0} \pm \rho\mathcal{L}_1)\Psi_2(P) & \left(\rho^2\frac{q_0^2}{y_0^2} \pm \frac{q_0}{y_0}\mathcal{L}_1 + \rho\frac{1}{2}(\mathcal{L}_1^2 \pm \mathcal{L}_2)\right)\Psi_2(P) \\ \rho^2\Psi_3(P) & (\frac{q_0}{y_0} \pm \rho^2\mathcal{L}_1)\Psi_3(P) & \left(\rho\frac{q_0^2}{y_0^2} \pm \frac{q_0}{y_0}\mathcal{L}_1 + \rho^2\frac{1}{2}(\mathcal{L}_1^2 \pm \mathcal{L}_2)\right)\Psi_3(P) \end{pmatrix}, \tag{6.41}$$

$$\Lambda_b^{\pm}(P) = \frac{q_0}{3y_0}\begin{pmatrix} \Psi_1(P) & (\frac{y_0}{q_0} \pm \mathcal{L}_2)\Psi_1(P) & \left(\frac{y_0^2}{q_0^2} \pm \frac{y_0}{q_0}\mathcal{L}_2 + \frac{1}{2}(\mathcal{L}_2^2 \pm \mathcal{L}_1)\right)\Psi_1(P) \\ \rho^2\Psi_1(P) & (\frac{y_0}{q_0} \pm \rho^2\mathcal{L}_2)\Psi_1(P) & \left(\rho\frac{y_0^2}{q_0^2} \pm \frac{y_0}{q_0}\mathcal{L}_2 + \rho^2\frac{1}{2}(\mathcal{L}_2^2 \pm \mathcal{L}_1)\right)\Psi_1(P) \\ \rho\Psi_1(P) & (\frac{y_0}{q_0} \pm \rho\mathcal{L}_2)\Psi_1(P) & \left(\rho^2\frac{y_0^2}{q_0^2} \pm \frac{y_0}{q_0}\mathcal{L}_2 + \rho\frac{1}{2}(\mathcal{L}_2^2 \pm \mathcal{L}_1)\right)\Psi_1(P) \end{pmatrix}, \tag{6.42}$$

where $q_0 = \lambda_0 - t$, $y_0 = \sqrt[3]{\lambda_0(\lambda_0-1)(\lambda_0-t)^2}$ and

$$\Psi_r(P) = \frac{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right]\left(\int_\infty^P \mathrm{d}\boldsymbol{v} - \int_\infty^{P_0^{(r)}} \mathrm{d}\boldsymbol{v};\Pi\right)}{\theta\left(\int_\infty^P \mathrm{d}\boldsymbol{v} - \int_\infty^{P_0^{(r)}} \mathrm{d}\boldsymbol{v};\Pi\right)}\frac{\theta(\mathbf{0};\Pi)}{\theta\left[\begin{smallmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{smallmatrix}\right](\mathbf{0};\Pi)}, \quad r = 1,2,3, \quad P \text{ on the first sheet of } \mathcal{C}_{3,1}.$$

In the above it is supposed that the operators $\mathcal{L}_i$ and $\mathcal{L}_i^2$, $i = 1,2$, are acting on the θ-function depending in $\int_\infty^P \mathrm{d}\boldsymbol{v}$. Then

$$\begin{aligned} \Lambda_\infty &= \Lambda_b^-(P)|_{P=(\infty,\infty)}, \quad \Lambda_t = \Lambda_b^+(P)|_{P=(t,0)}, \\ \Lambda_1 &= \Lambda_a^+(P)|_{P=(1,0)}, \quad \Lambda_0 = \Lambda_a^-(P)|_{P=(0,0)}. \end{aligned} \tag{6.43}$$

From the expansions (6.36)-(6.39) and from (2.12), the solution of the Schlesinger equations can be derived in a straightforward manner

$$\begin{aligned} A_\infty &= \Lambda_\infty\sigma_3\Lambda_\infty^{-1}, \quad A_t = \Lambda_t\sigma_3\Lambda_t^{-1}, \\ A_1 &= \Lambda_1\sigma_3\Lambda_1^{-1}, \quad A_0 = \Lambda_0\sigma_3\Lambda_0^{-1}. \end{aligned} \tag{6.44}$$

The matrices A_0, A_1, A_t and A_∞ satisfy the constraint

$$A_0 + A_1 + A_t + A_\infty = 0,$$

which follows from the fact that the point at infinity is a Fuchsian singularity for the equation (1.3). The substitution (6.44) into the above relation leads to nontrivial θ-functional equivalence.

Combining the reduction formula (6.32) with (6.35)-(6.41), it is possible to express the solution (6.44) of the Schlesinger system in terms of elliptic functions.

In what follows we express the τ function corresponding to the solution (6.44) of the Schlesinger system in terms of Jacobi's ϑ-functions. According to the formula (5.27), (6.29) and (6.23) we obtain

$$\tau(t,\boldsymbol{\delta},\boldsymbol{\epsilon}) = \mathrm{e}^{\pi\imath\langle\boldsymbol{\delta},\Pi\boldsymbol{\delta}\rangle+2\pi\imath\langle\boldsymbol{\epsilon},\boldsymbol{\delta}\rangle}(t(1-t))^{-\frac{2}{9}}\times \tag{6.45}$$
$$\times\,\frac{\sum_{k=2}^{3}\vartheta_k(\frac{1}{2\pi\imath}\log c_1c_2+\frac{3T}{2\pi\imath}\log d_1d_2;\,6T)\vartheta_k(\frac{1}{2\pi\imath}\log\frac{c_1}{c_2}+\frac{T}{2\pi\imath}\log\frac{d_1}{d_2};\,2T)}{\vartheta_3(0;\,6T)\vartheta_3(0;\,2T)+\vartheta_2(0;\,6T)\vartheta_2(0;\,2T)},$$

If $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ are shifted by $1/3$ integers, the corresponding constants d_i and c_i are shifted by third roots of unity and the corresponding τ function is expressed by the above formula with the Jacobi's ϑ-function shifted by $1/6$ periods. As an example we consider the shift

$$\boldsymbol{\epsilon}+\boldsymbol{\delta}\Pi\to\boldsymbol{\epsilon}+\boldsymbol{\delta}\Pi+\mathfrak{A}(P_2+P_3-2P_2),\;\; P_2=(t,0),\; P_3=(1,0),$$

where the vector $\mathfrak{A}(P_2+P_3-2P_2)=\left(-\frac{2}{3},\frac{1}{3}\right)\Pi$ is non-singular. The corresponding constants d_i and c_i, $i=1,2$ transform to

$$d_1\to d_1\mathrm{e}^{-\frac{4\pi\imath}{3}},\quad d_2\to d_2\mathrm{e}^{\frac{2\pi\imath}{3}},\;\; c_i\to c_i,\; i=1,2.$$

7. Conclusion

In this manuscript we have studied the solution of the R-H problem for a particular class of quasi-permutation monodromy matrices and for a given set of $2m+2$ singular points. The dimension of the space of monodromy matrices is $2m(N-1)$. Inspired by [3] we have solved the problem using the Szegö kernel of a Riemann surface. The Riemann surface is obtained from the reduction of the monodromy representation to a permutation representation of the symmetric group S_N. The form of the monodromy matrices considered, is such that the permutation representation obtained, generates the cyclic subgroup Z_N of the permutation group. For this reason the family of Riemann surfaces we have studied have Z_N symmetry and genus $N(m-1)$. This symmetry in our problem has enabled us to write the entries of the $N\times N$ matrix solution of the R-H problem as a product of an algebraic function and θ-quotients. The algebraic function turns out to be related to the Szegö kernel with zero characteristics. The $2N(m-1)$ monodromy parameters are in one to one correspondence with the $2N(m-1)$ characteristics of the θ-quotients. The R-H problem is solvable if the corresponding characteristics is non-singular.

We have studied the set of non-singular divisors supported on the branch points and we have shown that the corresponding non-singular characteristics are rational numbers of the form k/N, $k=1,\dots,N-1$. We have shown that the solution of the R-H problem for reducible monodromy representation is expressed in terms of θ- quotients with k/N characteristics. Furthermore we have shown that if two monodromy representations are equivalent up to multiplication by N-th roots of unity, then the corresponding solutions of the R-H problem have characteristics that differ by $1/N$.

From the solution of the R-H problem we have straightforwardly obtained a particular solution of the Schlesinger equations. The Jimbo-Miwa-Ueno τ-function corresponding to this particular solution of the Schlesinger system is derived in a complete form by the explicit evaluation of the projective connection associated to the Riemann surfaces $\mathcal{C}_{N,m}$.

Finally we have investigated in detail the case of a 3×3 matrix R-H problem with four singular points, $\lambda_1=0,\ \lambda_2=t,\ \lambda_3=1,\ \lambda_4=\infty$. The monodromy space is four-dimensional. The R-H problem is solved in terms of the Szegö kernel defined on a trigonal curve of genus two admitting the dihedral group D_3 of automorphisms. For this reason the trigonal curve is a covering over two elliptic curves which are 3-isogenous. This fact enables us to write the solution of the R-H problem in terms of Jacobi's ϑ-functions with modulus $T=T(t)$. The inverse function $t=t(T)$ is in general not single valued. For T belonging to Siegel half space $\mathcal{H}_1$ modulo the group $\Gamma_0(3)$, the function $t=t(T)$ is single valued and the explicit formula is given in (6.28). From this formula we have derived an expression for the solution of the corresponding general Halphen system equivalent to the one derived in [22]. From the solution of the

R-H problem we have derived a four parameter family of solutions of the Schlesinger system. We suppose that these solutions would be the analogous of the elliptic solution of the Painlevé VI equation [56]. The study of the analytic continuation of the solutions of the above 3×3 Schlesinger system in the spirit of [57, 58] remains one of the subjects of our further investigations. In particular we are interested to single out algebraic solutions and derive explicit algebraic expression for the 3×3 Schlesinger system as in [59],[60]. Another interesting point of future investigations concerns the case of the rank N problem with four singular points $\{0,1,t,\infty\}$. The Riemann surfaces associated to the problem are the curves $\mathcal{C}_{N,1}$ of genus $N-1$. It would be desirable to understand whether the period matrix of the curve $\mathcal{C}_{N,1}$ depends just on one parameter T belonging to the Siegel upper half space modulo the group $\Gamma_0(N)$. If this is the case, the structure of the analytic continuation of the solution of the corresponding $N\times N$ Schlesinger system should be described by an induced action of $\Gamma_0(N)$ on the symplectic group $\mathrm{Sp}(2(N-1);\mathbb{Z})$. The corresponding algebraic solutions of the Schlesinger system should be singled out as well.

8. Appendix : Derivation of the Thomae formula

We prove here Theorem 5.7, that is the formula

$$\theta^8(\mathbf{0};\Pi)=\frac{\prod_{s=1}^{N-1}\det\mathcal{A}_s^4}{(2\pi\imath)^{4m(N-1)}}\prod_{i<j}(\lambda_{2i}-\lambda_{2j})^{2(N-1)}\prod_{k<l}(\lambda_{2k+1}-\lambda_{2l+1})^{2(N-1)}, \tag{8.1}$$

where the matrices $\mathcal{A}_s$, $s=1,\dots,N-1$, are defined in (4.8).

Proof. The proof of the theorem, consists of several steps. First we use Fay relation (3.26) for zero characteristics, namely

$$S(P,Q)^2=\mathrm{d}\omega(P,Q)+\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z_k\partial z_l}\log\theta(\mathbf{0};\Pi)\mathrm{d}v_k(P)\mathrm{d}v_l(Q). \tag{8.2}$$

We derive Thomae formula by evaluating the residues of (8.2) at $P=Q=(\lambda_i,0)$. The residue of the term containing the derivatives of $\theta(\mathbf{0};\Pi)$, can be obtained combining the heat equation (3.12), Rauch variation formula (3.4) and the fact that the function $\theta(\mathbf{0};\Pi)$ is even, which gives

$$\begin{aligned}&\operatorname*{Res}_{P=(\lambda_i,0)}\left[\sum_{s=1}^{N}\sum_{k,l=1}^{(N-1)m}\frac{\partial^2}{\partial z_k\partial z_l}\log\theta(\mathbf{0};\Pi)\frac{\mathrm{d}v_k(P^{(s)})\mathrm{d}v_l(P^{(s)})}{(\mathrm{d}z(P))^2}\right]\\&=\sum_{k,l=1}^{(N-1)m}(1+2\delta_{kl})\frac{\partial}{\partial\Pi_{k,l}}\log\theta(\mathbf{0};\Pi)\frac{\partial\Pi_{k,l}}{\partial\lambda_i}=2\frac{\partial}{\partial\lambda_i}\log\theta(\mathbf{0};\Pi).\end{aligned} \tag{8.3}$$

From the expansion of the Szegö kernel given in (3.23) we obtain

$$\operatorname*{Res}_{P=Q=(\lambda_i,0)}\left[\sum_{s=1}^{N}\frac{(S[0](P^{(s)},Q^{(s)}))^2}{dz(P)dz(Q)}\right]=\frac{N^2-1}{12N}\operatorname*{Res}_{\lambda=\lambda_i}\left[\frac{p'(\lambda)}{p(\lambda)}-\frac{q'(\lambda)}{q(\lambda)}\right]^2. \tag{8.4}$$

Now let us consider the Bergmann kernel $\mathrm{d}\omega(P,Q)$. In order to write the explicit expression for $\mathrm{d}\omega(P,Q)$, we follow [61]. The first step consists of constructing the normalized meromorphic differential of the third kind $\mathrm{d}\Omega_{Q,Q_0}(P)$ with simple poles at the points $Q=(\nu,w)$ and $Q=(\nu_0,w_0)$, with residues

± 1 respectively, that is, for $P = (\lambda, y)$,

$$
\begin{aligned}
\mathrm{d}\Omega_{Q,Q_0}(P) &= \frac{d\lambda}{N(\lambda-\nu)}\left(1+\sum_{s=1}^{N-1}\frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}}\right) - \frac{d\lambda}{N(\lambda-\nu_0)}\left(1+\sum_{s=1}^{N-1}\frac{w_0^s q(\lambda)^{s-1}}{y^s q(\nu_0)^{s-1}}\right)\\
&\quad - \frac{1}{N}\sum_{j=1}^{(N-1)m}\mathrm{d}v_j(\lambda)\oint_{\alpha_j} d\xi\left[\frac{\left(1+\sum_{s=1}^{N-1}\frac{w^s q(\xi)^{s-1}}{y_0^s q(\nu)^{s-1}}\right)}{(\xi-\nu)} - \frac{\left(1+\sum_{s=1}^{N-1}\frac{w_0^s q(\xi)^{s-1}}{y_0^s q(\nu_0)^{s-1}}\right)}{(\xi-\nu_0)}\right]
\end{aligned} \tag{8.5}
$$

where $\mathrm{d}v_j$, $j = 1, \ldots, (N-1)m$ is the basis of normalized holomorphic differentials and the point $(\xi, y_0) \in \mathcal{C}_{N,m}$. The differential $\mathrm{d}\Omega_{Q,Q_0}(P)$ as a function of Q is an Abelian integral with periods given by the relations

$$
\oint_{\alpha_j} \mathrm{d}_\nu \mathrm{d}\Omega_{Q,Q_0}(P) = 0, \quad \oint_{\beta_j} \mathrm{d}_\nu \mathrm{d}\Omega_{Q,Q_0}(P) = 2\pi\imath\, \mathrm{d}v_j(P), \;\; j = 1, \ldots, (N-1)m. \tag{8.6}
$$

Furthermore the differential $\mathrm{d}\Omega_{Q,Q_0}(P)$ satisfies the symmetry property $\mathrm{d}_\nu\mathrm{d}\Omega_{Q,Q_0}(P) = \mathrm{d}_\lambda\mathrm{d}\Omega_{P,P_0}(Q)$, for $P_0 \neq P$. Therefore the 2-differential, $\mathrm{d}\omega(P,Q) := \mathrm{d}_\nu\mathrm{d}\Omega_{Q,Q_0}(P)$,

(1) is symmetric in P and Q;
(2) is holomorphic everywhere except for a double pole along $P = Q$, where

$$
\mathrm{d}\omega(P,Q) = \mathrm{d}\lambda\,\mathrm{d}\nu\left(\frac{1}{(\lambda-\nu)^2} + \text{regular terms}\right);
$$

(3) for any fixed P, it satisfies (8.6).

Therefore, $\mathrm{d}\omega(P,Q)$ is the Bergmann kernel given alternatively in the form (3.19).

In order to write more explicitly the Bergmann kernel, let us introduce the Abelian differentials $\mathrm{d}\sigma_{r,j}(\nu, w)$ of the second kind having the only pole at infinity of order $N(j+1) - r + 1$, that is

$$
\mathrm{d}\sigma_{r,j}(\nu, w) = \frac{q(\nu)^{r-1}}{w^r}\mathcal{Q}_{r,j}(\nu)\mathrm{d}\nu, \;\; r = 1, \ldots, N-1, \;\; j \geq 0, \tag{8.7}
$$

where $\mathcal{Q}_{r,j}(\nu)$ are polynomials in ν of degree $m + j$. The coefficients of the polynomials $\mathcal{Q}_{r,j}(\nu)$, $r = 1, \ldots, N-1$, $j \geq 0$, are uniquely determined by the conditions

$$
\int_{\alpha_s} \mathrm{d}\sigma_{r,j}(\nu, w) = 0, \;\; s = 1, \ldots, m,
$$

$$
\mathrm{d}\sigma_{r,j}(\nu, w) \simeq \left(\nu^{j-\frac{r}{N}} + O\left(\frac{1}{\nu^{1+\frac{r}{N}}}\right)\right)\mathrm{d}\nu, \;\; (\nu, w) \to (\infty, \infty).
$$

From the Riemann bilinear relations we obtain the identities

$$
\int_{Q_0}^{Q} \mathrm{d}\sigma_{r,j}(P) + \operatorname*{Res}_{P=(\infty,\infty)}[\mathrm{d}\Omega_{Q,Q_0}(P)\sigma_{r,j}(P)] = 0, \quad r = 1, \ldots, N-1, \; j = 0, \ldots m-1,
$$

so that we can reduce the expression of $\mathrm{d}\omega(P,Q) = \mathrm{d}_\nu\mathrm{d}\Omega_{Q,Q_0}(P)$ to the form

$$
\begin{aligned}
\mathrm{d}\omega(P,Q) &= \frac{\mathrm{d}\lambda\mathrm{d}\nu}{N(\lambda-\nu)^2}\left(1+\sum_{s=1}^{N-1}\frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}}\right) + \frac{\mathrm{d}\lambda\mathrm{d}\nu}{N(\lambda-\nu)}\frac{d}{d\nu}\left(\sum_{s=1}^{N-1}\frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}}\right) +\\
&\quad - \frac{1}{N}\sum_{s=1}^{N-1}\sum_{j=1}^{m}\lambda^{j-1}\frac{q(\lambda)^{s-1}}{y^s}\frac{q^{N-s-1}(\nu)\widetilde{\mathcal{Q}}_{s,j}(\nu)}{w^{N-s}}\mathrm{d}\lambda\mathrm{d}\nu,
\end{aligned} \tag{8.8}
$$

where $\widetilde{\mathcal{Q}}_{s,j}(\nu)$ is a polynomial depending on $\mathcal{Q}_{N-s,0}(\nu), \mathcal{Q}_{N-s,1}(\nu), \ldots, \mathcal{Q}_{N-s,m-j}(\nu)$, $j = 1, \ldots, m$, $s = 1, \ldots, N-1$.

Proposition 8.1. *For $s = 1, \ldots, N-1$, the following identities are satisfied:*

$$\frac{\partial}{\partial \lambda_i} \log \det \mathcal{A}_s = \frac{1}{\prod_{\substack{l=1 \\ l\neq i}}^{2m+1} (\lambda_i - \lambda_l)} \sum_{j=1}^{m} \lambda_i^{j-1} \widetilde{\mathfrak{Q}}_{s,j}(\lambda_i), \quad i = 1, \ldots, 2m+1, \tag{8.9}$$

where the matrix $\mathcal{A}_s$ is defined in (4.8).

Sketch of the proof. The integral of $\mathrm{d}\omega(P,Q)$ in the P variable along the α_j periods is identically zero. Therefore, substituting the local coordinate $\nu - \lambda_i = t^N$ in $\mathrm{d}\omega(P,Q)$ and imposing that the terms of order dt, $tdt, \ldots, t^{N-2}dt$ of the integral

$$\oint_{\alpha_j} \mathrm{d}\omega(P,Q) = 0,$$

are identically zero, we obtain the statement.
Combining all the above relations we can derive the explicit expression of the projective connection (3.21)

$$\frac{1}{6} R(z(P)) = \lim_{P\to Q} \left[\frac{\mathrm{d}\omega(P,Q)}{\mathrm{d}z(P)\, \mathrm{d}z(Q)} - \frac{1}{(z(P)-z(Q))^2} \right].$$

Proposition 8.2. *The projective connection $R(z(P))$ can be obtained from (8.8) and reads*

$$\begin{aligned} \frac{1}{6} R(z(P)) = &-\frac{1}{N} \sum_{s=1}^{N-1} \sum_{j=1}^{m} \frac{z(P)^{j-1} \widetilde{\mathfrak{Q}}_{s,j}(z(P))}{p(z(P)) q(z(P))} + \frac{N^2-1}{12N^2} \left[\frac{p'(z(P))}{p(z(P))} - \frac{q'(z(P))}{q(z(P))} \right]^2 - \\ &- \frac{N-1}{4N} \left[\frac{q''(z(P))}{q(z(P))} + \frac{p''(z(P))}{p(z(P))} \right], \end{aligned} \tag{8.10}$$

where the prime denotes the derivative $\dfrac{d}{dz(P)}$.

Combining (8.9) and (8.10), we evaluate the residue of the Bergmann kernel at the branch points, namely,

$$\begin{aligned} \operatorname*{Res}_{P=Q=(\lambda_i,0)} \left[\sum_{s=1}^{N} \frac{\mathrm{d}\omega(P^{(s)}, Q^{(s)})}{\mathrm{d}z(P) \mathrm{d}z(Q)} \right] = &\frac{N^2-1}{12N} \operatorname*{Res}_{\lambda=\lambda_i} \left[\frac{p'(\lambda)}{p(\lambda)} - \frac{q'(\lambda)}{q(\lambda)} \right]^2 - \\ &- \frac{N-1}{4} \operatorname*{Res}_{\lambda=\lambda_i} \left[\frac{q''(\lambda)}{q(\lambda)} + \frac{p''(\lambda)}{p(\lambda)} \right] - \sum_{k=1}^{N-1} \frac{\partial}{\partial \lambda_i} \log \det \mathcal{A}_k. \end{aligned} \tag{8.11}$$

Substituting (8.3), (8.4) and (8.11) in (8.2) and simplifying, we obtain

$$2 \frac{\partial}{\partial \lambda_i} \log \theta(\mathbf{0}; \Pi) = \frac{N-1}{4} \operatorname*{Res}_{\lambda=\lambda_i} \left[\frac{q''(\lambda)}{q(\lambda)} + \frac{p''(\lambda)}{p(\lambda)} \right] + \sum_{k=1}^{N-1} \frac{\partial}{\partial \lambda_i} \log \det \mathcal{A}_k, \tag{8.12}$$

which gives (8.1) up to a constant C.

To compute C we pinch the branch points in the following way

$$\lambda_{2k} = e_k + \epsilon, \quad \lambda_{2k-1} = e_k - \epsilon \quad k = 1, \ldots, m, \quad 0 < \epsilon \ll 1.$$

In this case the l.h.s of (8.1) becomes equal to one as $\epsilon \to 0$, more precisely $\theta(\mathbf{0};\Pi) = 1 + O(\epsilon)$. Regarding the r.h.s the following relations are needed:

$$\lim_{\epsilon\to 0}(\mathcal{A}_s)_{ij} = 2\pi\imath \frac{e_i^{j-1}}{\prod_{\substack{k\neq i\\ k=1}}^{m}(e_i - e_k)(e_i-\lambda_{2m+1})^{\frac{s}{N}}}$$

so that

$$\lim_{\epsilon\to 0}(\det\mathcal{A}_s) = (2\pi\imath)^m \frac{1}{\prod_{\substack{k<j\\ k,j=1}}^{m}(e_k-e_j)}\frac{1}{\prod_{k=1}^{m}(e_k-\lambda_{2m+1})^{\frac{s}{N}}}. \tag{8.13}$$

Substituting (8.13) into (8.1) and letting $\epsilon \to 0$ in all the terms of (8.1), we obtain

$$1 = C\,(2\pi\imath)^{4m(N-1)}, \tag{8.14}$$

and the expression for C follows. □

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