A characterization of halved cubes

Wilfried Imrich
Department of Mathematics and Applied Geometry
Montanuniversität Leoben
A-8700 Leoben, Austria

Sandi Klavžar* and Aleksander Vesel
Department of Mathematics, PEF
University of Maribor
Koroška cesta 160
62000 Maribor, Slovenia

Abstract

The vertex set of a halved cube $Q_d'$ consists of a bipartition vertex set of a cube $Q_d$ and two vertices are adjacent if they have a common neighbour in the cube. Let $d \geq 5$. Then it is proved that $Q_d'$ is the only connected, $\binom{d}{2}$-regular graph on $2^d - 1$ vertices in which every edge lies in two $d$-cliques and two $d$-cliques do not intersect in a vertex.

1 Introduction

Let $G$ be a bipartite graph with bipartition $V(G) = X \cup Y$. A halved graph $G'$ of $G$ is defined as follows. $V(G') = X$ and $uv \in E(G')$ whenever $u$ and $v$ have a common neighbour in $G$. $G$ has another halved graph with vertex set $Y$. When we consider the $d$-cube $Q_d$ both halved graphs are isomorphic and we talk about the halved $d$-cube $Q_d'$.

Partial Hamming graphs are exactly those graphs which can be isometrically embedded into a Cartesian product of complete graphs, cf. [9]. We refer also to [2, 8] where these graphs are called Hamming graphs. In case every one of the factors is the complete graph $K_2$ on two vertices one obtains an isometric embedding into a hypercube and speaks of a partial

*This work was supported in part by the Ministry of Science and Technology of Slovenia under the grant J1-7036/94.
binary Hamming graph. By a scale embedding of a graph $G$ into a graph $H$ we mean a mapping

$$\psi : V(G) \to V(H)$$

for which there exists a positive integer $\lambda$ such that

$$d_H(\psi(u), \psi(v)) = \lambda d_G(u, v)$$

for all $u, v \in V(G)$, where $d_H$ and $d_G$ denote the usual path distance in $G$ and $H$, respectively. If one relaxes the condition of isometry and considers so-called scale embeddings into hypercubes a class larger than that of partial Hamming graphs arises. It has been characterized by Assouad and Deza [1] as the class of graphs isometrically embeddable into the metric space $\ell_1$. These graphs have in turn been characterized by Deza and Grishukhin [3] and Shpectorov [14] as isometric subgraphs of Cartesian products of complete graphs, cocktail party graphs and halved cubes.

It was this recent study of $\ell_1$-graphs that motivated us to consider halved cubes. As it is clear from the above, halved cubes play an important role in the characterization of $\ell_1$-graphs. In fact, without going into details, by a result of Graham and Winkler [6] about so-called canonical isometric embeddings of graphs into Cartesian products together with an algorithm of Feder [5], a good algorithm for recognizing isometric subgraphs of halved cubes would suffice for a good algorithm for recognizing $\ell_1$-graphs. An $O(mn)$ algorithm for recognizing isometric subgraphs of halved cubes and thus of $\ell_1$-graphs was recently obtained by Deza and Shpectorov, [4]. Here $n$ denotes the number of vertices and $m$ the number of edges of a given graph.

We also wish to recall that Aurenhammer, Formann, Idury, Schäffer and Wagner [2] and Imrich and Klavžar [8] proved that it can be decided in $O(mn)$ time whether a given graph is a partial Hamming graph.

As usual, for a vertex $u \in V(G)$ let $N(u) = \{v ; uv \in E(G)\}$. A clique is a maximal complete subgraph. If $Q$ is a clique we will also use $Q$ to denote its vertex set. A clique on $d$ vertices will be called a $d$-clique. The cocktail party graph on $2n$ vertices is the complete graph $K_{2n}$ minus a complete matching.

In this note we first study the structure of halved cubes and then give a characterization of these graphs. A halved cube on $2^{d-1}$ vertices is the only connected, $\binom{d-1}{2}$-regular graph in which every edge lies in two $d$-cliques and two $d$-cliques do not intersect in a single vertex.

2 The characterization

We will first summarize several properties of halved cubes. Then we will prove that some of these properties already imply that a given graph is a halved cube thus obtaining the desired characterization.
The vertex set of the $d$-cube $Q_d$ may be represented by all sequences of length $d$ over $\{0, 1\}$ where two vertices are adjacent if they differ in exactly one position. We may henceforth consider vertices of the halved $d$-cube $Q'_d$ as sequences of length $d$ over $\{0, 1\}$. In the sequel we will, without loss of generality, assume that a vertex of $Q'_d$ is such a sequence with an even number of 1’s. In particular, $(0, 0, \ldots, 0) \in Q'_d$. Then two vertices of $Q'_d$ are adjacent if and only if they differ in two positions.

Clearly, $Q'_d$ has $2^{d-1}$ vertices. In addition, from the coordinate representation of $Q'_d$ it follows immediately that $Q'_d$ is a $\binom{d}{2}$-regular graph. (We also recall that halved cubes are distance-regular graphs, cf. [7].)

Note that $Q'_3$ is isomorphic to the complete graph $K_4$ on four vertices and that $Q'_4$ is isomorphic to the cocktail party graph on 8 vertices. To simplify the presentation we may henceforth assume that $d \geq 5$.

**Proposition 1**

(i) There are only two types of cliques of $Q'_d$, namely $4$-cliques and $d$-cliques.

(ii) Every vertex of $Q'_d$ lies in $d$ $d$-cliques.

(iii) $Q'_d$ has $2^{d-1}$ $d$-cliques.

**Proof.** (i) We include the proof of (i) for the sake of completeness although it can be found in [7].

Let $u, v$ and $w$ be distinct vertices of a clique $Q$ of $Q'_d$. We may, without loss of generality, assume that $u = (0, 0, 0, 0, \ldots)$, $v = (1, 1, 0, 0, \ldots)$, and $w = (1, 0, 1, 0, \ldots)$, where all three vertices agree in the remaining coordinates.

Let $z$ be another vertex of $Q$. It must have exactly one 1 in its first two coordinates for otherwise it would not be adjacent to at least one of $u$ and $v$.

If $z = (0, 1, \ldots)$, it must agree with $w$ in coordinates 3, 4, $\ldots$, $d$ and there is only one such vertex. Clearly the vertices $u$, $v$, $w$ and $z$ induce a clique.

If $z = (1, 0, \ldots)$ it must be of the form $(1, 0, 0, \ldots, 1, \ldots)$. Clearly these $d - 3$ vertices, together with $u$, $v$ and $w$ form a $d$-clique.

(ii) By the argument from (i), the $d$-cliques of $Q'_d$ are induced by the neighborhoods of vertices of $Q_d$ with an odd number of 1’s. Now, since every vertex of $Q'_d$ is in $d$ such neighborhoods, it is contained in precisely $d$ such cliques.

(iii) This follows by the same argument as (ii).

We next give properties of halved cubes with respect to a given edge.

**Proposition 2** Let $uv$ be an edge of $Q'_d$. Then

(i) $|N(u) \cap N(v)| = 2(d - 2)$. 

3
(ii) \( uv \) belongs to precisely two \( d \)-cliques of \( Q'_d \), say \( Q \) and \( Q' \).
(iii) \( Q \cap Q' = \{ u, v \} \).
(iv) \( Q - \{ u, v \} \) and \( Q' - \{ u, v \} \) are joined by a matching.

**Proof.** We may without loss of generality assume \( u = (0, 0, 0, \ldots, 0) \) and \( v = (1, 1, 0, 0, \ldots, 0) \). Let \( w \) be a vertex adjacent to both \( u \) and \( v \). Then \( w \) starts out \((1, 0, \ldots)\) or \((0, 1, \ldots)\) and it has exactly one 1 in the remaining \( d-2 \) coordinates. Thus there are \( 2(d-2) \) vertices in \( N(u) \cap N(v) \).

Furthermore, the vertex sets
\[
\{u, v, (1, 0, 1, 0, \ldots, 0), (0, 1, 0, 1, 0, \ldots, 0), \ldots, (0, 1, 0, 0, 1, 0, \ldots, 0)\}
\]
and
\[
\{u, v, (0, 1, 1, 0, \ldots, 0), (0, 1, 0, 1, \ldots, 0), \ldots, (0, 1, 0, 0, 1, 0)\}
\]
induce the two cliques containing \( uv \). All the rest now easily follows. \( \Box \)

A connected graph \( G \) is a \((0,2)\)-graph if any two distinct vertices in \( G \) have exactly two common neighbors or none at all, cf. [12, 13]. Note that in bipartite graphs this condition applies only to pairs of vertices at distance two.

We will need the following result due to Mulder [13, page 55], cf. also [11].

**Theorem 3** Let \( G \) be a \( d \)-regular \((0,2)\)-graph. Then \( |V(G)| = 2^d \) if and only if \( G \) is \( Q_d \).

We are ready now to characterize halved cubes.

**Theorem 4** Let \( d \geq 5 \). Let \( G \) be a connected, \( \binom{d}{2} \)-regular graph on \( 2^d - 1 \) vertices. Then \( G \) is the halved cube \( Q'_d \) if and only if

(i) every edge of \( G \) is contained in exactly two \( d \)-cliques,
(ii) for any \( d \)-cliques \( Q \) and \( Q' \), \( |Q \cap Q'| \neq 1 \).

**Proof.** If \( G \) is a halved cube then Proposition 2 yields (i) and (ii). Conversely, suppose that (i) and (ii) hold. Since \( G \) is a \( \binom{d}{2} \)-regular graph on \( 2^d - 1 \) vertices, \( |E(G)| = d(d-1)2^{d-3} \). Thus, because of (i), there are \( 2|E(G)| = 2^{d-1} \) \( d \)-cliques of \( G \). In addition, since \( G \) is \( \binom{d}{2} \)-regular and every edge is in two \( d \)-cliques, every vertex of \( G \) belongs to \( \frac{2\binom{d}{2}}{d-1} = d \) \( d \)-cliques.

Let \( Q \) and \( Q' \) be \( d \)-cliques of \( G \) with \( |Q \cap Q'| = s \) for \( s \geq 1 \). Then by (ii), \( s \geq 2 \). Let \( u \in Q \cap Q' \) and let \( Q, Q', Q_1, Q_2, \ldots, Q_{d-2} \) be the \( d \)-cliques
containing \( u \). Note first that for any \( i, Q_i \cap (Q \cap Q') = \{ u \} \), for otherwise an edge of this intersection would belong to at least three \( d \)-cliques. Thus by (ii), \( Q_i \) must intersect \( Q \setminus Q' \) for \( i = 1, 2, \ldots, d - 2 \). Furthermore, if for \( w \in Q \setminus Q' \) we have \( w \in Q_i \cap Q_j, i \neq j \), then the edge \( uw \) would not satisfy (i). If follows that \( d - s \geq d - 2 \), thus \( s = 2 \). Hence if \( Q \cap Q' \neq \emptyset \) then \( |Q \cap Q'| = 2 \).

Let \( n = 2^d - 1 \) and denote the vertices of \( G \) by \( V(G) = \{ u_1, u_2, \ldots, u_n \} \). Let \( H \) be a graph which we get from \( G \) in the following way. To every \( d \)-clique \( Q \) of \( G \) we add a new vertex and join it to every vertex of \( Q \). These are the newly defined edges of \( H \). The original edges of \( G \) are all removed. Note that \( H \) is bipartite. Since \( G \) contains \( n \) \( d \)-cliques we may write \( V(H) = \{ u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \} \). By construction, \( d_H(u_i) = d \), for every \( i = 1, 2, \ldots, n \), and since every \( u_i \) is in \( d \)-\( d \)-cliques, we conclude that \( H \) is \( d \)-regular.

We claim that \( H \) is a \((0, 2)\)-graph. \( H \) is connected because \( G \) is connected and every edge of \( G \) lies in a \( d \)-clique. Let \( d_H(u_i, u_j) = 2 \) and let \( v_k \) be a common neighbor of \( u_i \) and \( u_j \). Then \( u_iu_j \) must be an edge of \( G \) and since it is contained in two \( d \)-cliques, there is another common neighbor of \( u_i \) and \( u_j \), say \( v_k \). Furthermore, \( v_k \) and \( u_k \) are their only common neighbors for otherwise \( u_iu_j \) would lie in more than two \( d \)-cliques of \( G \). Now, let \( u_k \) be a common neighbor of vertices \( v_i \) and \( v_j \) and let \( Q_i \) and \( Q_j \) be the cliques of \( G \) corresponding to \( v_i \) and \( v_j \). Since \( u_k \in Q_i \cap Q_j \) we have \( |Q_i \cap Q_j| = 2 \). But this means that \( v_i \) and \( v_j \) have precisely two common neighbors and the claim is proved.

We have seen that \( H \) is a \( d \)-regular \((0, 2)\)-graph on \( 2^d \) vertices. Thus \( H \) is \( Q_d \) by Theorem 3. To complete the proof we are going to show that \( G \) is the halved graph of \( H \). More precisely, we need to show that \( u_iu_j \in E(G) \) if and only if \( d_H(u_i, u_j) = 2 \). Let \( u_iu_j \in E(G) \). Then \( u_iu_j \) belongs to a \( d \)-clique \( Q \) and by construction there is a vertex of \( H \) adjacent to every vertex of \( Q \). In particular, \( d_H(u_i, u_j) = 2 \). Conversely, let \( d_H(u_i, u_j) = 2 \). Because in \( H \) all the edges of \( G \) are removed there is a vertex \( v_k \) (not in \( G \)) such that \( u_iv_k \in E(H) \) and \( v_ku_j \in E(H) \). But this implies that \( u_i \) and \( u_j \) belong to a common clique of \( G \), hence \( u_iu_j \in E(G) \).

We note that condition (ii) of Theorem 4 can be replaced by the following equivalent condition:

(iii') for any \( d \)-cliques \( Q \) and \( Q' \), \( |Q \cap Q'| \leq 2 \).

In the proof of Theorem 4 we have shown that (ii) implies (iii'). Suppose now that (iii') holds and assume that \( |Q \cap Q'| = 1 \) for \( d \)-cliques \( Q \) and \( Q' \). Let \( u \in Q \cap Q' \). Let \( V(Q) = \{ u, w_1, w_2, \ldots, w_{d-1} \} \). Clearly, \( uw_i \in Q \) for \( i = 1, 2, \ldots, d - 1 \). Let \( Q_i \neq Q \) be the second \( d \)-clique containing \( uw_i \), \( i = 1, 2, \ldots, d - 1 \). Then \( Q_i \neq Q' \). Furthermore, if \( i \neq j \) then \( Q_i \neq Q_j \),
for otherwise $|Q_1 \cap Q| \geq 3$. It follows that $u$ is contained in at least $d + 1$ $d$-cliques, a contradiction.

References


[12] H.M. Mulder, $(0, \lambda)$-graphs and $n$-cubes, Discrete Math. 28 (1979) 179–188.
