On some cycles in Wenger graphs

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Contents

Definition of Wenger graphs $W_n(q)$

- Let q be a prime power, and let $\mathbb{F}_q$ be the finite field of q elements. For any integer n with $n \geq 1$, the vertex set $V(W_n(q))$ is the disjoint union of two copies of the $n+1$ dimensional vector space $\mathbb{F}_q^{n+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{n+1} and the other by L_{n+1}.
- Elements of P_{n+1} will be called points and those of L_{n+1} lines.
- The point $p = (p_1, p_2, \ldots, p_{n+1})$ is adjacent to the line $l = [l_1, l_2, \ldots, l_{n+1}]$ if and only if

$$p_i + l_i = p_1 l_{i-1}.$$

for $i = 2, 3, \ldots, n+1$.

Isomorphism of Wenger graph $W_n(q)$

- The graph $W_n(q)$ has $2q^{n+1}$ vertices, q^{n+2} edges and is q-regular.
- $W_1(q)$, $W_2(q)$ and $W_4(q)$ are magnitude extremal graphs of a fixed order and without C_4, C_6 and C_{10}, respectively. $W_1(q)$ and $W_2(q)$ are isomorphic to large induced subgraphs of $PG(2,q)$.
- They are examples of edge-transitive but not vertex-transitive graphs.
- Another useful presentation of Wenger graphs was suggested by Lazebnik and Viglione (2002). It was also presented in Cioabă, Lazebnik and Li (2014). For $W_n(q)$, the point p is adjacent to the line l if and only if

$$p_i + l_i = p_1 l_1^{i-1},$$

for $i = 2, 3, \ldots, n+1$.

Cycles in sparse graphs

- What are the lengths of the cycles in Wenger graphs?
- Are they Hamiltonian?
- Usual sufficient conditions of Hamiltonicity of a graph G, like Dirac's condition, require a graph to have the average degree $d_{ave}(G) \geq c \cdot v$, i.e., G to be dense.
- For $W_n(q)$, $d_{ave}(G) = c \cdot v^{\frac{1}{n+1}}$, and so they are sparse, and, hence, usual methods do not apply.
- A successful example is the work of Lazebnik, Mellinger and Vega (2013), where this was done for cycles of all even length more than 4 in Levi graphs of finite projective planes ($d_{ave} \sim c \cdot v^{1/2}$).

Cycles in Wenger graphs

Theorem 1 (Shao, He and Shan (2008))

For any integer n with $n \geq 2$ and for any integer k with $k \neq 5$, $4 \leq k \leq 2char(\mathbb{F}_q)$, there are cycles of length $2k$ in $W_n(q)$.

Theorem 2 (Lazebnik and Wang (2014))

Let n be an integer with $n \geq 2$ and let q be a prime power with $q \geq 5$ and $char(\mathbb{F}_q) \geq 3$. For any integer k with $6 \leq k \leq 4q+1$, $W_n(q)$ contains cycles of length $2k$.

Theorem 3 (Lazebnik and Wang (2014))

For any integer n with $n \geq 2$ and let q be a prime power with $q \geq 5$ and $char(\mathbb{F}_q) \geq 3$. $W_n(q)$ contains cycles of length at least $\lfloor 2.37(q-4)^{1.58} \rfloor = 2.37(1-o(1))q^{1.58}, q \to \infty$.

The idea of constructing cycles in $W_n(q)$

Constructing cycles of lengths up to $8k+4$ with $1 \leq k \leq q-1$ by using special patterns on the first coordinates of points and lines of these cycles. → Cycles of lengths up to $8q+2$. → Connecting these cycles constructed by using some 8-cycles.

Main framework: notations and assumptions

Lemma 4

Let n be an integer with $n \geq 2$ and q be a prime power with $char(\mathbb{F}_q) \geq 3$. For any integer k with $1 \leq k \leq q-1$, $W_n(q)$ contains cycles of length $8k+4$.

- Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \in \mathbb{F}_q$ with $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\beta_2 \neq \beta_3$ and $\beta_1 \neq \beta_3$.
- For any integer n with $n \geq 2$ and any integer k with $1 \leq k \leq q-1$, let $P'_{n+1} = \{p_1, p_2, \cdots, p_{4k+2}\}$ and $L'_{n+1} = \{l_1, l_2, \ldots, l_{4k+2}\}$ be the subsets of P_{n+1} and L_{n+1} in $W_n(q)$.
- For $p_i \in P'_{n+1}$ and $l_i \in L'_{n+1}$, denote $p_i(j)$ the jth coordinate of point p_i and $l_i(j)$ the jth coordinate of line l_i.

Main framework: how to construct cycles in Wenger graphs

- We can describe the first coordinates of p_i and l_i in table form.

i	1	2	3	$\dots$	$2k+2$	$2k+3$	$2k+4$	$\dots$
$p_i(1)$	α_1	α_2	α_1	$\dots$	α_2	α_1	α_2	$\dots$
$l_i(1)$	β_1	β_2	β_3	$\dots$	β_1	β_2	β_3	$\dots$

Table 1: The first coordinates of p_i and l_i in C_{8k+4}

Main framework: joining these cycles in Wenger graphs

Lemma 5

Let n be an integer with $n \geq 1$ and q be a prime power with $q \geq 3$. If there is a path P' of length 4 with endpoints p_1 and p_3 in $W_n(q)$, then there is another path P'' of length 4 with the same endpoints p_1 and p_3 with $P' \cap P'' = \emptyset$ in $W_n(q)$.

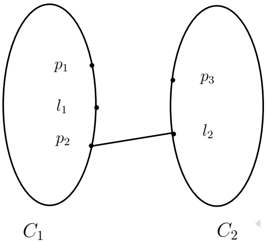
Joining cycles by C_8

- We can describe the first coordinates of p_i and l_i in table form.

i	1	2	3	4
$p_i(1)$	α_1	α_2	α_3	$\alpha_1+\alpha_3-\alpha_2$
$l_i(1)$	β_1	β_2	β_1	β_2

Table 5: The first coordinates of p_i and l_i in C_8

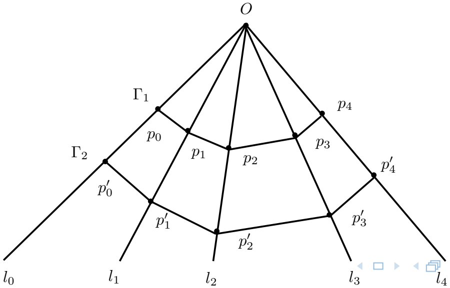
Figure 1: The construction of joining cycles by using C_8



Construction of cycles of length $2(p^2 - p)$

- Let q be a prime power and $\mathbb{F}_q$ be the finite field of q elements. Let O be the point $(0,0)$ of the plane, and let $l_0, l_1, ..., l_{q-1}$ be the lines through point O. Denote l_i by $[\beta_i, 0]$ with $\beta_i \in \mathbb{F}_q$, for $i = 0, 1, \ldots, q-1$.

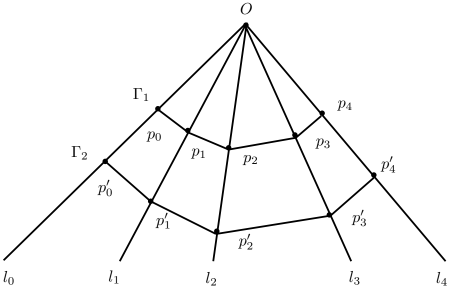
Figure 2: Two paths in $G(5)$



Construction of cycles of length $2(p^2 - p)$

- For any given point $p = (x, y)$ with $x, y \in \mathbb{F}_q$, we use $l_i + p$ to denote the line parallel to l_i that passes through p. The point (x, y) is on the line l_i if and only if $y = \beta_i x$.

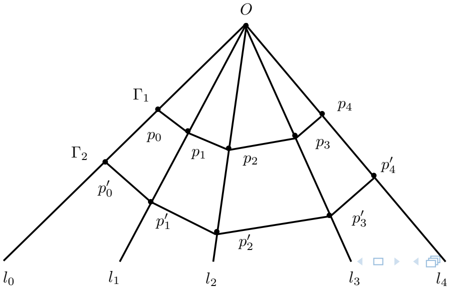
Figure 3: Two paths in $G(5)$



Construction of cycles of length $2(p^2 - p)$

- $G(q)$ is the graph with the vertex set of $G(q)$ being all the vertices on lines $l_0, l_1, ..., l_{q-1}$.
- Pick any point p_0 on l_0, different from O. Let p_{i+1} be the point of intersection of $l_{i+2 \pmod q} + p_i$ and l_{i+1}, for all $i = 0, 1, ..., q-2$.

Figure 4: Two paths in $G(5)$

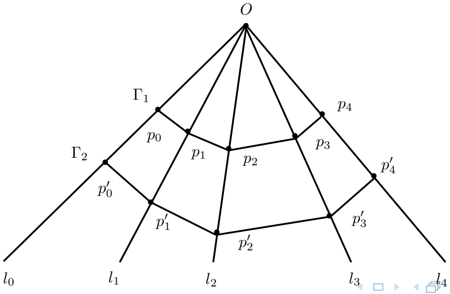


Construction of cycles of length $2(p^2 - p)$

Lemma 6

Let $p_0 \neq p_0' \in l_0$ and let Γ_1, Γ_2 be two distinct paths with $p_0 \in \Gamma_1, p_0' \in \Gamma_2$, then Γ_1, Γ_2 share neither points nor lines.

Figure 5: Two paths in $G(5)$



Theorem 7

Let q be a prime power. For any integer k with $3 \leq k \leq q^2 - q + 1$, $G(q)$ contains cycles of length k.

Theorem 8

Let q be a prime power, $p = char(\mathbb{F}_q)$ and $n \geq 1$, $W_n(q)$ contains cycles of length $2p^2 - 2p$.

Summary

1. $W_1(q)$ contains cycles of length $2k$, where $k = 3, 4, \ldots, q^2 - q + 1$.
2. Let $p = char(\mathbb{F}_q)$. For $n \geq 2$, $q \geq 5$ and $p \geq 3$, $W_n(q)$ contains cycles of length $2k$, where $k = 4, 6, 7, \ldots, 4q + 1$ and $k = p^2 - p$.
3. For $n \geq 2$ and $q \geq 3$, $W_n(q)$ contains cycles of length at least $\lfloor 2.37(q-4)^{1.58} \rfloor = 2.37(1-o(1))q^{1.58}$, $q \to \infty$.

Conjecture 1

For any $n \geq 5$, and any prime power q, Wenger graph $W_n(q)$ contains cycles of ALL even lengths greater than 10.

Thank you!