p-ADIC NORMS AND QUADRATIC EXTENSIONS, II

CHRISTOPHE CORNUT

Abstract. We investigate the graph theoretical aspects and combinatorics of the action of a unitary group in n variables over a local non-archimedian field on the Bruhat-Tits building of an odd orthogonal group in $2n+1$ variables.

1. Introduction

1.1. This paper is part of an ongoing project whose final aim is to construct and study a new Euler System for certain motives over totally real fields. These motives, which are pure of weight -1 and endowed with a symplectic duality, should occur in suitable twists of the middle cohomology of Shimura varieties attached to certain odd orthogonal groups. As for the Euler System, it should essentially be the image by suitable Abel-Jacobi maps of special cycles of unitary type in the ambient Shimura variety. One of the first – but key step – in this program is to obtain a reasonable understanding of the *distribution relations* that intertwine the Hecke and Galois actions on these cycles.

1.2. This mostly reduces to the following local question. Let E/F be a quadratic extension of p-adic local fields, and let $(V, \phi) = (W, \mathrm{tr}_{E/F}\psi)\perp(D, \phi)$ be a quadratic F-space which is the orthogonal sum of the quadratic space that underlies an Hermitian E-space (W, ψ) and an anisotropic F-line D. Put

$$SU = SU(W, \psi) \subset U = U(W, \psi) \subset SO = SO(V, \phi)$$

and let K be a compact open subgroup of SO. Let $\mathcal{S} = \mathrm{ind}_{SU}^{SO}\mathbf{1}$ be the space of all locally constant and compactly supported functions $SU\backslash SO \to \mathbf{C}$, on which SO and $T = U/SU$ act respectively by right and left multiplication. The local model of our space of cycles with its commuting Hecke and Galois actions turns out to be the space $\mathcal{S}^K$ of right K-invariant functions in $\mathcal{S}$, with the commuting actions of the local Hecke algebra $\mathcal{H} = \mathcal{H}(SO, K)$ and the torus T.

1.3. We thus want to have some understanding on the structure of this specific $\mathcal{H}[T]$-module, especially in the case where E/F is unramified, SO is split, and K is hyperspecial. More generally, the results of this paper pertain to the case where E/F is unramified and K is the stabilizer of a vertex in the Bruhat-Tits building $\mathcal{I}$ of SO (with no further assumptions on the odd orthogonal group SO).

1.4. In a previous paper [1], we have described the orbits of U on the building $\mathcal{I}$, merely viewing the latter as a set. We now also consider the simplicial structure of that building. The main result is a formula (or perhaps more something like an algorithm) that computes, for each vertex or edge in the building, the volume of its stabilizer in U and the image of this stabilizer in the compact torus T.

1.5. Notations & Conventions.

1.5.1. We let F is a non-archimedian local field of *odd* residue characteristic p with ring of integers $\mathcal{O}$, maximal ideal $\mathcal{P}$, and finite residue field $\mathbf{F}=\mathcal{O}/\mathcal{P}$ of order q (a power of p). We let π be a uniformizing element of F, so that $\mathcal{P}=\mathcal{O}\pi$. We denote by $|\bullet|:F\to\mathbf{R}_{\geq 0}$ the normalized discrete valuation of F for which $|\pi|=q^{-1}$.

1.5.2. We fix once and for all an *unramified* quadratic extension E of F, with ring of integers $\mathcal{O}_E$, maximal ideal $\mathcal{P}\mathcal{O}_E$ and residue field $\mathbf{E}=\mathcal{O}_E/\mathcal{P}\mathcal{O}_E$, a quadratic extension of $\mathbf{F}$. For $n\geq 0$, we denote by $\mathcal{O}_n\overset{def}{=}\mathcal{O}_E+\mathcal{P}^n\mathcal{O}_E$ the order of conductor n in $\mathcal{O}_E=\mathcal{O}_0$. It is a local, complete, Gorenstein, 1-dimensional ring with maximal ideal $\mathcal{R}_n=\mathcal{P}\mathcal{O}_{n-1}$ and residue field $\mathcal{O}_n/\mathcal{R}_n\simeq\mathbf{F}$ if $n>0$ (and maximal ideal $\mathcal{R}_0=\mathcal{P}\mathcal{O}_0$ and residue field $\mathbf{E}=\mathcal{O}_0/\mathcal{R}_0$ if $n=0$). We denote by $\lambda\mapsto\overline{\lambda}$ the non-trivial automorphism of E/F and let $\mathrm{tr}:E\to F$ and $N:E\to F$ be the usual trace and norm maps. We let η be any generator of the free, rank one $\mathcal{O}$-module $\mathcal{O}_E\cap\ker\mathrm{tr}$. Therefore η belongs to $\mathcal{O}_E^\times$, $\overline{\eta}=-\eta$, $\eta^2\in\mathcal{O}^\times$ and $\mathcal{O}_n=\mathcal{O}\oplus\mathcal{O}\pi^n\eta$ for all $n\geq 0$.

1.5.3. An F-norm on an F-vector space V is a map $\alpha:V\to\mathbf{R}_{\geq 0}$ such that

$$\alpha(x+y)\leq\max\{\alpha(x),\alpha(y)\},\quad \alpha(ax)=|a|\,\alpha(x)\quad\text{and}\quad\alpha(x)=0\iff x=0$$

for all $x,y\in V$ and $a\in F$. If α is such a norm and $\lambda\in\mathbf{R}$, we denote by

$$B^0(\alpha,\lambda)\overset{def}{=}\left\{x\in V:\alpha(x)<q^\lambda\right\}\quad\text{and}\quad B(\alpha,\lambda)\overset{def}{=}\left\{x\in V:\alpha(x)\leq q^\lambda\right\}$$

its open and closed balls of radius q^λ. These are $\mathcal{O}$-sublattices in V for which $\mathcal{P}B(\alpha,\lambda)\subset B^0(\alpha,\lambda)\subset B(\alpha,\lambda)$. We abusively refer to the quotient $\mathbf{F}$-vector space

$$S(\alpha,\lambda)\overset{def}{=}B(\alpha,\lambda)/B^0(\alpha,\lambda)$$

as the *sphere* of radius q^λ.

2. The linear case

2.1. Classification.

2.1.1. Let $\mathcal{N}$ be the $\mathcal{O}$-linear and additive category whose objects are pairs (V,α) where V is a finite dimensional E-vector space and α is an F-norm on V. A morphism $f:(V_1,\alpha_1)\to(V_2,\alpha_2)$ is an E-linear morphism $f:V_1\to V_2$ such that $\alpha_2(f(v))\leq\alpha_1(v)$ for all $v\in V_1$. The category $\mathcal{N}$ has an internal Hom [1, §3.1.3], a tensor product [1, §3.5] and a duality [1, §3.3] which are compatible with the analogous operations on the category of E-vector spaces:

$$\begin{aligned}\underline{\mathrm{Hom}}\left((V_1,\alpha_1),(V_2,\alpha_2)\right)&=(\mathrm{Hom}_E(V_1,V_2),\mathrm{Hom}(\alpha_1,\alpha_2))\\(V_1,\alpha_1)\otimes(V_2,\alpha_2)&=(V_1\otimes_E V_2,\alpha_1\otimes\alpha_2)\\(V,\alpha)^\star&=(V^\star,\alpha^\star)\end{aligned}$$

2.1.2. Let (V,α) be a 1-dimensional object of $\mathcal{N}$. There exists a non-zero element e of V and a pair $(\rho,c)\in\mathbf{R}\times\mathbf{R}_{\geq 0}$ such that for all $\lambda\in E$, $\alpha(\lambda e)=q^\rho\,\|\lambda\|_c$ where the F-norm $\|\bullet\|_c:E\to\mathbf{R}_{\geq 0}$ is defined by

$$\forall x,y\in F:\qquad\|x+y\eta\|_c=\max\left\{q^{-c}\,|x|\,,|y|\right\}.$$

We say that α has type (ρ,c) with respect to e. The possible types of α form a class $[\rho,c]$ in $\mathbf{R}/\mathbf{Z}\times\mathbf{R}_{\geq 0}$ and the map $(V,\alpha)\mapsto[\rho,c]$ induces a bijection

$$\mathcal{L}\ni[V,\alpha]\overset{\simeq}{\longmapsto}[\rho,c]\in\mathbf{R}/\mathbf{Z}\times\mathbf{R}_{\geq 0}$$

where $\mathcal{L}$ is the set of isomorphism classes of 1-dimensional objects in $\mathcal{N}$.

2.1.3. For $i \in \{1,2\}$, let (V_i, α_i) be a 1-dimensional object of type (ρ_i, c_i) with respect to some E-basis e_i of V_i. Then $\alpha_1 \otimes \alpha_2$ is of type $(\rho_1 + \rho_2, \min(c_1, c_2))$ with respect to the E-basis $e_1 \otimes e_2$ of $V_1 \otimes V_2$ [1, §3.5].

2.1.4. Similarly, if (V, α) is of type (ρ, c) with respect to some E-basis e of V, then $(V, \alpha)^\star = (V^\star, \alpha^\star)$ is of type $(c - \rho, c)$ with respect to the dual E-basis $e^\star$ of $V^\star$ [1, §3.3]. This second formula suggests the change of variables

$$(\tfrac{\theta+c}{2}, c) = (\rho, c) \leftrightarrow (\theta, c) = (2\rho - c, c).$$

In these new (θ, c)-coordinates, we have $\mathcal{L} \simeq \mathbf{R}/2\mathbf{Z} \times \mathbf{R}_{\geq 0}$ and the formula for the duality becomes just $(\theta, c) \mapsto (-\theta, c)$.

2.1.5. Any object (V, α) of $\mathcal{N}$ decomposes (non-uniquely) as a direct sum of 1-dimensional subobjects $(V, \alpha) = \oplus(V_i, \alpha_i)$, and the map

$$(V, \alpha) \mapsto [V, \alpha] \stackrel{def}{=} \sum [V_i, \alpha_i]$$

defines a bijection between the set of isomorphism classes of objects in $\mathcal{N}$ and the commutative monoid $\mathbf{N}[\mathcal{L}]$ of effective divisors on $\mathcal{L}$.

2.1.6. For $(V, \alpha) \in \mathcal{N}$, we put $\det(V, \alpha) = (\det V, \det \alpha)$ where

$$\det V \stackrel{def}{=} \Lambda_E^n V \subset V^{\otimes n} \qquad \text{and} \qquad \det \alpha \stackrel{def}{=} \alpha^{\otimes n} | \det V$$

with $n = \dim_E V$. If $(V, \alpha) = \oplus_{i=1}^n (V_i, \alpha_i)$ with α_i of type (ρ_i, c_i) with respect to some E-basis e_i of V_i, then $\det V$ is of type $(\sum \rho_i, \min\{c_i\})$ with respect to the E-basis $e_1 \wedge \cdots \wedge e_n = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}$ of $\det V$ (by 2.1.3).

2.1.7. Fix a finite dimensional E-vector space V and let $\mathcal{N}(V)$ be the set of all F-norms on V: this is the Bruhat-Tits building of $GL_F(V)$. Let $n = \dim_E V$ and

$$\mathbf{N}[\mathcal{L}]^n \stackrel{def}{=} \{D \in \mathbf{N}[\mathcal{L}] : \deg D = n\}.$$

Then by 2.1.5, the map $\alpha \mapsto [\alpha] = [V, \alpha]$ induces a bijection

$$[\bullet] : GL_E(V) \backslash \mathcal{N}(V) \xrightarrow{\simeq} \mathbf{N}[\mathcal{L}]^n.$$

For $[\alpha] = \sum_{i=1}^n [\rho_i, c_i]$, we set $c_{\min}(\alpha) = \min\{c_i\}$ and $c_{\max}(\alpha) = \max\{c_i\}$.

2.1.8. The following lemma should have been included in [1].

Lemma. *Let $GL(\alpha) \subset GL_E(V)$ be the stabilizer of $\alpha \in \mathcal{N}(V)$. Then*

$$\det GL(\alpha) = \mathcal{O}^\times_{\lceil c_{\min}(\alpha) \rceil}.$$

Proof. Fix a decomposition $(V, \alpha) = \oplus(V_i, \alpha_i)$ with (V_i, α_i) of type (ρ_i, c_i). Then

$$\mathcal{O}^\times_{\lceil c_i \rceil} = GL(\alpha_i) = \det GL(\alpha_i) \subset \det GL(\alpha) \subset GL(\det \alpha) = \mathcal{O}^\times_{\lceil c_{\min}(\alpha) \rceil}$$

using 2.1.6 for the last equality. This proves our claim since $c_{\min}(\alpha) = \min\{c_i\}$. □

2.2. **Filtrations.** Fix an F-norm α on a finite dimensional E-vector space V.

2.2.1. For any real number $c \geq 0$, let α_c be the F-norm on $V \simeq V \otimes_E E$ defined by $\alpha_c = \alpha \otimes \|\bullet\|_c$, so that $\alpha_c = \alpha$ for all $c \geq c_{\max}(\alpha)$ [1, §3.5]. Since the identity map on E is a morphism $(E, \|\bullet\|_{c_1}) \to (E, \|\bullet\|_{c_2})$ for any $c_1 \leq c_2$, we actually obtain a family of compatible norms on V,

$$\alpha_0 \to \cdots \to \alpha_{c_1} \to \cdots \to \alpha_{c_2} \to \cdots \to \alpha.$$

Applying the same construction to the dual norm $\alpha^\star$ on $V^\star$ [1, §3.3] and dualizing back to V, we obtain a dual compatible family of F-norms on V,

$$\alpha \to \cdots \to \alpha^{c_2} \to \cdots \to \alpha^{c_1} \to \cdots \to \alpha^0$$

where $\alpha^c = ((\alpha^\star)_c)^\star$ with again $\alpha^c = \alpha$ for $c \geq c_{\max}(\alpha)$.

2.2.2. For any $c \geq 0$ and $\lambda \in \mathbf{R}$, we define $\mathbf{F}$-subspaces of $S(\alpha, \lambda)$ by

$$S_c(\alpha, \lambda) = \ker\left(S(\alpha, \lambda) \to S(\alpha^c, \lambda)\right) \quad \text{and} \quad S^c(\alpha, \lambda) = \operatorname{im}\left(S(\alpha_c, \lambda) \to S(\alpha, \lambda)\right).$$

Therefore $S_c(\alpha, \lambda)$ is decreasing from $S_0(\alpha, \lambda)$ to $S_\infty(\alpha, \lambda) = 0$, while $S^c(\alpha, \lambda)$ is increasing from $S^0(\alpha, \lambda)$ to $S^\infty(\alpha, \lambda) = S(\alpha, \lambda)$. From the study of the 1-dimensional case below (together with 2.1.5), we find that $S_c(\alpha, \lambda) \subset S^c(\alpha, \lambda)$ for all $c \geq 0$. We have thus obtained a canonical filtration of $S(\alpha, \lambda)$ by $\mathbf{F}$-subspaces

$$0 \subset \cdots \subset S_c(\alpha, \lambda) \subset \cdots \subset S_0(\alpha, \lambda) \subset S^0(\alpha, \lambda) \subset \cdots \subset S^c(\alpha, \lambda) \subset \cdots \subset S(\alpha, \lambda).$$

2.2.3. For any $c \geq 0$ and $\lambda \in \mathbf{R}$, we define a subquotient of $S(\alpha, \lambda)$ by

$$S(\alpha, \lambda, c) = S^c(\alpha, \lambda)/S_c(\alpha, \lambda).$$

We view this subquotient as the middle term (that is, the image or coimage) in the canonical factorization of the composite map $S(\alpha_c, \lambda) \to S(\alpha, \lambda) \to S(\alpha^c, \lambda)$:

$$\begin{array}{ccccc} & & S(\alpha, \lambda) & & \\ & \nearrow & & \searrow & \\ S(\alpha_c, \lambda) & \twoheadrightarrow & S(\alpha, \lambda, c) & \hookrightarrow & S(\alpha^c, \lambda) \end{array}$$

Since multiplication by η on E is a morphism $(E, \|\bullet\|_c) \to (E, q^{-c}\|\bullet\|_c)$ in the category $\mathcal{N}$, multiplication by η on V yields a commutative diagram

$$\begin{array}{ccc} \alpha_c & \xrightarrow{\times 1} & \alpha^c \\ \times\eta \downarrow & & \downarrow \times\eta \\ q^{-c}\alpha_c & \xrightarrow{\times 1} & q^{-c}\alpha^c \end{array}$$

Passing to the spheres, we obtain the outer commutative square in

$$\begin{array}{ccccc} S(\alpha_c, \lambda) & \twoheadrightarrow & S(\alpha, \lambda, c) & \hookrightarrow & S(\alpha^c, \lambda) \\ [\eta] \downarrow & & [\eta] \downarrow & & [\eta] \downarrow \\ S(\alpha_c, \lambda + c) & \twoheadrightarrow & S(\alpha, \lambda + c, c) & \hookrightarrow & S(\alpha^c, \lambda + c) \end{array}$$

in which the middle vertical map is uniquely determined by the commutativity of the whole diagram. From the analysis of the 1-dimensional case below (together with 2.1.5), we find that for $c > 0$,

$$\ker\left([\eta] : S(\alpha, \lambda, c) \to S(\alpha, \lambda + c, c)\right) = S^c_-(\alpha, \lambda)/S_c(\alpha, \lambda) \tag{2.1}$$

$$\operatorname{im}\left([\eta] : S(\alpha, \lambda - c, c) \to S(\alpha, \lambda, c)\right) = S_c^+(\alpha, \lambda)/S_c(\alpha, \lambda) \tag{2.2}$$

where $S^c_-(\alpha, \lambda) = \cup_{d<c} S^d(\alpha, \lambda)$ and $S_c^+(\alpha, \lambda) = \cap_{d<c} S_d(\alpha, \lambda)$. For $c = 0$, $[\eta]$ is an isomorphism of $S(\alpha, \lambda, 0)$ which defines an $\mathbf{E} = \mathbf{F}[\eta]$-vector space structure on this $\mathbf{F}$-vector space.

2.2.4. For all $c \geq 0$ and $\lambda \in \mathbf{R}$, we finally define

$$\mathrm{Gr}^c S(\alpha,\lambda) = S^c(\alpha,\lambda)/S^c_-(\alpha,\lambda) \quad \text{and} \quad \mathrm{Gr}_c S(\alpha,\lambda) = S^+_c(\alpha,\lambda)/S_c(\alpha,\lambda)$$

where $S^0_-(\alpha,\lambda) = S_0(\alpha,\lambda)$ and $S^+_0(\alpha,\lambda) = S^0(\alpha,\lambda)$, and still denote by

$$[\eta] : \mathrm{Gr}^c S(\alpha,\lambda) \to \mathrm{Gr}_c S(\alpha,\lambda+c)$$

the isomorphism induced by $[\eta] : S(\alpha,\lambda,c) \to S(\alpha,\lambda+c,c)$.

2.2.5. Suppose that $\dim_E V = 1$ and α has type (ρ,d) with respect to the E-basis e of V. Then $\alpha^\star$ is of type $(d-\rho,d)$ with respect to the dual basis $e^\star$ of $V^\star$. For $c \geq d$, $\alpha_c = \alpha = \alpha^c$ and thus for any $\lambda \in \mathbf{R}$,

$$S_c(\alpha,\lambda) = \{0\} \subset S^c(\alpha,\lambda) = S(\alpha,\lambda) = S(\alpha,\lambda,c).$$

For $c < d$, α_c is of type (ρ,c) with respect to e, $(\alpha^\star)_c$ is of type $(d-\rho,c)$ with respect to $e^\star$ and $\alpha^c = ((\alpha^\star)_c)^\star$ is of type $(\rho+c-d,c)$ with respect to e. We have

$$\begin{aligned} S(\alpha,\lambda) = 0 \quad &\text{unless} \quad \lambda \equiv \rho - d \text{ or } \rho - \lceil d \rceil \bmod \mathbf{Z} \\ S(\alpha_c,\lambda) = 0 \quad &\text{unless} \quad \lambda \equiv \rho - c \text{ or } \rho - \lceil c \rceil \bmod \mathbf{Z} \\ S(\alpha^c,\lambda) = 0 \quad &\text{unless} \quad \lambda \equiv \rho - d \text{ or } \rho + c - d - \lceil c \rceil \bmod \mathbf{Z} \end{aligned}$$

Put $n = \lceil d \rceil$. Elementary computations lead to the following table, where the indicated subquotients X of $S(\alpha,\lambda)$ are described as $\mathcal{A}/\mathcal{B}$ if $X = \mathcal{A}e/\mathcal{B}e$.

$c < d$	$d = n$	$d < n$	
λ	$\rho - d$	$\rho - d$	$\rho - n$
$S(\alpha,\lambda)$	$\mathcal{O}_n/\mathcal{P}\mathcal{O}_n$	$\mathcal{O}_n/\mathcal{R}_n$	$\mathcal{R}_n/\mathcal{P}\mathcal{O}_n$
$S_c(\alpha,\lambda)$	$\mathcal{R}_n/\mathcal{P}\mathcal{O}_n$	0	$\mathcal{R}_n/\mathcal{P}\mathcal{O}_n$
$S^c(\alpha,\lambda)$	$\mathcal{R}_n/\mathcal{P}\mathcal{O}_n$	0	$\mathcal{R}_n/\mathcal{P}\mathcal{O}_n$
$S(\alpha,\lambda,c)$	0	0	0

Given the table, it is also easily verified that the map

$$[\eta] : S(\alpha,\lambda,c) \to S(\alpha,\lambda+c,c)$$

is trivial unless $c = d$ and $\lambda \equiv \rho - d \bmod \mathbf{Z}$, in which case it has kernel $S^d_-(\alpha,\lambda)$ and image $S^+_d(\alpha,\lambda+c)$ if $c = d > 0$, and yields the obvious $\mathbf{E} = \mathbf{F}[\eta]$-vector space structure on $S(\alpha,\lambda,0) = S(\alpha,\lambda) = \mathcal{P}^{\rho-\lambda}\mathcal{O}_E \cdot e/\mathcal{P}^{\rho-\lambda+1}\mathcal{O}_E \cdot e \simeq \mathbf{E}$ if $c = d = 0$.

2.2.6. Returning to the general case, the above table shows that the multiplicity $m_{[\rho,c]}(\alpha)$ of $[\rho,c] \in \mathcal{L}$ in $[\alpha] \in \mathbf{N}[\mathcal{L}]$ is given by

$$m_{[\rho,c]}(\alpha) = \begin{cases} \dim_{\mathbf{E}} S(\alpha,\rho,0) & \text{if } c = 0, \\ \dim_{\mathbf{F}} \mathrm{Gr}^c S(\alpha,\rho-c) = \dim_{\mathbf{F}} \mathrm{Gr}_c S(\alpha,\rho) & \text{if } c \neq 0. \end{cases}$$

3. Buildings of orthogonal Groups

Let V be a finite dimensional F-vector space with a non-degenerate symmetric pairing $\phi : V \times V \to F$ and let $G = O(V,\phi)$ be the corresponding orthogonal group.

3.1. The dual of a lattice L in V is the lattice $\widetilde{L} = \{v \in V | \phi(v, L) \subset \mathcal{O}\}$. We say that L is *almost self-dual* if and only if $P\widetilde{L} \subset L \subset \widetilde{L}$. The dual of a π-periodic flag of lattices $\mathcal{B}$ in V is the π-periodic flag of lattices $\widetilde{\mathcal{B}} = \{\widetilde{L} | L \in \mathcal{B}\}$. We say that $\mathcal{B}$ is *self -dual* if and only if $\widetilde{\mathcal{B}} = \mathcal{B}$. The dual of a norm $\alpha : V \to \mathbf{R}$ is the norm $\widetilde{\alpha} : V \to \mathbf{R}$ defined by $\widetilde{\alpha}(v) = \sup_{w \in V} \frac{|\phi(v,w)|}{\alpha(w)}$. We say that α is *self-dual* if and only if $\widetilde{\alpha} = \alpha$. These various concepts are related as follows.

3.2. For a self-dual π-periodic flag of lattices $\mathcal{B}$ in V, the almost self-dual lattices in $\mathcal{B}$ form a non-empty finite segment $|\mathcal{B}| = \{L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_r\}$ of

$$\mathcal{B} = \left\{ \cdots \subsetneq \mathcal{P}\widetilde{L}_0 \subset L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_r \subset \widetilde{L}_r \subsetneq \widetilde{L}_{r-1} \cdots \subsetneq \widetilde{L}_0 \subset \mathcal{P}^{-1} L_0 \subsetneq \cdots \right\}.$$

The map $\mathcal{B} \mapsto |\mathcal{B}|$ yields a G-equivariant bijection between the set of all π-periodic self-dual flags of lattices in V and the set of all non-empty finite chains of almost self-dual lattices in V.

3.3. For a self-dual norm α on V, the flag of balls $B(\alpha) = \{B(\alpha, \lambda) | \lambda \in \mathbf{R}\}$ is a self-dual π-periodic flag of lattices in V and $|B(\alpha)| = \left\{B(\alpha, \lambda) | \lambda \in [0, \tfrac{1}{2}[\right\}$. Indeed

$$\widetilde{B(\alpha, \lambda)} = B^0(\alpha, 1 - \lambda) \quad \text{and} \quad \widetilde{B^0(\alpha, \lambda)} = B(\alpha, 1 - \lambda) \quad \text{for all } \lambda \in \mathbf{R}. \tag{3.1}$$

The map $\alpha \mapsto B(\alpha)$ is a G-equivariant *surjection* between the set of all self-dual norms on V and the set of all π-periodic self-dual flags of lattices in V.

3.4. We say that two self-dual norms α and β on V belong to the same facet $\mathcal{F}$ if and only if $B(\alpha) = B(\beta) = B(\mathcal{F})$. This defines a partition of the set $\mathcal{I}$ of all self-dual norms on V into facets, which may be identified with π-periodic self-dual flags of lattices in V, and also with non-empty finite chains of almost self-dual lattices in V. We define a G-equivariant partial order on the set of facets by

$$\mathcal{F} \prec \mathcal{G} \iff B(\mathcal{F}) \subset B(\mathcal{G}) \iff |B(\mathcal{F})| \subset |B(\mathcal{G})| \,.$$

3.5. The dimension and type of a facet $\mathcal{F}$ with $|B(\mathcal{F})| = \{L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_r\}$ are respectively defined by $\dim \mathcal{F} = r$ and

$$\operatorname{type} \mathcal{F} = \dim_{\mathbf{F}} \left(L_0/\mathcal{P}\widetilde{L}_0, L_1/L_0, \cdots, L_r/L_{r-1}, \widetilde{L}_r/L_r \right) \in \mathbf{N}^{r+2}.$$

These invariants depend only upon the G-orbit of $\mathcal{F}$, and we shall see that the type in fact classifies these G-orbits. The dimension and type of a self-dual norm α are those of its facet $\mathcal{F}(\alpha)$.

3.6. Let $V = V_0 \perp H_1 \perp \cdots \perp H_m$ be a Witt decomposition: V_0 is an anisotropic F-subspace of V and the H_i's are hyperbolic F-planes in V. The apartment $\mathcal{A} \subset \mathcal{I}$ defined by such a decomposition is the set of all self-dual norms α on V such that for all $v_0 \in V_0$ and $h_i \in H_i$,

$$\alpha\left(v_0 + h_1 + \cdots + h_m\right) = \max\left\{\alpha(v_0), \alpha(h_1), \cdots, \alpha(h_m)\right\}.$$

The choice of a Witt basis (e_i, e_{-i}) of H_i for each i defines an isometry

$$\operatorname{coord}_{(e_{\pm i})} : \mathcal{A} \xrightarrow{\simeq} \mathbf{R}^m \quad \text{given by} \quad \operatorname{coord}_{(e_{\pm i})}(\alpha) = (\lambda_1, \cdots, \lambda_m)$$

where $\alpha(e_{\pm i}) = q^{\pm \lambda_i}$ and for all $v_0 \in V_0$ and $x_{\pm i} \in F$,

$$\alpha\left(v_0 + \sum \left(x_i e_i + x_{-i} e_{-i}\right)\right) = \max\left\{ |Q(v_0)|^{1/2}, |x_{\pm i}|\, q^{\pm \lambda_i} \right\}.$$

3.7. Any self-dual norm α is contained in some apartment $\mathcal{A}$ as above, and it is possible to choose the Witt basis (e_i, e_{-i}) of the H_i's so that α has coordinates

$$\mathrm{coord}_{(e_{\pm i})}(\alpha) = (\lambda_1, \cdots, \lambda_m) \quad \text{with } 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \tfrac{1}{2}.$$

These coordinates are then uniquely determined by the G-orbit of α. Indeed, let

$$\delta_0 = \dim_{\mathbf{F}} S(\alpha|V_0, 0) \quad \text{and} \quad \delta_{\frac{1}{2}} = \dim_{\mathbf{F}} S(\alpha|V_0, \tfrac{1}{2}).$$

These integers depend only upon (V, ϕ) and sum to the F-dimension of V_0, the "anisotropic part" of V. For $\lambda \in [0, \frac{1}{2}]$, put $m(\lambda) = |\{i \in 1, \cdots, m | \lambda_i = \lambda\}|$. Then

$$\mathrm{type}\,\mathcal{F}(\alpha) = \left(\delta_0 + 2m(0), m(s_1), \cdots, m(s_r), \delta_{\frac{1}{2}} + 2m(\tfrac{1}{2})\right)$$

where $\{s_1 < \cdots < s_r\} = \{\lambda_1, \cdots, \lambda_m\} \cap]0, \frac{1}{2}[= \{\lambda \in]0, \frac{1}{2}[| S(\alpha, \lambda) \neq 0\}$. We refer to

$$\mathrm{rtype}\,\mathcal{F}(\alpha) \overset{def}{=} \left(m(0), m(s_1), \cdots, m(s_r), m(\tfrac{1}{2})\right) \in \mathbf{N}^{r+2}$$

as the reduced type of the facet and call $\mathrm{inv}(\alpha) \overset{def}{=} (\lambda_1, \cdots, \lambda_n)$ the invariant of α.

3.8. Since all datum $(V_0, (e_{\pm i}))$ as above are conjugated by G, we see that two self-dual norms belong to the same G-orbit if and only if they have the same invariant. The same argument also shows that two facets belong to the same G-orbit if and only if they have the same type (or reduced type). If the anisotropic part V_0 of V is non-trivial, any self-dual norm α on V is fixed by an element $g \in G$ with $\det g = -1$ and we may replace $G = O(V, \phi)$ by $SO(V, \phi)$ in the above statements.

3.9. Let $\mathcal{F}$ be a facet of reduced type $(m_0, m_1, \cdots, m_r, m_{r+1})$. Let α be any self-dual norm in $\mathcal{F}$ and choose an apartment $\mathcal{A}$ containing α, together with a datum $(V_0, (e_{\pm i}))$ as above such that $\mathrm{coord}_{(e_i)}\alpha = \mathrm{inv}\alpha = (\lambda_1, \cdots, \lambda_m)$. Define $\Delta_r = \{0 < t_1 < \cdots < t_r < \frac{1}{2}\}$ and map each element of Δ_r to the norm $\gamma \in \mathcal{A}$ with coordinates $(\nu_1, \cdots, \nu_m)$ characterized by $0 \le \nu_1 \le \cdots \le \nu_m \le \frac{1}{2}$ and

$$m_0 = |\{i|\nu_i = 0\}|, \quad m_{r+1} = \left|\{i|\nu_i = \tfrac{1}{2}\}\right| \quad \text{and} \quad m_j = |\{i|\nu_i = t_j\}|$$

for all $j \in \{1, \cdots, r\}$. Let $\Delta_r((e_{\pm i})) \subset \mathcal{A}$ be the image of this map. An elementary computation shows that for any norm γ in $\mathcal{A}$, $B(\gamma) = B(\alpha) \iff \gamma \in \Delta_r((e_{\pm i}))$. Now let β be any norm in $\mathcal{F}$ and choose $g \in G$ so that $\gamma = g\beta$ lies in $\mathcal{A}$ with coordinates $0 \le \nu_1 \le \cdots \le \nu_m \le \frac{1}{2}$. Since γ and α have the same type, it must be that $\gamma \in \Delta_r((e_{\pm i}))$, so that $B(\gamma) = B(\alpha)$. However $B(\gamma) = g \cdot B(\beta)$ and $B(\alpha) = B(\beta)$, so that $g \cdot B(\beta) = B(\beta)$, hence $g \cdot |B(\beta)| = |B(\beta)|$ and g actually fixes each of the balls of β, i.e. $\gamma = g\beta = \beta$. This shows that $\mathcal{F} = \Delta_r((e_{\pm i}))$.

3.10. A *vertex* is given by one of the following equivalent objects:

- a self-dual norm α on V with $\log_q \alpha(V - \{0\}) \subset \frac{1}{2}\mathbf{Z}$,
- a 0-dimensional facet $\mathcal{F}$,
- an almost self-dual lattice L in V.

The equivalences are given by $\mathcal{F} = \{\alpha\}$ and $|B(\mathcal{F})| = |B(\alpha)| = \{L\}$. We typically denote by α_L the self-dual norm with $|B(\alpha_L)| = \{L\}$. The type of the vertex is

$$\dim_{\mathbf{F}}\left(L/P\widetilde{L}, \widetilde{L}/L\right) = \dim_{\mathbf{F}}\left(S(\alpha, 0), S(\alpha, \tfrac{1}{2})\right) \in \mathbf{N}^2.$$

We denote by $\mathcal{I}(\frac{1}{2})$ the set of all vertices.

3.11. An *edge* is given by one of the following equivalent objects

- a self-dual norm α on V with $\log_q \alpha(V - \{0\}) \subset \frac{1}{4}\mathbf{Z}$,
- a facet $\mathcal{F}$ of dimension ≤ 1,
- a pair of almost self-dual lattices $(L_0 \subset L_1)$ in V,

The equivalences are given by $\mathcal{F} = \mathcal{F}(\alpha)$, $\{L_0, L_1\} = |B(\mathcal{F})| = |B(\alpha)|$, $\mathcal{F}$ is the segment $[\alpha_{L_0}, \alpha_{L_1}]$ joining α_{L_0} and α_{L_1} in the building $\mathcal{I}$, and α is the middle of that segment. We typically write $\alpha = \alpha_{L_0 L_1}$ for this middle point. By convention, the source and target of the edge are the vertices corresponding respectively to L_0 and L_1. The type of the edge is the triple

$$\dim_{\mathbf{F}} \left(L_0/\mathcal{P}\widetilde{L}_0, L_1/L_0, \widetilde{L}_1/L_1 \right) = \dim_{\mathbf{F}} \left(S(\alpha, 0), S(\alpha, \tfrac{1}{4}), S(\alpha, \tfrac{1}{2}) \right) \in \mathbf{N}^3.$$

We denote by $\mathcal{I}(\frac{1}{4})$ the set of all edges.

3.12. For any self-dual norm α on V and $\lambda \in \mathbf{R}$, the formula (3.1) shows that the pairing $\pi\phi : B(\alpha, \lambda) \times B(\alpha, 1 - \lambda) \to \mathcal{O}_F$ induces a perfect pairing

$$\langle \bullet, \bullet \rangle_\lambda : S(\alpha, \lambda) \times S(\alpha, 1 - \lambda) \to \mathbf{F}.$$

These pairings are symmetric and π-periodic in the following sense:

$$\langle x, y \rangle_\lambda = \langle y, x \rangle_{1-\lambda} \quad \text{and} \quad \langle [\pi] X, Y \rangle_\lambda = \langle X, [\pi] Y \rangle_{1+\lambda}$$

for $(x, y) \in S(\alpha, \lambda) \times S(\alpha, 1 - \lambda)$ and $(X, Y) \in S(\alpha, 1 + \lambda) \times S(\alpha, 1 - \lambda)$. If $\lambda \in \frac{1}{2}\mathbf{Z}$, the formula $\phi_\lambda(x, y) = \left\langle x, [\pi^{2\lambda - 1}] y \right\rangle_\lambda$ thus defines a perfect symmetric pairing

$$\phi_\lambda : S(\alpha, \lambda) \times S(\alpha, \lambda) \to \mathbf{F}$$

and $\phi_{\lambda-1}([\pi]x, [\pi]y) = \phi_\lambda(x, y)$.

3.13. With notations as above, an edge of reduced type (a, b, c) is also given by

- a pair (α_0, H_0) where α_0 is a vertex of type $(a, b + c)$ and H_0 is a b-dimensional totally isotropic $\mathbf{F}$-subspace of $(S(\alpha_0, \frac{1}{2}), \phi_{\frac{1}{2}})$;
- a pair (α_1, H_1) where α_1 is a vertex of type $(a + b, c)$ and H_1 is a b-dimensional totally isotropic $\mathbf{F}$-subspace of $(S(\alpha_1, 0), \phi_0)$;

They correspond to the edge $\mathcal{F} = [\alpha_0, \alpha_1]$. If $|B(\mathcal{F})| = \{L_0 \subset L_1\}$, then

$$H_0 = L_1/L_0 \subset \widetilde{L}_0/L_0 = S(\alpha_0, \tfrac{1}{2}) \quad \text{and} \quad H_1 = \mathcal{P}\widetilde{L}_0/\mathcal{P}\widetilde{L}_1 \subset L_1/\mathcal{P}\widetilde{L}_1 = S(\alpha_1, 0).$$

3.14. We need a variant of the above notions involving 2-dimensional facets of reduced type $(a, 1, b, c)$ or $(a, b, 1, c)$, linking edges of reduced type $((a + 1, b, c)$ and $(a, b + 1, c))$ or $((a, b + 1, c)$ and $(a, b, c + 1))$ respectively. In view of 3.9, the distance between the related two edges (viewed as norms) will thus be equal to $\frac{1}{4}$. The following *ad-hoc* definition is hand-picked for our purposes. For any $\lambda \in \frac{1}{4}\mathbf{Z}$, we define a λ-corner to be either one of the following equivalent objects:

- a pair (α_1, D_1) where $\alpha_1 \in \mathcal{I}(\frac{1}{4})$ is an edge and $D_1 \subset S(\alpha_1, \lambda + \frac{1}{4})$ is an $\mathbf{F}$-line such that $D_1 \subset D_1^\perp$ if $\lambda + \frac{1}{4}$ belongs to $\frac{1}{2}\mathbf{Z}$, where $D_1^\perp$ is the orthogonal complement of D_1 for $\phi_{\lambda + \frac{1}{4}}$;
- a pair (α_2, H_2) where $\alpha_2 \in \mathcal{I}(\frac{1}{4})$ is an edge and $H_2 \subset S(\alpha_2, \lambda)$ is an $\mathbf{F}$-hyperplane such that $H_2^\perp \subset H_2$ if λ belongs to $\frac{1}{2}\mathbf{Z}$, where $H_2^\perp$ is the orthogonal complement of H_2 for ϕ_λ.

The equivalence is now given by the following rules. First of all, α_1 and α_2 have the same closed ball at $\lambda + \frac{\epsilon}{4}$, where $\epsilon = 1$ if $\lambda \in \frac{1}{2}\mathbf{Z}$ and $\epsilon = -1$ otherwise. Let next $B_1^0 \subset B_1$ and $B_2^0 \subset B_2$ be the open and closed balls of α_1 at $\lambda + \frac{1}{4}$ and α_2 at λ. Then $B_2^0 \subset B_1^0 \subsetneq B_2 \subset B_1$ and

$$D_1 = B_2/B_1^0 \subset B_1/B_1^0 = S(\alpha_1, \lambda + \tfrac{1}{4})), \quad H_2 = B_1^0/B_2^0 \subset B_2/B_2^0 = S(\alpha_2, \lambda).$$

We denote such a λ-corner by $(\alpha_1, D_1) \mapsto (\alpha_2, H_2)$.

4. The even orthogonal case

4.1. **Classification.** We recall the following results of [1, §4].

4.1.1. Let $\mathcal{M}$ be the additive category of triples (V, ψ, α) where (V, ψ) is a finite dimensional E-hermitian space and α is a self-dual F-norm on the underlying symmetric F-space (V, ϕ), with $\phi = \mathrm{tr}\psi$. A morphism $f : (V_1, \psi_1, \alpha_1) \to (V_2, \psi_2, \alpha_2)$ is an E-linear morphism $f : V_1 \to V_2$ such that $\psi_2(f(v), f(w)) = \psi_1(v, w)$ and $\alpha_2(f(v)) \leq \alpha_1(v)$ for all $v, w \in V_1$.

4.1.2. There are four isomorphism classes of anisotropic E-hermitian space, namely the classes $\mathcal{A}_0$, $\mathcal{A}_1, \mathcal{A}'_1$ and $\mathcal{A}_2$ which are respectively represented by

$$A_0 = (0, 0), \quad A_1 = (E, \overline{x}y), \quad A'_1 = (E, \pi\overline{x}y) \quad \text{and} \quad A_2 = A_1 \perp A'_1.$$

Each of these spaces carries a unique self-dual F-norm, namely the F-norm defined by $x \mapsto |Q(x)|^{1/2}$ where $Q(x) = \frac{1}{2}\phi(x, x) = \psi(x, x)$ is the underlying quadratic form. The invariant of these norms are respectively equal to

$$a_0 = 0, \quad a_1 = [0, 0], \quad a'_1 = [1, 0] \quad \text{and} \quad a_2 = a_1 + a'_1 \quad \text{in } \mathbf{N}[\mathcal{L}]$$

where we have used the (θ, c)-coordinates for a_1 and a'_1. The pair of integers $(\delta_0, \delta_{1/2})$ attached to the underlying anisotropic symmetric spaces in section 3.7 equals $2(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_2})$ with $(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_2})$ respectively given by

$$(0, 0), \quad (1, 0), \quad (0, 1) \quad \text{and} \quad (1, 1).$$

4.1.3. For any object (V, ψ, α) of $\mathcal{M}$, there exists a Witt decomposition

$$(V, \alpha) = (V_0, \alpha_0) \perp (V_+, \alpha_+) \oplus (V_-, \alpha_-)$$

where V_0 and $V_\pm$ are respectively anisotropic and totally isotropic E-subspaces of (V, ψ). For any such decomposition, α_0 is a self-dual norm on V_0 and $x \mapsto \eta\psi(x, \bullet)$ defines an isomorphism between (V_-, α_-) and $(V_+, \alpha_+)^\star$. The isomorphism class of (V, ψ, α) is uniquely determined by the isomorphism class of $(V_0, \psi|V_0)$ (itself uniquely determined by the isomorphism class of (V, ψ)) together with the invariant

$$\omega(\alpha) \overset{def}{=} [V_+, \alpha_+] = [V_-, \alpha_-] \in \mathbf{N}[\overline{\mathcal{L}}]$$

where $\overline{\mathcal{L}}$ is the quotient of $\mathcal{L}$ induced by the duality $\star$ on $\mathcal{N}$. Note that using the (θ, c)-coordinates, we have $\overline{\mathcal{L}} \simeq \mathbf{R}/\sim \times \mathbf{R}_{\geq 0}$ where the equivalence relation $\sim$ on $\mathbf{R}$ is defined by $\theta_1 \sim \theta_2 \iff \theta_1 = \pm\theta_2 \bmod 2\mathbf{Z}$.

4.1.4. Fix a finite dimensional E-hermitian space (V,ψ) and let $\mathcal{I}(V,\phi)$ be the set of all self-dual norms on the underlying symmetric F-space (V,ϕ): this is the Bruhat-Tits building of $SO(V,\phi)$. Let $n = \mathrm{witt}_E(V,\psi)$ be the Witt index and set

$$\mathbf{N}[\overline{\mathcal{L}}]^n \overset{def}{=} \left\{D \in \mathbf{N}[\overline{\mathcal{L}}] : \deg D = n\right\}.$$

Then by 4.1.3, the map $\alpha \mapsto \omega(\alpha)$ induces a bijection

$$\omega : U(V,\psi)\backslash\mathcal{I}(V,\phi) \xrightarrow{\simeq} \mathbf{N}[\overline{\mathcal{L}}]^n.$$

If $\omega(\alpha) = \sum_{i=1}^n [\theta_i, c_i]$, then $c_{\max}(\alpha) = \max\{c_i\}$ but

$$c_{\min}(\alpha) = \begin{cases} 0 & \text{if } \dim_E V \neq 2n, \\ \min\{c_i\} & \text{if } \dim_E V = 2n. \end{cases}$$

4.1.5. For $r \in \mathbf{N}$, let $U_r = \{z/\overline{z} : z \in \mathcal{O}_r^\times\}$. Thus U_0 is the group of norm 1 elements in $E^\times$ and $U_r \subset \mathcal{O}_r^\times \cap U_0$. In fact, $U_r \times \{\pm 1\} = \mathcal{O}_r^\times \cap U_0$ for all $r > 0$, but we will not use this equality.

Lemma. *Let $U(\alpha) \subset U(V,\psi)$ be the stabilizer of $\alpha \in \mathcal{I}(V,\phi)$. Then*

$$\det U(\alpha) = U_r \quad \textit{with } r = \lceil c_{\min}(\alpha) \rceil.$$

Proof. Fix a Witt decomposition $(V,\alpha) = (V_0,\alpha_0) \perp\!\!\perp_{i=1}^n (V_i,\alpha_i) \oplus (V_{-i},\alpha_{-i,})$ where V_0 is an anisotropic E-subspace of V and the $V_{\pm i}$'s are isotropic E-lines, with α_i of type (ρ_i, c_i) on V_i. Then

$$\det U(\alpha_0) \text{ and } \det U(\alpha_i \oplus \alpha_{-i}) \subset \det U(\alpha) \subset \det GL_E(\alpha) \cap U_0$$

where $GL_E(\alpha)$ is the stabilizer of α in $GL_E(V)$. Recall from lemma 2.1.8 that $\det GL_E(\alpha) = \mathcal{O}_r^\times$. On the other hand, it is easy to see that $U_{r_i} \subset \det U(\alpha_i \oplus \alpha_{-i})$ for $r_i = \lceil c_i \rceil$ $(i \in \{1, \cdots, n\})$. Since $U(\alpha_0) = U(V_0, \psi|V_0)$, we also have

$$\det U(\alpha_0) = \begin{cases} \{1\} & \text{if } V_0 = 0, \\ U_0 & \text{if } V_0 \neq 0. \end{cases}$$

Therefore $U_r \subset \det U(\alpha) \subset \mathcal{O}_r^\times \cap U_0$, which unfortunately only proves the lemma when $r = 0$. For the general case, see the appendix. □

4.2. **Vertices and edges.** Fix a finite dimensional E-hermitian space (V,ψ) with underlying symmetric form ϕ and let $\mathcal{I} = \mathcal{I}(V,\phi)$ be the building of $SO(V,\phi)$.

4.2.1. For any self-dual norm α on V with $\omega(\alpha) = \sum_{i=1}^n [\rho_i, c_i] = \sum_{i=1}^n [\theta_i, c_i]$,

$$\begin{array}{lcccc} \alpha \text{ is a vertex} & \Leftrightarrow & \forall i : \rho_i, c_i \in \frac{1}{2}\mathbf{Z} & \Leftrightarrow & \forall i : \theta_i, c_i \in \frac{1}{2}\mathbf{Z} \text{ and } \theta_i \equiv c_i \bmod \mathbf{Z} \\ \alpha \text{ is an edge} & \Leftrightarrow & \forall i : \rho_i, c_i \in \frac{1}{4}\mathbf{Z} & \Leftrightarrow & \forall i : \theta_i, c_i \in \frac{1}{4}\mathbf{Z} \text{ and } \theta_i \equiv c_i \bmod \frac{1}{2}\mathbf{Z} \end{array}$$

In the following tables, we list and name the various elements of $\overline{\mathcal{L}}$ which may occur in the ω-invariant of vertices or edges. For each of them, we describe the almost self-dual balls $L_0 \subset L_1$ of a self-dual norm α on an hyperbolic E-plane $\mathbf{H}$ which has the prescribed ω-invariant. The norm α splits according to a Witt decomposition $\mathbf{H} = Ee_+ \oplus Ee_-$ with $\eta\psi(e_+, e_-) = 1$, and its restriction to Ee_+ has the indicated type with respect to e_+. The parameter is an integer $r \in \mathbf{N}$, and the ball $L = \mathcal{A}_+ e_+ \oplus \mathcal{A}_- \pi^{-r} e_-$ is described as $\mathcal{A}_+ \oplus \mathcal{A}_-$. The first three lines corresponds to vertices, and the remaining seven lines to edges.

Name	(θ, c)	L_0	L_1	$\widetilde{L}_1$	$\widetilde{L}_0$
$\mathbf{0}_r$	(r, r)	$\mathcal{O}_r \oplus \mathcal{O}_r$			
$\mathbf{1}_{r+1}$	$(r+\frac{1}{2}, r+\frac{1}{2})$	$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$		$\mathcal{P}^{-1}\mathcal{O}_{r+1} \oplus \mathcal{O}_r$	
$\mathbf{2}_r$	$(r+1, r)$	$\mathcal{P}\mathcal{O}_r \oplus \mathcal{O}_r$		$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_r$	
$\mathbf{m}_r$	$(r+\frac{1}{2}, r)$	$\mathcal{P}\mathcal{O}_r \oplus \mathcal{O}_r$	$\mathcal{O}_r \oplus \mathcal{O}_r$		$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_r$
$\mathbf{01}_{r+1}$	$(r+\frac{1}{4}, r+\frac{1}{4})$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_r$	$\mathcal{O}_r \oplus \mathcal{O}_r$		$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$
$\mathbf{21}_{r+1}$	$(r+\frac{3}{4}, r+\frac{1}{4})$	$\mathcal{P}\mathcal{O}_r \oplus \mathcal{O}_r$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_r$	$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$	$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_r$
$\mathbf{02}_{r+1}$	$(r, r+\frac{1}{2})$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_{r+1}$	$\mathcal{O}_r \oplus \mathcal{O}_r$		$\mathcal{P}^{-1}\mathcal{O}_{r+1} \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$
$\mathbf{20}_{r+1}$	$(r+1, r+\frac{1}{2})$	$\mathcal{P}\mathcal{O}_r \oplus \mathcal{O}_r$	$\mathcal{O}_{r+1} \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$		$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_r$
$\mathbf{10}_{r+1}$	$(r+\frac{3}{4}, r+\frac{3}{4})$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_r$	$\mathcal{O}_{r+1} \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$		$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$
$\mathbf{12}_{r+1}$	$(r+\frac{1}{4}, r+\frac{3}{4})$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_{r+1}$	$\mathcal{O}_{r+1} \oplus \mathcal{O}_r$	$\mathcal{O}_r \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$	$\mathcal{P}^{-1}\mathcal{O}_{r+1} \oplus \mathcal{P}^{-1}\mathcal{O}_{r+1}$

4.2.2. Let $\mathcal{S}$ be the monoid of all finitely supported functions $s : \mathbf{N} \to \mathbf{N}$. Define

$$\mathbf{Ver} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \quad \text{and} \quad \mathbf{Edg} = \mathbf{Ver} \cup \{\mathbf{m}, \mathbf{01}, \mathbf{10}, \mathbf{02}, \mathbf{20}, \mathbf{12}, \mathbf{21}\}.$$

For any symbol $\mathbf{X} \in \mathbf{Edg}$ and $s \in \mathcal{S}$, define

$$s \cdot \mathbf{X} = \begin{cases} \sum_{r \geq 0} s(r) \cdot \mathbf{X}_r & \text{if } \mathbf{X} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{m}\} \\ \sum_{r \geq 0} s(r) \cdot \mathbf{X}_{r+1} & \text{if } \mathbf{X} \in \{\mathbf{01}, \mathbf{10}, \mathbf{02}, \mathbf{20}, \mathbf{12}, \mathbf{21}\} \end{cases} \quad \text{in } \mathbf{N}[\overline{\mathcal{L}}].$$

Let $\mathcal{V}$ and $\mathcal{E}$ be the set of all functions $\mathbf{X} \mapsto s_{\mathbf{X}}$ on respectively $\mathbf{Ver}$ and $\mathbf{Edg}$ with values in $\mathcal{S}$, and view $\mathcal{V}$ and $\mathcal{E}$ as embedded in $\mathbf{N}[\overline{\mathcal{L}}]$ by $s_\star \mapsto \sum_{\mathbf{X}} s_{\mathbf{X}} \cdot \mathbf{X}$. Define functions $I : \mathcal{S} \to \mathbf{N}$, $I : \mathcal{V} \to \mathbf{N}$ and $I : \mathcal{E} \to \mathbf{N}$ by $I(s) = \sum_{r \geq 0} s(r)$ and $I(s_\star) = \sum_{\mathbf{X}} I(s_{\mathbf{X}})$. Let $\mathcal{S}^m \subset \mathcal{S}$, $\mathcal{V}^m \subset \mathcal{V}$ and $\mathcal{E}^m \subset \mathcal{E}$ be the fibers of I above $m \in \mathbf{N}$. Define the reduced type of an element $s_\star \in \mathcal{V}$ by the formula

$$\operatorname{rtype}(s_\star) = (I(2s_{\mathbf{0}} + s_{\mathbf{1}}), I(s_{\mathbf{1}} + 2s_{\mathbf{2}})) \in \mathbf{N}^2.$$

Define also $[1] : \mathcal{S} \to \mathcal{S}$ by $([1]s)(0) = 0$ and $([1]s)(r+1) = s(r)$. Finally, define projections $\operatorname{source}, \operatorname{target} : \mathcal{E} \to \mathcal{V}$ by the following formulas:

$$\operatorname{source}(s_\star) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} s_{\mathbf{0}} \\ s_{\mathbf{1}} + s_{\mathbf{01}} + s_{\mathbf{10}} \\ s_{\mathbf{2}} + s_{\mathbf{m}} + s_{\mathbf{02}}[1] + s_{\mathbf{20}} + s_{\mathbf{12}}[1] + s_{\mathbf{21}} \end{pmatrix}$$

$$\operatorname{target}(s_\star) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} s_{\mathbf{0}} + s_{\mathbf{m}} + s_{\mathbf{01}} + s_{\mathbf{10}}[1] + s_{\mathbf{02}} + s_{\mathbf{20}}[1] \\ s_{\mathbf{1}} + s_{\mathbf{12}} + s_{\mathbf{21}} \\ s_{\mathbf{2}} \end{pmatrix}$$

Note that $I \circ \operatorname{source} = I = I \circ \operatorname{target} : \mathcal{V} \to \mathbf{N}$.

4.2.3. Let $U = U(V, \psi)$. Then the ω-invariant $\alpha \in \mathcal{I} \mapsto \omega(\alpha) \in \mathbf{N}[\overline{\mathcal{L}}]$ induces

- a bijection between the U-orbits of vertices $U \backslash \mathcal{I}(\frac{1}{2})$ and $\mathcal{V}^n$,
- a bijection between the U-orbits of edges $U \backslash \mathcal{I}(\frac{1}{4})$ and $\mathcal{E}^n$.

Moreover, $\operatorname{rtype}(\alpha) = \operatorname{rtype} \circ \omega(\alpha)$ for α in $\mathcal{V}^n$ and

$$U \backslash \mathcal{I}(\tfrac{1}{4}) \xrightarrow{\text{source}} U \backslash \mathcal{I}(\tfrac{1}{2}) \xrightarrow{\omega} \mathcal{V}^n = U \backslash \mathcal{I}(\tfrac{1}{4}) \xrightarrow{\omega} \mathcal{E}^n \xrightarrow{\text{source}} \mathcal{V}^n,$$

$$U \backslash \mathcal{I}(\tfrac{1}{4}) \xrightarrow{\text{target}} U \backslash \mathcal{I}(\tfrac{1}{2}) \xrightarrow{\omega} \mathcal{V}^n = U \backslash \mathcal{I}(\tfrac{1}{4}) \xrightarrow{\omega} \mathcal{E}^n \xrightarrow{\text{target}} \mathcal{V}^n.$$

This follows from the above tables.

4.3. **Filtrations.** Fix a self-dual F-norm α on a finite dimension E-hermitian space (V, ψ) with underlying symmetric space (V, ϕ). We denote by $(\Delta_{\mathbf{0}}, \Delta_{\mathbf{2}})$ the pair of integers attached to the anisotropic part of (V, ψ) in section 4.1.2.

4.3.1. For $c \geq 0$, the F-norms α_c and α^c are exchanged by the involution $\beta \mapsto \widetilde{\beta}$ of section 3.1, see [1, 4.1.2]. Thus for any $\lambda \in \mathbf{R}$,

$$\widetilde{B(\alpha_c, \lambda)} = B^0(\alpha^c, 1-\lambda) \quad \text{and} \quad \widetilde{B^0(\alpha_c, \lambda)} = B(\alpha^c, 1-\lambda).$$

The pairing $\pi\phi : B(\alpha_c, \lambda) \times B(\alpha^c, 1-\lambda) \to \mathcal{O}_F$ therefore induces a perfect pairing on the corresponding spheres, which fits into the following commutative diagram:

$$\begin{array}{ccccc} S(\alpha_c, \lambda) & \times & S(\alpha^c, 1-\lambda) & \longrightarrow & \mathbf{F} \\ \downarrow & & \uparrow & & \| \\ S(\alpha, \lambda) & \times & S(\alpha, 1-\lambda) & \xrightarrow{\langle\cdot,\cdot\rangle_\lambda} & \mathbf{F} \end{array}$$

It follows that for any $\lambda \in \mathbf{R}$ and $c \geq 0$,

$$S_c(\alpha, \lambda)^\perp = S^c(\alpha, 1-\lambda) \quad \text{and} \quad S^c(\alpha, \lambda)^\perp = S_c(\alpha, 1-\lambda)$$

where the orthogonal complements are taken with respect to $\langle \bullet, \bullet \rangle_\lambda$.

4.3.2. We thus obtain a family of derived perfect pairings

$$\langle \bullet, \bullet \rangle_{\lambda,c} : S(\alpha, \lambda, c) \times S(\alpha, 1-\lambda, c) \to \mathbf{F}$$

which are symmetric and π-periodic in the sense that

$$\langle x, y \rangle_{\lambda,c} = \langle y, x \rangle_{1-\lambda,c} \quad \text{and} \quad \langle [\pi]X, Y \rangle_{\lambda,c} = \langle X, [\pi]Y \rangle_{1+\lambda,c}$$

for all $(x, y) \in S(\alpha, \lambda, c) \times S(\alpha, 1-\lambda, c)$ and $(X, Y) \in S(\alpha, 1+\lambda, c) \times S(\alpha, 1-\lambda, c)$. But now also for $(x, y) \in S(\alpha, \lambda, c) \times S(\alpha, 1-\lambda-c, c)$,

$$\langle [\eta]x, y \rangle_{\lambda+c,c} + \langle x, [\eta]y \rangle_{\lambda,c} = 0$$

because $\phi(\eta x, y) + \phi(x, \eta y) = 0$ since $\mathrm{tr}_{E/F}\eta = 0$.

4.3.3. The formula $[x, y]_{\lambda,c} = \langle [\eta]x, y \rangle_{\lambda-c,c}$ thus defines a family of perfect pairings

$$[\bullet, \bullet]_{\lambda,c} : \mathrm{Gr}_c S(\alpha, \lambda) \times \mathrm{Gr}_c S(\alpha, 1+c-\lambda) \to \mathbf{F}$$

which are now skew-symmetric and π-periodic in the sense that

$$[x, y]_{\lambda,c} + [y, x]_{1+c-\lambda,c} = 0 \quad \text{and} \quad [[\pi]X, Y]_{\lambda,c} = [X, [\pi]Y]_{1+\lambda,c}$$

for all x, y, X and Y in the relevant spaces.

4.3.4. If $\lambda \in \frac{1}{2}\mathbf{Z}$, then $S_c(\alpha, \lambda)^\perp = S^c(\alpha, \lambda)$ for the symmetric pairing ϕ_λ on $S(\alpha, \lambda)$. The $S_c(\alpha, \lambda)$'s are therefore totally isotropic $\mathbf{F}$-subspaces of $(S(\alpha, \lambda), \phi_\lambda)$. The induced perfect symmetric pairings

$$\phi_{\lambda,c} : S(\alpha, \lambda, c) \times S(\alpha, \lambda, c) \to \mathbf{F}$$

may also be defined by $\phi_{\lambda,c}(x, y) = \left\langle x, [\pi^{2\lambda-1}]y \right\rangle_{\lambda,c}$. If also c belongs to $\mathbf{Z}$, then

$$\phi_{\lambda,c}([\pi^c\eta]x, y) + \phi_{\lambda,c}(x, [\pi^c\eta]y) = 0$$

for all $x, y \in S(\alpha, \lambda, c)$.

4.3.5. If similarly $2\lambda \in c+\mathbf{Z}$, we obtain a perfect symplectic pairing

$$\theta_{\lambda,c} : \mathrm{Gr}_c S(\alpha,\lambda) \times \mathrm{Gr}_c S(\alpha,\lambda) \to \mathbf{F}$$

defined by $\theta_{\lambda,c}(x,y) = \left[x,[\pi^{2\lambda-1-c}]y\right]_{\lambda,c}$. If also c belongs to $\mathbf{Z}$ (i.e. $\lambda \in \frac{1}{2}\mathbf{Z}$), the pairings $\phi_{\lambda,c}$ and $\theta_{\lambda,c}$ are related by the following commutative diagram:

$$\begin{array}{ccccccc} \phi_{\lambda,c}: & \mathrm{Gr}^c S(\alpha,\lambda) & \times & \mathrm{Gr}_c S(\alpha,\lambda) & \longrightarrow & \mathbf{F} \\ & [\pi^c\eta]\downarrow\simeq & & \| & & \| \\ \theta_{\lambda,c}: & \mathrm{Gr}_c S(\alpha,\lambda) & \times & \mathrm{Gr}_c S(\alpha,\lambda) & \longrightarrow & \mathbf{F} \end{array}$$

For $c=0$ and $\lambda \in \frac{1}{2}\mathbf{Z}$, these pairings together define a perfect hermitian pairing

$$\psi_{\lambda,0} : S(\alpha,\lambda,0) \times S(\alpha,\lambda,0) \to \mathbf{E} = \mathbf{F}[\eta]$$

on the $\mathbf{E}$-vector space $S(\alpha,\lambda,0)$ by $\psi_{\lambda,0}(x,y) = \phi_{\lambda,0}(x,y) - \theta_{\lambda,0}(x,y)\cdot\eta$.

4.3.6. Suppose now that $\alpha \in \mathcal{I}(\frac{1}{2})$, i.e. α is a vertex. Then $\mathrm{Gr}_c S(\alpha,\lambda) = 0$ unless $(\lambda,c) \in \frac{1}{2}\mathbf{Z} \times \frac{1}{2}\mathbf{N}$, in which case $S_c(\alpha,\lambda)^{\perp} = S^c(\alpha,\lambda)$ for the perfect symmetric pairing ϕ_λ on $S(\alpha,\lambda)$. If $\omega(\alpha) = s_\star \in \mathcal{V}$, then

$$\Delta_{\mathbf{0}} + 2s_{\mathbf{0}}(0) = \dim_{\mathbf{E}} S(\alpha,0,0) \quad \text{and} \quad \Delta_{\mathbf{2}} + 2s_{\mathbf{2}}(0) = \dim_{\mathbf{E}} S(\alpha,\tfrac{1}{2},0)$$

and for all $r \in \mathbf{N}$,

$$s_{\mathbf{1}}(r) = \dim_{\mathbf{F}} \mathrm{Gr}_{r+\frac{1}{2}} S(\alpha,0) = \dim_{\mathbf{F}} \mathrm{Gr}_{r+\frac{1}{2}} S(\alpha,\tfrac{1}{2})$$

while for $r>0$,

$$2s_{\mathbf{0}}(r) = \dim_{\mathbf{F}} \mathrm{Gr}_r S(\alpha,0) \quad \text{and} \quad 2s_{\mathbf{2}}(r) = \dim_{\mathbf{F}} \mathrm{Gr}_r S(\alpha,\tfrac{1}{2}).$$

This follows from the definitions by inspection of the tables in 4.2.1 and 2.2.5.

4.3.7. Let still $\alpha \in \mathcal{I}(\frac{1}{2})$ be a vertex with $\omega(\alpha) = t_\star \in \mathcal{V}$, and recall from section 3.13 that the edges $\beta \in \mathcal{I}(\frac{1}{4})$ with target α correspond to totally isotropic $\mathbf{F}$-subspaces $\mathcal{H}$ of $(S(\alpha,0),\phi_0)$. We shall now relate $\omega(\beta) = e_\star \in \mathcal{E}$ to the relative positions of $\mathcal{H}$ and the above filtrations on $S(\alpha,0)$. We already know that

$$\begin{array}{rcl} t_{\mathbf{0}} & = & e_{\mathbf{0}} + e_{\mathbf{m}} + e_{\mathbf{01}} + e_{\mathbf{10}}[1] + e_{\mathbf{02}} + e_{\mathbf{20}}[1] \\ t_{\mathbf{1}} & = & e_{\mathbf{1}} + e_{\mathbf{12}} + e_{\mathbf{21}} \\ t_{\mathbf{2}} & = & e_{\mathbf{2}} \end{array}$$

For $c \in \frac{1}{2}\mathbf{N}$, define

$$\begin{array}{rclcrcl} \mathcal{H}_c & = & \mathcal{H} \cap S_c(\alpha,0) & & \mathcal{H}^c & = & \mathcal{H} \cap S^c(\alpha,0) \\ \mathcal{H}_c^+ & = & \mathcal{H} \cap S_c^+(\alpha,0) & \text{and} & \mathcal{H}_-^c & = & \mathcal{H} \cap S_-^c(\alpha,0) \\ \mathrm{Gr}_c\mathcal{H} & = & \mathcal{H}_c^+/\mathcal{H}_c & & \mathrm{Gr}^c\mathcal{H} & = & \mathcal{H}^c/\mathcal{H}_-^c \end{array}$$

so that $\mathrm{Gr}_c\mathcal{H}$ and $\mathrm{Gr}^c\mathcal{H}$ are $\mathbf{F}$-subspaces of $\mathrm{Gr}_c S(\alpha,0)$ and $\mathrm{Gr}^c S(\alpha,0)$ which are orthogonal with respect to ϕ_0^c, while $\mathcal{H}(c) = \mathcal{H}^c/\mathcal{H}_c$ is a totally isotropic $\mathbf{F}$-subspace of $(S(\alpha,0,c),\phi_{0,c})$. If moreover $c=r>0$ is a positive integer, define orthogonal $\mathbf{F}$-subspaces of the symplectic $\mathbf{F}$-space $(\mathrm{Gr}_r S(\alpha,0),\theta_{0,r})$ by

$$t_r\mathcal{H} = [\pi^r\eta]\mathrm{Gr}^r\mathcal{H} \quad \text{and} \quad b_r\mathcal{H} = \mathrm{Gr}_r\mathcal{H}.$$

Let $t_r^{\perp}\mathcal{H} \supset b_r\mathcal{H}$ and $b_r^{\perp}\mathcal{H} \supset t_r\mathcal{H}$ be their orthogonal complements. Elementary computations then show that for all $r \in \mathbf{N}$,

$$e_{\mathbf{21}}(r) = \dim_{\mathbf{F}} \mathrm{Gr}_{r+\frac{1}{2}}\mathcal{H} \quad \text{and} \quad e_{\mathbf{12}}(r) = \dim_{\mathbf{F}} \mathrm{Gr}^{r+\frac{1}{2}}\mathcal{H}$$

while for $r>0$,

$$\dim_{\mathbf{F}}\begin{pmatrix} b_r\mathcal{H}\cap t_r\mathcal{H} \\ b_r\mathcal{H}\cap b_r^{\perp}\mathcal{H} \\ t_r\mathcal{H}\cap t_r^{\perp}\mathcal{H} \\ b_r\mathcal{H} \\ t_r\mathcal{H}\end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 0 & 2 & \\ 1 & 0 & 1 & 0 & 2\end{pmatrix}\begin{pmatrix} e_{\mathbf{m}}(r) \\ e_{\mathbf{10}}(r-1) \\ e_{\mathbf{01}}(r) \\ e_{\mathbf{20}}(r-1) \\ e_{\mathbf{02}}(r)\end{pmatrix}.$$

and finally for $r=0$,

$$\dim_{\mathbf{E}}\begin{pmatrix} \mathcal{H}(0)\cap\eta\mathcal{H}(0) \\ \mathcal{H}(0)\cap\eta\mathcal{H}(0)^{\perp}+\eta\mathcal{H}(0)\cap\mathcal{H}(0)^{\perp} \\ \mathcal{H}(0)+\eta\mathcal{H}(0)\end{pmatrix} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix} e_{\mathbf{m}}(0) \\ e_{\mathbf{01}}(0) \\ e_{\mathbf{02}}(0)\end{pmatrix}.$$

In this last formula, the orthogonal complements are taken with respect to the symmetric form $\phi_{0,0}$ underlying the $\mathbf{E}$-hermitian form $\psi_{0,0}$ on $S(\alpha,0,0)$.

4.3.8. Suppose similarly that $\mathcal{H}$ is a totally isotropic $\mathbf{F}$-subspace of $(S(\alpha,\frac{1}{2}),\phi_{\frac{1}{2}})$ corresponding to an edge $\beta\in\mathcal{I}(\frac{1}{4})$ with source $\alpha\in\mathcal{I}(\frac{1}{2})$, and put $\omega(\beta)=e_\star$ and $\omega(\alpha)=s_\star$, so that

$$\begin{aligned} s_{\mathbf{0}} &= e_{\mathbf{0}} \\ s_{\mathbf{1}} &= e_{\mathbf{1}}+e_{\mathbf{01}}+e_{\mathbf{10}} \\ s_{\mathbf{2}} &= e_{\mathbf{2}}+e_{\mathbf{m}}+e_{\mathbf{02}}[1]+e_{\mathbf{20}}+e_{\mathbf{12}}[1]+e_{\mathbf{21}} \end{aligned}$$

Define $\mathcal{H}_c$, $\mathcal{H}^c$, $\mathrm{Gr}^c\mathcal{H}$, $\mathrm{Gr}_c\mathcal{H}$ and $\mathcal{H}(c)$ just as above, with $\lambda=0$ replaced by $\lambda=\frac{1}{2}$. For $c=r>0$ a positive integer, put again $t_r\mathcal{H}=[\pi^r\eta]\mathrm{Gr}^r\mathcal{H}$ and $b_r\mathcal{H}=\mathrm{Gr}_r\mathcal{H}$, which are orthogonal subspaces of the symplectic $\mathbf{F}$-space $(\mathrm{Gr}_rS(\alpha,\frac{1}{2}),\theta_{\frac{1}{2},r})$ with orthogonal complements $t_r^{\perp}\mathcal{H}\supset b_r\mathcal{H}$ and $b_r^{\perp}\mathcal{H}\supset t_r\mathcal{H}$. Then for all $r\in\mathbf{N}$,

$$e_{\mathbf{01}}(r)=\dim_{\mathbf{F}}\mathrm{Gr}_{r+\frac{1}{2}}\mathcal{H}\quad\text{and}\quad e_{\mathbf{10}}(r)=\dim_{\mathbf{F}}\mathrm{Gr}^{r+\frac{1}{2}}\mathcal{H}$$

while for $r>0$,

$$\dim_{\mathbf{F}}\begin{pmatrix} b_r\mathcal{H}\cap t_r\mathcal{H} \\ b_r\mathcal{H}\cap b_r^{\perp}\mathcal{H} \\ t_r\mathcal{H}\cap t_r^{\perp}\mathcal{H} \\ b_r\mathcal{H} \\ t_r\mathcal{H}\end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 0 & 2 & \\ 1 & 0 & 1 & 0 & 2\end{pmatrix}\begin{pmatrix} e_{\mathbf{m}}(r) \\ e_{\mathbf{12}}(r-1) \\ e_{\mathbf{21}}(r) \\ e_{\mathbf{02}}(r-1) \\ e_{\mathbf{20}}(r)\end{pmatrix}$$

and finally for $r=0$,

$$\dim_{\mathbf{E}}\begin{pmatrix} \mathcal{H}(0)\cap\eta\mathcal{H}(0) \\ \mathcal{H}(0)\cap\eta\mathcal{H}(0)^{\perp}+\eta\mathcal{H}(0)\cap\mathcal{H}(0)^{\perp} \\ \mathcal{H}(0)+\eta\mathcal{H}(0)\end{pmatrix} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix} e_{\mathbf{m}}(0) \\ e_{\mathbf{21}}(0) \\ e_{\mathbf{20}}(0)\end{pmatrix}$$

where the orthogonal complements are now taken with respect to $\phi_{\frac{1}{2},0}$.

4.3.9. For all $d\in\mathbf{N}$, define

$$\pi(d)=\textstyle\prod_{i=1}^d(q^i-1),\quad \sigma(d)=\prod_{i=1}^d(q^i+1)\quad\text{and}\quad \tau(d)=\prod_{i=1}^d(q^i-(-1)^i).$$

For $\gamma\in\{\pi,\sigma,\tau\}$, $s\in\mathcal{S}$ and $s_\star:\mathbf{X}\to\mathcal{S}$ (where $\mathbf{X}$ is a finite set), define

$$\gamma(s)=\textstyle\prod_{r\in\mathbf{N}}\gamma(s(r))\quad\text{and}\quad\gamma(s_\star)=\prod_{x\in\mathbf{X}}\gamma(s_x).$$

Proposition. *Fix $e_\star\in\mathcal{E}$ with $\mathrm{source}(e_\star)=s_\star$ and $\mathrm{target}(e_\star)=t_\star$ in $\mathcal{V}$. For $x\in\{s,t\}$, let α_x be a vertex with $\omega(\alpha_x)=x_\star$. Then:*

(1) *The number of edges α with* target$(\alpha) = \alpha_t$ *and* $\omega(\alpha) = e_\star$ *equals*

$$q^{\Lambda_t(e_\star)} \cdot \frac{\pi(t_\star)}{\pi(e_\star)} \cdot \frac{\sigma(t_{\mathbf{0}})}{\sigma(e_{\mathbf{0}}, e_{\mathbf{02}}, e_{\mathbf{20}})} \cdot \frac{1}{\sigma(e_{\mathbf{m}}(0))} \cdot \frac{\tau(\Delta_{\mathbf{0}} + 2t_{\mathbf{0}}(0)) \cdot \pi(e_{\mathbf{0}}(0)) \cdot \sigma(e_{\mathbf{0}}(0))}{\pi(t_{\mathbf{0}}(0)) \cdot \sigma(t_{\mathbf{0}}(0)) \cdot \tau(\Delta_{\mathbf{0}} + 2e_{\mathbf{0}}(0))}$$

for some explicit constant $\Lambda_t(e_\star) \in \mathbf{N}$.

(2) *The number of edges α with* source$(\alpha) = \alpha_s$ *and* $\omega(\alpha) = e_\star$ *equals*

$$q^{\Lambda_s(e_\star)} \cdot \frac{\pi(s_\star)}{\pi(e_\star)} \cdot \frac{\sigma(s_{\mathbf{2}})}{\sigma(e_{\mathbf{2}}, e_{\mathbf{02}}, e_{\mathbf{20}})} \cdot \frac{1}{\sigma(e_{\mathbf{m}}(0))} \cdot \frac{\tau(\Delta_{\mathbf{2}} + 2s_{\mathbf{2}}(0)) \cdot \pi(e_{\mathbf{2}}(0)) \cdot \sigma(e_{\mathbf{2}}(0))}{\pi(s_{\mathbf{2}}(0)) \cdot \sigma(s_{\mathbf{2}}(0)) \cdot \tau(\Delta_{\mathbf{2}} + 2e_{\mathbf{2}}(0))}$$

for some explicit constant $\Lambda_s(e_\star) \in \mathbf{N}$.

Proof. These are the numbers of totally isotropic $\mathbf{F}$-subspaces $\mathcal{H}$ of $S = S(\alpha_t, 0)$ and $S = S(\alpha_s, \frac{1}{2})$ which satisfy the conditions of section 4.3.7 and 4.3.8, respectively. With the notations of these sections, let $N(0)$ be the number of possible choices for $\mathcal{H}(0)$. For each $c \in \frac{1}{2}\mathbf{N}$ with $c > 0$, let $N(c)$ be the number of possible choices for the pair $(\mathrm{Gr}^c\mathcal{H}, \mathrm{Gr}_c\mathcal{H})$. Put also

$$b(c) = \dim_{\mathbf{F}} \mathrm{Gr}_c\mathcal{H}, \quad t(c) = \dim_{\mathbf{F}} \mathrm{Gr}^c\mathcal{H}, \quad s(c) = \dim_{\mathbf{F}} \mathrm{Gr}_c S = \dim_{\mathbf{F}} \mathrm{Gr}^c S$$

and define $e(c)$ by $s(c) = b(c) + e(c) + t(c)$. Let finally

$$\mathcal{M}(c) = \dim_{\mathbf{F}} \mathcal{H}(c - \tfrac{1}{2}) \quad \text{and} \quad \mathcal{D}(c) = \dim_{\mathbf{F}} \mathcal{H}(c - \tfrac{1}{2})^\perp / \mathcal{H}(c - \tfrac{1}{2})$$

where the orthogonal complement is taken within $S(c - \frac{1}{2})$, so that

$$\begin{aligned} \mathcal{M}(c) &= \mathcal{M}(\tfrac{1}{2}) + \textstyle\sum_{d<c} (b(d) + t(d)) \\ \text{and} \quad \mathcal{D}(c) &= \mathcal{D}(\tfrac{1}{2}) + 2\textstyle\sum_{d<c} e(d) \end{aligned}$$

with $\mathcal{M}(\frac{1}{2}) = \dim_{\mathbf{F}} \mathcal{H}(0)$ and $\mathcal{D}(\frac{1}{2}) = \dim_{\mathbf{F}} \mathcal{H}(0)^\perp / \mathcal{H}(0)$. Now given

$$\mathrm{Gr}_c\mathcal{H} \subset \mathrm{Gr}_c S, \quad \mathcal{H}(c - \tfrac{1}{2}) \subset S(c - \tfrac{1}{2}) \quad \text{and} \quad \mathrm{Gr}^c\mathcal{H} \subset \mathrm{Gr}^c S,$$

the number of possible choices for $\mathcal{H}(c) \subset S(c)$ equals $q^{\lambda(c)}$ where

$$\lambda(c) = \mathcal{M}(c) \cdot e(c) + t(c) \cdot \left(\mathcal{D}(c) + \mathcal{M}(c) + e(c) + \tfrac{t(c)-1}{2}\right)$$

by lemma 4 of the appendix. The number of possible choices for $\mathcal{H}$ thus equals

$$q^{\sum_{c>0} \lambda(c)} \cdot \textstyle\prod_{c \geq 0} N(c).$$

We now compute the above constants for the second formula, i.e. using the conditions of section 4.3.8 above. First of all, for $r \in \mathbf{N}$ and $c = r + \frac{1}{2}$,

$$b(c) = e_{\mathbf{01}}(r), \quad t(c) = e_{\mathbf{10}}(r), \quad e(c) = e_{\mathbf{1}}(r) \quad \text{and} \quad s(c) = s_{\mathbf{1}}(r)$$

while for $c = r > 0$, $s(c) = 2s_{\mathbf{2}}(r)$ and

$$\begin{aligned} b(c) &= e_{\mathbf{m}}(r) + e_{\mathbf{12}}(r-1) + 2e_{\mathbf{02}}(r-1) \\ t(c) &= e_{\mathbf{m}}(r) + e_{\mathbf{21}}(r) + 2e_{\mathbf{20}}(r) \\ e(c) &= 2e_{\mathbf{2}}(r) + e_{\mathbf{12}}(r-1) + e_{\mathbf{21}}(r) \end{aligned}$$

and finally $\mathcal{M}(\frac{1}{2}) = 2e_{\mathbf{m}}(0) + 2e_{\mathbf{20}}(0) + e_{\mathbf{21}}(0)$ and $\mathcal{D}(\frac{1}{2}) = 2\Delta_{\mathbf{2}} + 4e_{\mathbf{2}}(0) + 2e_{\mathbf{21}}(0)$, which gives $\mathcal{M}(c)$, $\mathcal{D}(c)$ and $\lambda(c)$ as a function of $e_\star$ for all $c \in \frac{1}{2}\mathbf{N} - \{0\}$. Now lemma 1, 2 and 3 of the appendix give respectively

$$N(0) = \frac{q^{\lambda'(0)} \cdot \tau\left(\Delta_{\mathbf{2}} + 2s_{\mathbf{2}}(0)\right)}{\pi\left(e_{\mathbf{m}}(0), e_{\mathbf{21}}(0), e_{\mathbf{20}}(0)\right) \cdot \sigma\left(e_{\mathbf{m}}(0), e_{\mathbf{20}}(0)\right) \cdot \tau\left(\Delta_{\mathbf{2}} + 2e_{\mathbf{2}}(0)\right)},$$

$$N(r+\tfrac{1}{2}) = \frac{\pi\left(s_{\mathbf{1}}(r)\right)}{\pi\left(e_{\mathbf{1}}(r), e_{\mathbf{01}}(r), e_{\mathbf{10}}(r)\right)},$$

$$N(r) = \frac{q^{\lambda'(r)} \cdot \pi\left(s_{\mathbf{2}}(r)\right) \cdot \sigma\left(s_{\mathbf{2}}(r)\right)}{\pi\left(e_{\mathbf{m}}(r), e_{\mathbf{12}}(r-1), e_{\mathbf{21}}(r), e_{\mathbf{02}}(r-1), e_{\mathbf{20}}(r), e_{\mathbf{2}}(r)\right) \cdot \sigma\left(e_{\mathbf{02}}(r-1), e_{\mathbf{20}}(r), e_{\mathbf{2}}(r)\right)}$$

for $r \in \mathbf{N}$ (and $r \neq 0$ in the last formula) with

$$\lambda'(0) = 2e_{\mathbf{20}}(0)\left(e_{\mathbf{21}}(0) + \Delta_{\mathbf{2}} + 2e_{\mathbf{2}}(0)\right) + e_{\mathbf{20}}(0)\left(e_{\mathbf{20}}(0) - 1\right) + \tfrac{1}{2}e_{\mathbf{21}}(0)\left(e_{\mathbf{21}}(0) - 1\right),$$

$$\lambda'(r) = 2e_{\mathbf{02}}(r-1)\left(e_{\mathbf{21}}(r) + e_{\mathbf{20}}(r) + e_{\mathbf{2}}(r)\right) + 2e_{\mathbf{20}}(r)\left(e_{\mathbf{12}}(r-1) + e_{\mathbf{2}}(r)\right) + e_{\mathbf{12}}(r-1)e_{\mathbf{21}}(r)$$

for $0 \neq r \in \mathbf{N}$. This proves our second formula, with the exponent

$$\Lambda_s(e_\star) = \sum_{c \in \frac{1}{2}\mathbf{N}, c>0} \lambda(c) + \sum_{r \in \mathbf{N}} \lambda'(r).$$

The proof of the first one is entirely similar: simply switch **0** and **2** everywhere. □

4.4. **Volumes.** Let μ be a Haar measure on $U(V,\psi)$. Let $U(\alpha)$ be the stabilizer of $\alpha \in \mathcal{I}(V,\phi)$ in $U(V,\psi)$ and define $\mu(\alpha) = \mu(U(\alpha))$, so that $\mu(\alpha)$ depends only on the $U(V,\psi)$-orbit of α.

4.4.1. For a suitable (but unique) choice of μ, the following proposition holds.

Proposition. *For any edge $\alpha \in \mathcal{I}(\frac{1}{4})$ with $e_\star = \omega(\alpha) \in \mathcal{E}$,*

$$\mu(\alpha) = q^{-\Lambda(e_\star)} \cdot \pi\left(e_\star\right) \cdot \sigma\left(e_{\mathbf{0}}, e_{\mathbf{2}}, e_{\mathbf{02}}, e_{\mathbf{20}}\right) \cdot \sigma\left(e_{\mathbf{m}}(0)\right) \cdot \frac{\tau(\Delta_{\mathbf{0}} + 2e_{\mathbf{0}}(0), \Delta_{\mathbf{2}} + 2e_{\mathbf{2}}(0))}{\pi(e_{\mathbf{0}}(0), e_{\mathbf{2}}(0)) \cdot \sigma(e_{\mathbf{0}}(0), e_{\mathbf{2}}(0))}$$

for an exponent $\Lambda(e_\star) \in \mathbf{Z}$ described in section 4.4.2 below.

Proof. Let $\widetilde{\mu}(e_\star)$ be the right hand side of the above equation. Since $\mathcal{I}$ is connected, we just have to show that for $s_\star = \mathrm{source}(e_\star)$ and $t_\star = \mathrm{target}(e_\star)$, the ratios $\widetilde{\mu}(s_\star)/\widetilde{\mu}(e_\star)$ and $\widetilde{\mu}(t_\star)/\widetilde{\mu}(e_\star)$ match the numbers found in proposition 4.3.9. This is obvious for the prime-to-q part of these ratios, but we need to verify that

$$\Lambda_s(e_\star) = \Lambda(e_\star) - \Lambda(s_\star) \quad \text{and} \quad \Lambda_t(e_\star) = \Lambda(e_\star) - \Lambda(t_\star).$$

This is a routine, but highly tedious task. □

4.4.2. A closed formula for $\Lambda(e_\star)$ is given by

$$\Lambda(e_\star) = \sum_{0 \neq c \in \frac{1}{4}\mathbf{N}} X_c^t \left(A_c X_c + B_c Y_c + L_c\right)$$

where for $0 \neq c \in \frac{1}{4}\mathbf{N}$ and any $\alpha \in \mathcal{I}(\frac{1}{4})$ with $\omega(\alpha) = e_\star$,

$$Y_c = \begin{pmatrix} \dim_{\mathbf{F}} S_-^c(\alpha, 0)/S_c^+(\alpha, 0) \\ 2 \cdot \dim_{\mathbf{F}} S_-^c(\alpha, \frac{1}{4})/S_c^+(\alpha, \frac{1}{4}) \\ \dim_{\mathbf{F}} S_-^c(\alpha, \frac{1}{2})/S_c^+(\alpha, \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \dim_{\mathbf{F}} S(\alpha, 0, c - \frac{1}{4}) \\ 2 \cdot \dim_{\mathbf{F}} S(\alpha, \frac{1}{4}, c - \frac{1}{4}) \\ \dim_{\mathbf{F}} S(\alpha, \frac{1}{2}, c - \frac{1}{4}) \end{pmatrix}$$

while the matrices X_c, A_c, B_c and L_c are given by the following table, in which $k = \lfloor c \rfloor \in \mathbf{N}$ and $((\lambda))$ denotes a matrix all of whose coefficients are equal to λ (and whose size should be clear from the context).

c	X_c	$A_c - ((2k))$	$B_c - ((k))$	$L_c + ((k))$
$k+0$	$\begin{pmatrix} e_{\mathbf{0}}(k) \\ e_{\mathbf{m}}(k) \\ e_{\mathbf{2}}(k) \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$
$k+\frac{1}{4}$	$\begin{pmatrix} e_{\mathbf{01}}(k) \\ e_{\mathbf{21}}(k) \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$
$k+\frac{1}{2}$	$\begin{pmatrix} e_{\mathbf{02}}(k) \\ e_{\mathbf{1}}(k) \\ e_{\mathbf{20}}(k) \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -\frac{1}{2} \\ -1 \end{pmatrix}$
$k+\frac{3}{4}$	$\begin{pmatrix} e_{\mathbf{12}}(k) \\ e_{\mathbf{10}}(k) \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Note also that with these definitions,

$$Y_c = \begin{pmatrix} 2\Delta_{\mathbf{0}} \\ 0 \\ 2\Delta_{\mathbf{2}} \end{pmatrix} + \sum_{d \in \frac{1}{4}\mathbf{N}, 0 \leq d < c} T_d X_d$$

where T_d depends only upon $d \bmod \mathbf{Z}$ and

$$T_0 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \ T_{\frac{1}{4}} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \\ 0 & 2 \end{pmatrix}, \ T_{\frac{1}{2}} = \begin{pmatrix} 0 & 2 & 0 \\ 4 & 0 & 4 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } T_{\frac{3}{4}} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

5. The odd orthogonal case

5.1. **Classification.** We recall the following results from [1, §5].

5.1.1. A quasi-hermitian E-space is a triple $(\overline{V}, \psi, D)$ where $(\overline{V}, \psi)$ is a finite dimensional hermitian E-space and D is an anisotropic F-line in the symmetric F-space $(\overline{V}, \phi)$ underlying $(\overline{V}, \psi)$, with $\phi = \mathrm{tr}\psi$. Such a structure defines

$$\underline{V} = (E \cdot D)^{\perp} \subset V = \underline{V} \perp D \subset \overline{V} = V \perp \eta D$$

Thus $(\underline{V}, \psi)$ is a non-degenerate E-hyperplane of $(\overline{V}, \psi)$ while (V, ϕ) is a non-degenerate F-hyperplane in $(\overline{V}, \phi)$. The Witt indices are related by

$$\mathrm{witt}_F(V, \phi) = \mathrm{witt}_E(\underline{V}, \psi) + \mathrm{witt}_E(\overline{V}, \psi).$$

Note that this notion differs slightly from the one used in [1, §5].

5.1.2. A quasi-hermitian E-space $(\overline{V}, \psi, D)$ induces a commutative diagram

$$\begin{array}{ccccc} U(\underline{V}, \psi) & & \longrightarrow & & U(\overline{V}, \psi) \\ \downarrow & & & & \downarrow \\ SO(\underline{V}, \phi) & \longrightarrow & SO(V, \phi) & \longrightarrow & SO(\overline{V}, \psi) \end{array}$$

For $X \in \{\underline{V}, V, \overline{V}\}$, let $\mathcal{I}(X)$ be the set of self-dual F-norms on (X, ϕ): this is the Bruhat-Tits building of $SO(X, \phi)$. A self dual norm on $\underline{V}$ or V extends uniquely to a self-dual norm on $V = \underline{V} \perp D$ or $\overline{V} = V \perp \eta D$. This yields embeddings

$$\mathcal{I}(\underline{V}) \hookrightarrow \mathcal{I}(V) \quad \text{and} \quad \mathcal{I}(V) \hookrightarrow \mathcal{I}(\overline{V})$$

which are respectively $SO(\underline{V},\phi)$ and $SO(V,\phi)$-equivariant, respect the metrics and map each building onto a convex subset of the next one. Convex projection defines equivariant retractions of these embeddings, see [1, §2.2.8]. We denote by

$$\underline{\alpha}\in\mathcal{I}(\underline{V})\qquad\text{and}\qquad\overline{\alpha}\in\mathcal{I}(\overline{V})$$

the (restriction of) convex projection and unique self-dual extension of $\alpha\in\mathcal{I}(V)$.

5.1.3. Let $\mathcal{I}_0(V)$ be the image of $\mathcal{I}(\underline{V})$ in $\mathcal{I}(V)$ and define more generally

$$\mathcal{I}_s(V)\overset{def}{=}\{\alpha\in\mathcal{I}(V):d(\alpha)=s\}\quad\text{where}\quad d(\alpha)\overset{def}{=}\mathrm{dist}(\alpha,\mathcal{I}_0(V))=\mathrm{dist}(\alpha,\alpha_0)$$

for all $s\in\mathbf{R}_{\geq 0}$, where $\alpha_0\in\mathcal{I}_0(V)$ is the convex projection of $\alpha\in\mathcal{I}(V)$ (and thus also the unique self-dual extension of $\underline{\alpha}\in\mathcal{I}(\underline{V})$). For $d(\alpha)=s$ and $t\in[0,s]$, we let α_t be the point at distance t from α_0 on the segment joining α_0 and $\alpha_s=\alpha$ in $\mathcal{I}(V)$, i.e. $\{\alpha_t\}=\mathcal{I}_t(V)\cap[\alpha_0,\alpha_s]$. The map $\alpha_s\mapsto\alpha_t$ thus defines an $SO(\underline{V},\phi)$-equivariant map from $\mathcal{I}_s(V)$ to $\mathcal{I}_t(V)$.

5.1.4. For any non-degenerate E-subspace W of $\underline{V}$ with orthogonal complement $\overline{V}'$ in $\overline{V}$, the triple $(\overline{V}',\psi,D)$ is a quasi-hermitian E-subspace of $(\overline{V},\psi,D)$ for which

$$\underline{V}=W\perp\underline{V}',\quad V=W\perp V',\quad\text{and}\quad\overline{V}=W\perp\overline{V}'.$$

In this situation, for any self-dual F-norm α on V,

$$\alpha|W\in\mathcal{I}(W)\iff\alpha|V'\in\mathcal{I}(V')\iff\alpha=\alpha|W\perp\alpha|V'$$

in which case also for $\beta=\alpha|W$ and $\alpha'=\alpha|V'$, $d(\alpha)=d(\alpha')=s$ and

$$\overline{\alpha}=\beta\perp\overline{\alpha'},\quad\underline{\alpha}=\beta\perp\underline{\alpha'}\quad\text{and}\quad\alpha_t=\beta\perp\alpha'_t\text{ for all }t\in[0,s].$$

We say that α is *special* if and only if it does not admit such a splitting with $W\neq 0$. Any self-dual F-norm α on V therefore does have such a splitting for which α' is a special self-dual F-norm on V'.

5.1.5. A *special* basis $\mathcal{B}$ of V is an orthogonal F-basis

$$\mathcal{B}=(v_0,w_1,v_1,w_2,\cdots,w_n,v_n)$$

of V such that $Fv_n=D$, $v_{i-1}=\eta w_i$ and $Q(v_i)+Q(w_i)=0$ for all $1\leq i\leq n$. It defines two orthogonal decompositions of V, namely

$$V=Ew_1\perp\cdots\perp Ew_n\perp D=Fv_0\perp H_1\perp\cdots\perp H_n$$

where $H_i=Fw_i\perp Fv_i$ is an hyperbolic F-plane whose isotropic F-lines are spanned by $e_{\pm i}=\frac{1}{2}(v_i\pm w_i)$. The latter decomposition yields an apartment $\mathcal{A}(\mathcal{B})$ of $\mathcal{I}(V)$ and the special basis $\mathcal{B}$ of V gives an isometry $\mathrm{coord}_{\mathcal{B}}:\mathcal{A}(\mathcal{B})\xrightarrow{\simeq}\mathbf{R}^n$ mapping $\alpha\in\mathcal{A}(\mathcal{B})$ to the n-tuple $(s_1,\cdots,s_n)\in\mathbf{R}^n$ defined by

$$\alpha(e_{\pm i})=q^{r_0\pm s_i}\quad\text{with}\quad|Q(v_i)|=|Q(w_i)|=|\phi(e_i,e_{-i})|=q^{2r_0}.$$

The apartment $\mathcal{A}(\mathcal{B})$ is stable under the projection $\alpha\mapsto\underline{\alpha}$, which corresponds to the projection $(s_1,\cdots,s_n)\mapsto(s_1,\cdots,s_{n-1},0)$. Therefore $s_n=d(\alpha)$ and for all $t\in[0,s_n]$, α_t also belongs to $\mathcal{A}(\mathcal{B})$ with $\mathrm{coord}_{\mathcal{B}}(\alpha_t)=(s_1,\cdots,s_{n-1},t)$. Note that special basis do exist if and only if V is split, i.e. $\dim_F V=2\mathrm{witt}_F V+1$.

5.1.6. We consider n-tuple $(s_1, \cdots, s_n) \in \mathbf{R}^n$ satisfying one of the conditions

(SP) $0 < s_1 < s_3 < \cdots < s_{2\lfloor \frac{n-1}{2} \rfloor + 1}$ and $0 < s_2 < s_4 < \cdots < s_{2\lfloor \frac{n}{2} \rfloor}$,
(Sp) $0 \leq s_1 < s_3 < \cdots < s_{2\lfloor \frac{n-1}{2} \rfloor + 1}$ and $0 < s_2 < s_4 < \cdots < s_{2\lfloor \frac{n}{2} \rfloor}$,
(sp) $0 \leq s_1 \leq s_3 \leq \cdots \leq s_{2\lfloor \frac{n-1}{2} \rfloor + 1}$ and $0 \leq s_2 \leq s_4 \leq \cdots \leq s_{2\lfloor \frac{n}{2} \rfloor}$.

The formulas $s_0 = 0$ and $c_i = s_i + s_{i-1}$ for $i = 1, \cdots, n$ define a bijective correspondence between such n-tuples $(s_1, \cdots, s_n)$ and sequences $(c_1, \cdots, c_n)$ for which

(SP) $0 < c_1 < c_2 < \cdots < c_n$,
(Sp) $0 \leq c_1 < c_2 < \cdots < c_n$,
(sp) $0 \leq c_1 \leq c_2 \leq \cdots \leq c_n$.

For $i = 1, \cdots, n$, we put $\theta_i = (-1)^i(s_i - s_{i-1}) \in \mathbf{R}$ and define

$$[s_1, \cdots, s_n]_D = \textstyle\sum_{i=1}^n [\theta_i + \mathcal{Q}(D), c_i] \quad \text{in} \quad \mathbf{N}[\overline{\mathcal{L}}]$$

where we have used the (θ, c)-coordinates on $\overline{\mathcal{L}} \simeq \mathbf{R}/\sim \times \mathbf{R}_{\geq 0}$. Here

$$\mathcal{Q}(D) = \log_q Q(D - \{0\}) = 2\mathbf{Z} \quad \text{or} \quad 1 + 2\mathbf{Z}.$$

We also define

$$\begin{aligned} \lceil s_1, \cdots, s_n \rceil_D &= \textstyle\sum_{i \equiv n \bmod 2} [\theta_i + \mathcal{Q}(D), c_i], \\ \text{and} \quad \lfloor s_1, \cdots, s_n \rfloor_D &= \textstyle\sum_{i \not\equiv n \bmod 2} [\theta_i + \mathcal{Q}(D), c_i]. \end{aligned}$$

These two elements of $\mathbf{N}[\overline{\mathcal{L}}]$ therefore sum to $[s_1, \cdots, s_n]_D$, from which the sequences $(c_1, \cdots, c_n)$ and $(s_1, \cdots, s_n)$ are easily retrieved.

5.1.7. Any element a of $\mathbf{N}[\overline{\mathcal{L}}]$ has a unique decomposition $a = 2a_{reg} + a_{sp}$ with $a_{reg} \in \mathbf{N}[\overline{\mathcal{L}}]$ and $a_{sp} = \sum_{x \in X} x$ for a finite subset X of $\overline{\mathcal{L}}$. We say that a_{reg} and a_{sp} are the *regular* and *special* parts of a and define a subset $\mathbf{N}[\overline{\mathcal{L}}]_D$ of $\mathbf{N}[\overline{\mathcal{L}}]$ by

$$a \in \mathbf{N}[\overline{\mathcal{L}}]_D \iff \exists (s_1, \cdots, s_m) \text{ satisfying } (\mathbf{Sp}) \text{ such that } a_{sp} = [s_1, \cdots, s_m]_D\,.$$

Note that if $(s_1, \cdots, s_m)$ merely satisfies (**sp**), then still $[s_1, \cdots, s_m]_D \in \mathbf{N}[\overline{\mathcal{L}}]_D$.

5.1.8. For any self-dual F-norm α on V, we put

$$\omega(\alpha) \stackrel{def}{=} \omega(\underline{\alpha}) + \omega(\overline{\alpha}) = 2\omega_{reg}(\alpha) + \omega_{sp}(\alpha) \quad \text{in } \mathbf{N}[\overline{\mathcal{L}}].$$

This defines a map $\omega : U(\underline{V}, \psi)\backslash \mathcal{I}(V) \to \mathbf{N}[\overline{\mathcal{L}}]^n$ where $n = \text{witt}_F V$. We shall see below that the latter is a bijection onto $\mathbf{N}[\overline{\mathcal{L}}]^n_D = \mathbf{N}[\overline{\mathcal{L}}]_D \cap \mathbf{N}[\overline{\mathcal{L}}]^n$.

5.1.9. By Theorem 5.2 of [1], a self-dual norm α on V is special if and only if there exists a special basis $\mathcal{B}$ of V such that $\alpha \in \mathcal{A}(\mathcal{B})$ with $\text{coord}_{\mathcal{B}}(\alpha) = (s_1, \cdots, s_n)$ satisfying (**SP**) – in particular, special norms do exists precisely when V is split. On the other hand, Proposition 5.1 of [1] asserts that for any self-dual norm $\alpha \in \mathcal{A}(\mathcal{B})$ for which $\text{coord}_{\mathcal{B}}(\alpha) = (s_1, \cdots, s_n)$ merely satisfies (**sp**),

$$\omega(\underline{\alpha}) = \lfloor s_1, \cdots, s_n \rfloor_D \quad \text{and} \quad \omega(\overline{\alpha}) = \lceil s_1, \cdots, s_n \rceil_D\,.$$

Then $\omega(\alpha) = [s_1, \cdots, s_n]_D$, from which the sequences $(c_1, \cdots, c_n)$ and $(s_1, \cdots, s_n)$ are easily retrieved. Therefore $(s_1, \cdots, s_n)$ is a well-defined invariant attached to such norms.

5.1.10. Let now α be any self-dual norm on V and fix a splitting $\alpha = \beta \perp \alpha'$ corresponding to $V = W \perp V'$ where $\alpha' = \alpha | V'$ is a special self-dual norm on V'. Let $n = \mathrm{witt}_F V$, $m = \mathrm{witt}_F V'$ and $r = \mathrm{witt}_E W$, so that $n = 2r + m + \delta$ for some $\delta \in \{0,1\}$. Let $(s_1, \cdots, s_m)$ be the invariant of α' on V', as defined above (satisfying (**SP**)). Fix also Witt decompositions

$$\begin{array}{rcl} (W,\beta) & = & (W_0,\beta_0)\perp((W_+,\beta_+)\oplus(W_-,\beta_-)) \\ (\underline{V}',\underline{\alpha}') & = & (\underline{V}'_0,\underline{\alpha}'_0)\perp\left((\underline{V}'_+,\underline{\alpha}'_+)\oplus(\underline{V}'_-,\underline{\alpha}'_-)\right) \\ (\overline{V}',\overline{\alpha}') & = & (\overline{V}'_0,\overline{\alpha}'_0)\perp\left((\overline{V}'_+,\overline{\alpha}'_+)\oplus(\overline{V}'_-,\overline{\alpha}'_-)\right) \end{array}$$

where for $X \in \{W, \underline{V}', \overline{V}'\}$, X_0 and $X_\pm$ are suitable E-subspaces of X which are respectively anisotropic and totally isotropic. Then

$$\omega(\underline{\alpha}) = \omega(\beta) + \lfloor s_1, \cdots, s_m \rfloor_D + \underline{\Omega} \quad \text{and} \quad \omega(\overline{\alpha}) = \omega(\beta) + \lceil s_1, \cdots, s_m \rceil_D + \overline{\Omega}$$

where $\underline{\Omega} = \omega(\beta_0 \perp \underline{\alpha}'_0)$ and $\overline{\Omega} = \omega(\beta_0 \perp \overline{\alpha}'_0)$. Let $\mathcal{A} \in \{\mathcal{A}_1, \mathcal{A}'_1\}$ be the isomorphism class of $(E \cdot D, \psi)$, so that $\mathcal{A} = \mathcal{A}_1$ if $\mathcal{Q}(D) = 2\mathbf{Z}$ while $\mathcal{A} = \mathcal{A}'_1$ if $\mathcal{Q}(D) = 1 + 2\mathbf{Z}$. The definition of a special basis implies that

$$\begin{cases} \underline{V}'_0 = 0 \text{ and } (\overline{V}'_0, \psi) \in \mathcal{A} & \text{if } m \text{ is even,} \\ (\underline{V}'_0, \psi) \in \mathcal{A} \text{ and } \overline{V}'_0 = 0 & \text{if } m \text{ is odd.} \end{cases}$$

Therefore $\underline{\Omega} = 0$ if m is even, $\overline{\Omega} = 0$ if m is odd, and $\underline{\Omega} + \overline{\Omega} = \Omega$ is the ω-invariant of the canonical norm $x \mapsto |Q(x)|^{1/2}$ on $W_0 \perp E \cdot D$, namely

$$\Omega = \begin{cases} 0 & \text{if } \delta = 0 \\ [\mathcal{Q}(D), 0] & \text{if } \delta = 1 \end{cases} \quad \text{in } \mathbf{N}[\overline{\mathcal{L}}] \text{ with } \overline{\mathcal{L}} \simeq \mathbf{R}/\sim \times \mathbf{R}_{\geq 0}$$

since indeed $\delta = \mathrm{witt}_E (W_0 \perp E \cdot D)$. We thus obtain

$$(\omega(\underline{\alpha}), \omega(\overline{\alpha})) = (\omega(\beta), \omega(\beta)) + \begin{cases} \left(\lfloor s_1, \cdots, s_m \rfloor_D, \lceil s_1, \cdots, s_m \rceil_D\right) & \text{if } n \equiv m \bmod 2 \\ \left(\lfloor 0, s_1, \cdots, s_m \rfloor_D, \lceil 0, s_1, \cdots, s_m \rceil_D\right) & \text{if } n \not\equiv m \bmod 2 \end{cases}$$

so that $\omega(\alpha) = 2\omega_{reg}(\alpha) + \omega_{sp}(\alpha)$ with $\omega_{reg}(\alpha) = \omega(\beta)$ and

$$\omega_{sp}(\alpha) = \begin{cases} [s_1, \cdots, s_m]_D & \text{if } n \equiv m \bmod 2, \\ [0, s_1, \cdots, s_m]_D & \text{if } n \not\equiv m \bmod 2. \end{cases}$$

It follows that $\omega(\alpha)$ indeed belongs to $\mathbf{N}[\overline{\mathcal{L}}]_D$.

5.1.11. In these constructions, we may scale the special basis $\mathcal{B}' = (v_0, \cdots, v_m)$ of V' so that its last vector is equal to a fixed non-zero vector $v_m^0 \in D$. All such normalized sequences $\mathcal{B}'$ lie in the same orbit for the diagonal action of $U(\underline{V}, \psi)$ on V^{2m+1}. It follows that the ω-invariant defines a bijection (cf. Theorem 5.3 of [1])

$$\omega : U(\underline{V}, \psi) \backslash \mathcal{I}(V, \phi) \xrightarrow{\simeq} \mathbf{N}[\overline{\mathcal{L}}]_D^n.$$

5.1.12. If $\omega_{sp}(\alpha) = [s_1, \cdots, s_m]_D$, then $d(\alpha) = s_m$ and for all $t \in [s_{m-2}, s_m]$,

$$\omega(\alpha_t) = 2\omega_{reg}(\alpha) + [s_1, \cdots, s_{m-1}, t]_D \, .$$

Therefore $\omega_{reg}(\alpha_t) = \omega_{reg}(\alpha)$ and $\omega_{sp}(\alpha_t) = [s_1, \cdots, s_{m-1}, t]_D$ if $t > s_{m-2}$ while

$$\begin{array}{rcl} \omega_{reg}(\alpha_{s_{m-2}}) & = & \omega_{reg}(\alpha) + [(-1)^m (s_{m-2} - s_{m-1}) + \mathcal{Q}(D), s_{m-1} + s_{m-2}], \\ \text{and} \quad \omega_{sp}(\alpha_{s_{m-2}}) & = & [s_1, \cdots, s_{m-2}]_D \, . \end{array}$$

Note also that

$$\omega(\underline{\alpha}) = \omega_{reg}(\alpha) + \lfloor s_1, \cdots, s_m \rfloor_D \quad \text{and} \quad \omega(\overline{\alpha}) = \omega_{reg}(\alpha) + \lceil s_1, \cdots, s_m \rceil_D$$

5.1.13. If $\omega(\alpha) = \sum_{i=1}^n [\theta_i, c_i]$ with $\omega_{sp}(\alpha) = [s_1, \cdots, s_m]_D$, we set

$$c_{\min}(\alpha) \stackrel{def}{=} \begin{cases} \min\{c_i\} & \text{if } V \text{ splits} \\ 0 & \text{otherwise} \end{cases} \leq c^{sp}_{\max}(\alpha) \stackrel{def}{=} s_{m-1} + s_m \leq c_{\max}(\alpha) \stackrel{def}{=} \max\{c_i\}$$

with the convention that $s_0 = 0 = s_{-1}$ (for $m = 0$ or 1).

5.1.14. Recall from section 4.1.5 that $U_r = \{z/\overline{z} : z \in \mathcal{O}_r^\times\}$ for $r \in \mathbf{N}$.

Lemma. *Let $U(\alpha)$ be the stabilizer of α in $U(\underline{V}, \psi)$. Then*

$$\det U(\alpha) = U_r \qquad \textit{with } r = \lceil c_{\min}(\alpha) \rceil .$$

Proof. With the notations of section 5.1.10, we have

$$U(\beta) \times U(\alpha') \subset U(\alpha) \subset U(\underline{\alpha}) \quad \text{and} \quad U(\overline{\alpha})$$

so that

$$\det U(\beta) \cdot \det U(\alpha') \subset \det U(\alpha) \subset \det U(\underline{\alpha}) \cap \det U(\overline{\alpha}).$$

On the other hand, a case by case study gives

$$\min\{c_{\min}(\beta), c_{\min}(\alpha')\} = c_{\min}(\alpha) = \max\{c_{\min}(\underline{\alpha}), c_{\min}(\overline{\alpha})\} .$$

It is therefore sufficient to show that

$$\det U(\beta) = U_{\tilde{r}}, \quad \det U(\alpha') = U_{r'}, \quad \det U(\underline{\alpha}) = U_{\underline{r}} \quad \text{and} \quad \det U(\overline{\alpha}) = U_{\overline{r}}$$

where $\tilde{r} = \lceil c_{\min}(\beta) \rceil$, $r' = \lceil c_{\min}(\alpha') \rceil$, $\underline{r} = \lceil c_{\min}(\underline{\alpha}) \rceil$ and $\overline{r} = \lceil c_{\min}(\overline{\alpha}) \rceil$. The first, third and fourth of these equalities readily follow from lemma 4.1.5. As for the second one, it is itself the special case of the lemma. So suppose that $\alpha = \alpha'$ is special and let $\mathcal{B} = (v_0, w_1, v_1, w_2, \cdots, w_n, v_n)$ be a special basis of V such that $\alpha \in \mathcal{A}(\mathcal{B})$ with $\mathrm{coord}_{\mathcal{B}}(\alpha) = (s_1, \cdots, s_n)$ satisfying (**SP**). Thus $r = \lceil s_1 \rceil \geq 1$, and given the argument above, we just have to show that $U_r \subset \det U(\alpha)$. If $V' = Ew_1 \perp Fv_1$ and $\alpha' = \alpha|V'$, then α' is still a special self-dual F -norm on V' and $U(\alpha') \subset U(\alpha)$. We may thus also assume that $n = 1$.

We need to show that for all $v \in V$ and $u \in U_r$, $\alpha(uv) \leq \alpha(v)$. Let $|Q(v_1)| = q^{2r_0}$ and $e_\pm = \frac{1}{2}(v_1 \pm w_1)$, so that for all $x, y, z \in F$,

$$\alpha(xv_0 + ye_+ + ze_-) = \max\left\{|x|\, q^{r_0}, |y|\, q^{r_0+s_1}, |z|\, q^{r_0-s_1}\right\} .$$

For $u = a + \eta b \in E$ (with $a, b \in F$) acting as (u, Id) on $V' = Ew_1 \oplus Fv_1$, we have

$$uv_0 = av_0 + b\eta^2(e_+ - e_-) \quad \text{and} \quad \lambda e_\pm = \tfrac{1}{2}\left(\pm bv_0 + (1 \pm a)e_+ + (1 \mp a)e_-\right) .$$

Therefore $\alpha(ux) \leq \alpha(x)$ for all $x \in V$ if and only if

$$\begin{cases} \max\{|b|\, q^{r_0}, |1 \pm a|\, q^{r_0+s_1}, |1 \mp a|\, q^{r_0-s_1}\} & \leq \quad q^{r_0 \pm s_1} \\ \text{and} \qquad\qquad \max\{|a|\, q^{r_0}, |b|\, q^{r_0+s_1}\} & \leq \quad q^{r_0} \end{cases}$$

i.e. $|a| \leq 1$, $|1 - a| \leq q^{-2s_1}$ and $|b| \leq q^{-s_1}$. These conditions are met by any $u \in U_r$. Indeed, for $u = \lambda/\overline{\lambda}$ with $\lambda = a' + \eta b' \in \mathcal{O}_r^\times$, we have $a' \in \mathcal{O}^\times$ and $b' \in \mathcal{P}^r$. Therefore if $c = a'^{-1}b'$, then $c \in \mathcal{P}^r$ and

$$u = \frac{1 + \eta c}{1 - \eta c} = (1 + \eta c) \cdot \left(1 + \eta c + (\eta c)^2 + \cdots\right) = a + \eta b$$

with $a = 1 + 2\left((\eta c)^2 + (\eta c)^4 + \cdots\right)$ and $b = 2c\left(1 + (\eta c)^2 + (\eta c)^4 + \cdots\right)$. □

5.1.15. We define $\mathbf{D}(\alpha,\lambda) = S(\alpha|D,\lambda) \subset S(\alpha,\lambda)$. Thus $\mathbf{D}(\alpha,\lambda) = 0$ unless $\lambda \in d(\alpha) + \frac{1}{2}\mathcal{Q}(D)$, in which case it is an $\mathbf{F}$-line in $S(\alpha,\lambda)$. If moreover $d(\alpha) \in \frac{1}{2}\mathbf{N}$, so that $\lambda \in \frac{1}{2}\mathbf{Z}$, then $\mathbf{D}(\alpha,\lambda)$ is anisotropic for ϕ_λ if $d(\alpha) = 0$, and isotropic otherwise. For all $c \geq 0$ and $\lambda \in \mathbf{R}$, we put $S_c(\alpha,\lambda) = S_c(\overline{\alpha},\lambda) \cap S(\alpha,\lambda)$. Note that for $c \geq c^{sp}_{\max}(\alpha)$, the subspace $S_c(\overline{\alpha},\lambda)$ of $S(\overline{\alpha},\lambda)$ is already contained in $S(\alpha,\lambda)$.

5.2. **Vertices and edges.** Fix a quasi-hermitian space $(\overline{V},\psi,D)$ with underlying symmetric form $\phi = \mathrm{tr}_{E/F}\psi$. The pair of integers $(\delta_0,\delta_{\frac{1}{2}})$ of section 3.7 equals

$$(\delta_0,\delta_{\frac{1}{2}}) = 2(\Delta_{\mathbf{0}},\Delta_{\mathbf{2}}) \pm \begin{cases}(1,0) & \text{if } \mathcal{Q}(D) = 2\mathbf{Z}\\ (0,1) & \text{if } \mathcal{Q}(D) = 1+2\mathbf{Z}\end{cases}$$

where $(\Delta_{\mathbf{0}},\Delta_{\mathbf{2}}) \in \{(0,0),(0,1),(1,0),(1,1)\}$ is the pair attached to the hermitian E-space $(\underline{V},\psi)$ in section 4.1.2, and where the sign is uniquely determined by the conditions $0 \leq \delta_0,\delta_{\frac{1}{2}} \leq 2$. Put $\mathcal{I} = \mathcal{I}(V,\phi)$, $n = \mathrm{witt}_F(V,\phi)$ and $U = U(\underline{V},\psi)$.

5.2.1. If $\alpha \in \mathcal{I}$ is special and $\omega(\alpha) = [s_1,\cdots,s_n]_D = \sum_{i=1}^n [\theta_i,c_i]$, then

$$\begin{array}{lcccl}\alpha \text{ is a vertex} & \Leftrightarrow & \forall i:\ s_i \in \frac{1}{2}\mathbf{N} & \Leftrightarrow & \forall i:\ \theta_i,c_i \in \frac{1}{2}\mathbf{Z} \text{ and } \theta_i \equiv c_i \bmod \mathbf{Z}\\ \alpha \text{ is an edge} & \Leftrightarrow & \forall i:\ s_i \in \frac{1}{4}\mathbf{N} & \Leftrightarrow & \forall i:\ \theta_i,c_i \in \frac{1}{4}\mathbf{Z} \text{ and } \theta_i \equiv c_i \bmod \frac{1}{2}\mathbf{Z}\end{array}$$

Suppose that α is an edge with source $\alpha\{\frac{1}{2}\}$ and target $\alpha\{0\}$. Then for $x \in \{0,\frac{1}{2}\}$,

$$\omega(\alpha\{x\}) = [s_1\{x\},\cdots,s_n\{x\}]_D \in \mathbf{N}[\overline{\mathcal{L}}]^n_D$$

where for $s \in \frac{1}{4}\mathbf{N}$, we denote by $s\{x\}$ the unique element of $x + \frac{1}{2}\mathcal{Q}(D)$ which is closest to s if $s \notin \frac{1}{2}\mathbf{N}$ and put $s\{x\} = s$ otherwise. This follows from 5.1.9 by explicit computations in the apartment $\mathcal{A}(\mathcal{B})$ of $\mathcal{I}$ defined by a special basis $\mathcal{B}$ of V for which α belongs to $\mathcal{A}(\mathcal{B})$ with $\mathrm{coord}_{\mathcal{B}}\alpha = (s_1,\cdots,s_n)$. Note that while $(s_1,\cdots,s_n)$ does satisfy (**SP**), $(s_1\{x\},\cdots,s_n\{x\})$ may only satisfy (**sp**). And indeed, the source and target of a special edge need not be special anymore.

5.2.2. The ω-invariant $\alpha \in \mathcal{I} \mapsto \omega(\alpha) \in \mathbf{N}[\overline{\mathcal{L}}]^n_D$ therefore induces

- a bijection between $U\backslash\mathcal{I}(\frac{1}{2})$ and $\mathcal{V}^n_D = \mathcal{V}^n \cap \mathcal{V}_D$ where $\mathcal{V}_D = \mathcal{V} \cap \mathbf{N}[\overline{\mathcal{L}}]_D$,
- a bijection between $U\backslash\mathcal{I}(\frac{1}{4})$ and $\mathcal{E}^n_D = \mathcal{E}^n \cap \mathcal{E}_D$ where $\mathcal{E}_D = \mathcal{E} \cap \mathbf{N}[\overline{\mathcal{L}}]_D$.

Moreover, one checks using 5.2.1 that the source and target maps $\mathcal{E} \to \mathcal{V}$ of section 4.2.2 are compatible with the actual source and target maps $\mathcal{I}(\frac{1}{4}) \to \mathcal{I}(\frac{1}{2})$ under the above identifications, which means that they map $\mathcal{E}_D$ onto $\mathcal{V}_D$ and satisfy

$$\begin{array}{rcl}U\backslash\mathcal{I}(\frac{1}{4}) \xrightarrow{\text{source}} U\backslash\mathcal{I}(\frac{1}{2}) \xrightarrow{\omega} \mathcal{V}^n_D & = & U\backslash\mathcal{I}(\frac{1}{4}) \xrightarrow{\omega} \mathcal{E}^n_D \xrightarrow{\text{source}} \mathcal{V}^n_D,\\ U\backslash\mathcal{I}(\frac{1}{4}) \xrightarrow{\text{target}} U\backslash\mathcal{I}(\frac{1}{2}) \xrightarrow{\omega} \mathcal{V}^n_D & = & U\backslash\mathcal{I}(\frac{1}{4}) \xrightarrow{\omega} \mathcal{E}^n_D \xrightarrow{\text{target}} \mathcal{V}^n_D.\end{array}$$

5.3. **Projections and λ-corners.** For $s \in \frac{1}{4}\mathbf{N}$, we put $\mathcal{I}_s(\frac{1}{4}) = \mathcal{I}_s \cap \mathcal{I}(\frac{1}{4})$ and denote by $\mathrm{pr}_s : \mathcal{I}_{s+\frac{1}{4}}(\frac{1}{4}) \to \mathcal{I}_s(\frac{1}{4})$ the map induced by the projection $\mathcal{I}_{s+\frac{1}{4}} \to \mathcal{I}_s$ of section 5.1.3. It is computed as follows.

5.3.1. Suppose that α_1 belongs to $\mathcal{I}_{s+\frac{1}{4}}(\frac{1}{4})$ and use 5.1.9 to pick a decomposition $(V,\alpha_1)=(W,\beta)\perp(V',\alpha_1')$ and a special basis $\mathcal{B}=(v_0,w_1,v_1,\cdots,w_m,v_m)$ of V' such that $\alpha_1'=\alpha|V_1'$ belongs to the apartment $\mathcal{A}(\mathcal{B})$ defined by

$$V'=Fv_0\perp H_1\perp\cdots\perp H_m \quad \text{where} \quad H_i=Fe_i\oplus Fe_{-i}, \quad e_{\pm i}=\tfrac{1}{2}(v_i\pm w_i)$$

with $\mathrm{coord}_{\mathcal{B}}(\alpha_1')=(s_1,\cdots,s_{m-1},s+\frac{1}{4})$ satisfying **SP**. Then for $\alpha_2=\mathrm{pr}_s\alpha_1$, we have $(V,\alpha_2)=(W,\beta)\perp(V',\alpha_2')$ with α_2' in the same apartment and given by $\mathrm{coord}_{\mathcal{B}}(\alpha_2')=(s_1,\cdots,s_{m-1},s)$ (satisfying **sp**). Note that $m\geq 1$ since $s+\frac{1}{4}>0$.

5.3.2. Let α_i'' be the restriction of α_i' to the hyperbolic F-plane H_m and denote by α'' the common restriction of α_1' and α_2' to the orthogonal complement V'' of H_m in V'. Thus for $i\in\{1,2\}$ and any $\lambda\in\mathbf{R}$,

$$\begin{aligned} (V,\alpha_i) &= (W,\beta)\perp(V'',\alpha'')\perp(H_m,\alpha_i''),\\ B(\alpha_i,\lambda) &= B(\beta,\lambda)\perp B(\alpha'',\lambda)\perp B(\alpha_i'',\lambda),\\ B^0(\alpha_i,\lambda) &= B^0(\beta,\lambda)\perp B^0(\alpha'',\lambda)\perp B^0(\alpha_i'',\lambda),\\ \text{and}\quad S(\alpha_i,\lambda) &= S(\beta,\lambda)\oplus S(\alpha'',\lambda)\oplus S(\alpha_i'',\lambda). \end{aligned}$$

The latter decomposition is compatible with the pairings $\langle\bullet,\bullet\rangle_\lambda$. Since

$$\alpha_1''(e_{\pm m})=q^{r_0\pm(s+\frac{1}{4})} \quad \text{and} \quad \alpha_2''(e_{\pm m})=q^{r_0\pm s} \quad \text{with} \quad q^{2r_0}=|Q(v_m)|,$$

we find that for $\lambda=r_0+s$, $B(\alpha_1,\lambda+\frac{\epsilon}{4})=B(\alpha_2,\lambda+\frac{\epsilon}{4})$ where $\epsilon=1$ if $\lambda\in\frac{1}{2}\mathbf{Z}$ and $\epsilon=-1$ otherwise, while

$$B^0(\alpha_2,\lambda)\subset B^0(\alpha_1,\lambda+\tfrac{1}{4})=B(\alpha_1,\lambda)\subsetneq B(\alpha_2,\lambda)\subset B(\alpha_1,\lambda+\tfrac{1}{4})$$

with $\dim_{\mathbf{F}}B(\alpha_2,\lambda)/B(\alpha_1,\lambda)=1$. Moreover,

$$D_1\overset{def}{=}B(\alpha_2,\lambda)/B^0(\alpha_1,\lambda+\tfrac{1}{4})\subset S(\alpha_1'',\lambda+\tfrac{1}{4})\subset S(\alpha_1,\lambda+\tfrac{1}{4})$$

is the $\mathbf{F}$-line spanned by the image of $e_m\in B(\alpha_1'',\lambda+\frac{1}{4})$ in $S(\alpha_1'',\lambda+\frac{1}{4})$. It is isotropic for $\phi_{\lambda+\frac{1}{4}}$ if $\lambda+\frac{1}{4}$ belongs to $\frac{1}{2}\mathbf{Z}$. And indeed since $e_{-m}=\frac{1}{2}(v_m-w_m)$ belongs to $B^0(\alpha_1'',\lambda+\frac{1}{4})$ while $e_m=\frac{1}{2}(v_m+w_m)$ with $v_m,w_m\in B(\alpha_1'',\lambda+\frac{1}{4})$, this $\mathbf{F}$-line D_1 equals $\mathbf{D}(\alpha_1,\lambda+\frac{1}{4})$. Also,

$$H_2\overset{def}{=}B^0(\alpha_1,\lambda+\tfrac{1}{4})/B^0(\alpha_2,\lambda)\supset S(\beta,\lambda)\oplus S(\alpha'',\lambda)$$

is an $\mathbf{F}$-hyperplane of $S(\alpha_2,\lambda)$, a complement of which is the $\mathbf{F}$-line D_2 spanned by the image of $e_m\in B(\alpha_2,\lambda)$ in $S(\alpha_2,\lambda)$, and also the canonical $\mathbf{F}$-line $\mathbf{D}(\alpha_2,\lambda)$ (which is spanned by the image of $v_m\in B(\alpha_2,\lambda)$, and equals D_2 if $s>0$). If $\lambda\in\frac{1}{2}\mathbf{Z}$, then $H_2^\perp\subset H_2$ for the pairing ϕ_λ on $S(\alpha_2,\lambda)$.

5.3.3. For $i\in\{1,2\}$, we also have $(\overline{V},\overline{\alpha}_i)=(W,\beta)\perp(\overline{V}',\overline{\alpha}_i')$. Thus for all $c\geq 0$,

$$S(\overline{\alpha}_i,\lambda)=S(\beta,\lambda)\oplus S(\overline{\alpha}_i',\lambda) \quad \text{and} \quad S_c(\overline{\alpha}_i,\lambda)=S_c(\beta,\lambda)\oplus S_c(\overline{\alpha}_i',\lambda).$$

Lemma. *For the $\mathbf{F}$-subspace H_2 of $S(\overline{\alpha}_2,\lambda)\supset S(\alpha_2,\lambda)$ and any $c\geq 0$,*

$$S_c(\overline{\alpha}_2,\lambda)\subset H_2 \iff c\geq s_{m-1}+s.$$

Proof. For all $c\geq 0$, $S_c(\beta,\lambda)\subset S(\beta,\lambda)\subset H_2$. Since $\omega(\overline{\alpha}_2')=\lceil s_1,\cdots,s_{m-1},s\rceil_D$ by 5.1.9, $c_{\max}(\overline{\alpha}_2')=s_{m-1}+s$ and $S_c(\overline{\alpha}_2',\lambda)=0\subset H_2$ for all $c\geq s_{m-1}+s$. In view of (2.2), we thus have to show that

$$H_2+\mathrm{im}\,([\eta]:S(\overline{\alpha}_2',\lambda')\to S(\overline{\alpha}_2',\lambda))=S(\overline{\alpha}_2,\lambda)$$

where $\lambda' = \lambda - s_{m-1} - s = r_0 - s_{m-1}$. If $m = 1$, $s_{m-1} = 0$ and we use

$$v_0 \in B(\overline{\alpha}'_2, \lambda') \quad \text{with} \quad \eta v_0 = \eta^2 w_1 \in B(\overline{\alpha}'_2, \lambda) - B^0(\overline{\alpha}'_1, \lambda + \tfrac{1}{4}).$$

If $m > 1$, we consider instead $e_{-(m-1)} \in B(\overline{\alpha}'_2, \lambda')$, for which

$$\eta e_{-(m-1)} = \tfrac{1}{2}(\eta^2 w_m - v_{m-2}) \in B(\overline{\alpha}'_2, \lambda) - B^0(\overline{\alpha}'_1, \lambda + \tfrac{1}{4}).$$

This proves our claim in both cases. □

5.3.4. These computations show that the edges α_1 and $\alpha_2 = \mathrm{pr}_s \alpha_1$ are related by a λ-corner (cf. section 3.14) of the type considered in the following lemma.

Lemma. *Fix $\lambda \in \frac{1}{4}\mathbf{Z}$ and $s \in \frac{1}{4}\mathbf{N}$. Let $(\alpha_1, D_1) \mapsto (\alpha_2, H_2)$ be a λ-corner. The following conditions are equivalent:*

(1) *$d(\alpha_1) = s + \frac{1}{4}$ and $D_1 = \mathbf{D}(\alpha_1, \lambda + \frac{1}{4})$,*
(2) *$d(\alpha_2) = s$ and $H_2 \oplus \mathbf{D}(\alpha_2, \lambda) = S(\alpha_2, \lambda)$.*

If so, then $\lambda \in s + \frac{1}{2}\mathcal{Q}(D)$, $\mathrm{pr}_s(\alpha_1) = \alpha_2$ and

$$c^{sp}_{\max}(\alpha_1) = \tfrac{1}{4} + \min\left\{c \geq c^{sp}_{\max}(\alpha_2) : S_c(\alpha_2, s) \subset H_2\right\}.$$

Proof. Write $B^0_1 \subset B_1$ and $B^0_2 \subset B_2$ for the open and closed balls of α_1 at $\lambda + \frac{1}{4}$ and α_2 at λ, so that $B^0_2 \subset B^0_1 \subsetneq B_2 \subset B_1$ with $D_1 = B_2/B^0_1$ and $H_2 = B^0_1/B^0_2$. Note that

$$\mathbf{D}(\alpha_1, \lambda + \tfrac{1}{4}) = D \cap B_1 + B^0_1/B^0_1 \quad \text{and} \quad \mathbf{D}(\alpha_2, \lambda) = D \cap B_2 + B^0_2/B^0_2.$$

Therefore $D_1 = \mathbf{D}(\alpha_1, \lambda + \frac{1}{4}) \iff S(\alpha_2, \lambda) = H_2 \oplus \mathbf{D}(\alpha_2, \lambda)$, in which case

$$\lambda + \tfrac{1}{4} \in d(\alpha_1) + \tfrac{1}{2}\mathcal{Q}(D) \quad \text{and} \quad \lambda \in d(\alpha_2) + \tfrac{1}{2}\mathcal{Q}(D).$$

Since also $|d(\alpha_1) - d(\alpha_2)| \leq \mathrm{dist}(\alpha_1, \alpha_2) = \frac{1}{4}$, this shows that (1) $\iff$ (2). The remaining assertions follow from the above computations of $\mathrm{pr}_s \alpha_1$. □

5.4. **The fibers of convex projection.** Fix $\omega = 2\omega_{reg} + [s_1, \cdots, s_m]_D \in \mathcal{V}^n_D$ – with $(s_1, \cdots, s_m)$ satisfying **Sp**. Let $\beta \in \mathcal{I}(\underline{V})$ be any self-dual F-norm on $\underline{V}$ for which $\omega(\beta) = \omega_{reg} + \lfloor s_1, \cdots, s_m \rfloor_D$. We shall now compute the number

$$N(\omega) = \left|\left\{\alpha \in \mathcal{I}(\tfrac{1}{4}) : \omega(\alpha) = \omega \quad \text{and} \quad \underline{\alpha} = \beta\right\}\right|.$$

5.4.1. Fix $\alpha \in \mathcal{I}(\frac{1}{4})$ with $\omega(\alpha) = \omega$ and $\underline{\alpha} = \beta$. Put $s = d(\alpha) = s_m$ and for $t \in [0, s]$, let $\{\alpha_t\} = [\alpha_0 : \alpha_s] \cap \mathcal{I}_t$ where α_0 is the convex projection of $\alpha_s = \alpha$ to $\mathcal{I}_0(V)$. Put $s_0 = s_{-1} = 0$. Then for $t \in]s_{i-2}, s_i]$ with $1 \leq i \leq m$ and $i \equiv m \bmod 2$,

$$\omega_{sp}(\alpha_t) = [s_1, \cdots, s_{i-1}, t]_D \qquad \text{and} \qquad c^{sp}_{\max}(\alpha_t) = s_{i-1} + t$$

With these notations, $N(\omega) = \prod_{t \in [0, s[\cap \frac{1}{4}\mathbf{N}} N(\omega, t)$ where

$$N(\omega, t) = \left|\left\{\alpha' \in \mathcal{I}_{t+\frac{1}{4}}(\tfrac{1}{4}) : \omega(\alpha') = \omega(\alpha_{t+\frac{1}{4}}) \text{ and } \mathrm{pr}_t(\alpha') = \alpha_t\right\}\right|.$$

5.4.2. Fix $\lambda(0) \in \frac{1}{2}\mathcal{Q}(D)$ and set $\lambda(t) = t + \lambda(0)$. For $t \in [0, s[\cap \frac{1}{4}\mathbf{N}$, we know from section 5.3 that $N(\omega, t)$ is also the number of $\mathbf{F}$-hyperplane complements H of $\mathbf{D}(\alpha_t, \lambda(t))$ in $S(\alpha_t, \lambda(t))$such that $H^{\perp} \subset H$ for the symmetric pairing ϕ_{t+r_0} on $S(\alpha_t, \lambda(t))$ if $\lambda(t) \in \frac{1}{2}\mathbf{Z}$, and such that

$$\min\left\{c \geq c^{sp}_{\max}(\alpha_t) : S_c(\alpha_t, \lambda(t)) \subset H\right\} = c^{sp}_{\max}(\alpha_{t+\frac{1}{4}}) - \tfrac{1}{4}.$$

5.4.3. For all $t \in [0, s[$, there is a unique $1 \leq i \leq m$ with $i \equiv m \bmod 2$ such that $t \in [s_{i-2}, s_i[$. Put $c(t) = s_{i-1} + t$ and define

$$\nu(t) = \dim_{\mathbf{F}} S(\beta, \lambda(t)) - \dim_{\mathbf{F}} S_{c(t)}(\beta, \lambda(t)) - \begin{cases} 0 & \text{if } \lambda(t) \notin \mathbf{Z} \\ 1 & \text{if } \lambda(t) \in \mathbf{Z} \end{cases}$$

$$\text{and } R(t) = \begin{cases} 1 & \text{if } t \neq s_{i-2}, \\ 1 - q^{-\dim_{\mathbf{F}} \mathrm{Gr}_{c(t)} S(\beta, \lambda(t))} & \text{if } t = s_{i-2} \text{ and } c(t) \neq 0, \\ 1 - (-q)^{-\dim_{\mathbf{E}} S(\beta, \lambda(0), 0)} & \text{if } c(t) = 0. \end{cases}$$

Note that $c(t) = 0$ implies $t = s_{i-1} = s_{i-2} = 0$. Note also that for all $t \in [0, s[\cap \frac{1}{4}\mathbf{N}$, $c(t) = c^{sp}_{\max}(\alpha_{t+\frac{1}{4}}) - \frac{1}{4}$ by 5.4.1. We will prove that

Proposition. *For all $t \in [0, s[\cap \frac{1}{4}\mathbf{N}$, $N(\omega, t) = q^{\nu(t)} R(t)$.*

5.4.4. Suppose first that $t \in]s_{i-2}, s_i[\cap \frac{1}{4}\mathbf{N}$. Then $N(\omega, t)$ is just the number of $\mathbf{F}$-hyperplane complements H of $\mathbf{D}(\alpha_t, \lambda(t))$ in $S(\alpha_t, \lambda(t))$ with $S_{c(t)}(\alpha_t, \lambda(t)) \subset H$ and $H^\perp \subset H$ if $t \in \frac{1}{2}\mathbf{N}$, namely

$$N(\omega, t) = q \wedge \left(\dim_{\mathbf{F}} S(\alpha_t, \lambda(t)) - \dim_{\mathbf{F}} S_{c(t)}(\alpha_t, \lambda(t)) - \left\{ \begin{array}{ll} 2 & \text{if } t \in \frac{1}{2}\mathbf{N} \\ 1 & \text{if } t \notin \frac{1}{2}\mathbf{N} \end{array} \right\} \right)$$

by lemma 7 and 5 of the appendix, respectively. Using explicit decompositions of (V, α) as we did already several times, we find that

$$\dim_{\mathbf{F}} S(\alpha_t, \lambda(t)) = \dim_{\mathbf{F}} S(\beta, \lambda(t)) + \begin{cases} 1 & \text{if } t \in \mathbf{N} \\ 2 & \text{if } t \in \frac{1}{2}\mathbf{N} - \mathbf{N} \\ 1 & \text{if } t \in \frac{1}{4}\mathbf{N} - \frac{1}{2}\mathbf{N} \end{cases}$$

$$\text{and } \dim_{\mathbf{F}} S_{c(t)}(\alpha_t, \lambda(t)) = \dim_{\mathbf{F}} S_{c(t)}(\beta, \lambda(t))$$

since also $c(t) = c^{sp}_{\max}(\alpha_t)$. Therefore $N(\omega, t) = q^{\nu(t)} R(t)$ in this case.

5.4.5. Suppose next that $t = s_{i-2} \neq 0$. Thus $N(\omega, t)$ now equals $q^{\nu(t)}$ minus

$$q \wedge \left(\dim_{\mathbf{F}} S(\alpha_t, \lambda(t)) - \dim_{\mathbf{F}} S_{c(t)-\frac{1}{4}}(\alpha_t, \lambda(t)) - \left\{ \begin{array}{ll} 2 & \text{if } t \in \frac{1}{2}\mathbf{N} \\ 1 & \text{if } t \notin \frac{1}{2}\mathbf{N} \end{array} \right\} \right)$$

But $c(t) - \frac{1}{4} \geq c^{sp}_{\max}(\alpha_t)$, so that again

$$\dim_{\mathbf{F}} S_{c(t)-\frac{1}{4}}(\alpha_t, \lambda(t)) = \dim_{\mathbf{F}} S_{c(t)-\frac{1}{4}}(\beta, \lambda(t))$$

and $N(\omega, t) = q^{\nu(t)} R(t)$. If $t = 0$ but $c(0) \neq 0$, we obtain $N(\omega, 0) = q^{\nu(0)} R(0)$ similarly, using Lemma 6 instead of 7 in the appendix.

5.4.6. Suppose finally that $t = 0 = c(0)$. Then lemma 6 below shows that

$$N(\omega, 0) = q^{\nu(0)} (1 - \epsilon q^{\frac{1}{2} \dim_{\mathbf{F}} S(\beta, \lambda(0), 0)})$$

where $\epsilon = 1$ if $S(\beta, \lambda(0))$ is split and -1 otherwise. Note that $S(\beta, \lambda(0))$ splits if and only if $S(\beta, \lambda(0), 0)$ does, and recall from section 4.3.5 that the latter is the $\mathbf{F}$-quadratic space underlying an $\mathbf{E}$-hermitian space. Therefore

$$\epsilon q^{\frac{1}{2} \dim_{\mathbf{F}} S(\beta, \lambda(0), 0)} = (-q)^{\dim_{\mathbf{E}} S(\beta, \lambda(0), 0)}$$

and $N(\omega, 0) = q^{\nu(0)} \cdot R(0)$, which finishes the proof of proposition 5.4.3.

6. Appendix

6.1. **Proof of Lemma 4.1.5.** Let (V,ψ) be a finite dimensional E-hermitian space which is split and let α be an F-norm on V which is self-dual with respect to the underlying symmetric F-form $\phi = \mathrm{tr}\psi$. Fix a Witt decomposition

$$(V,\alpha) = \perp_{i=1}^{n} (V_i,\alpha_i) \oplus (V_{-i},\alpha_{-i})$$

in which all the $V_{\pm i}$'s are isotropic E-lines in V, so that $\eta\psi$ induces isomorphims $(V_i,\alpha_i) \simeq (V_{-i},\alpha_{-i})^\star = (V_{-i}^\star, \alpha_{-i}^\star)$. We let $(\rho_{\pm i}, c_{\pm i})$ be the type of $(V_{\pm i},\alpha_{\pm i})$ with respect to some E-basis $e_{\pm i}$ of $V_{\pm i}$, chosen such that $\eta\psi(e_{-i},e_i) = 1$ (and thus $\rho_{-i} = c_i - \rho_i$ and $c_{-i} = c_i$).

For i and j in $\{\pm 1, \cdots, \pm n\}$, we denote by $u_{i,j} : V_j \to V_i$ the (i,j)-component of an E-linear endomorphism $u : V \to V$. Thus for the adjoint endomorphism $u^\star : V \to V$ (with respect to ψ), we have

$$(u^\star)_{i,j} : V_j \to V_i \qquad = \qquad (u_{-j,-i})^\star : V_{-j}^\star \to V_{-i}^\star$$

under the above identifications $V_i \simeq V_{-i}^\star$ and $V_j \simeq V_{-j}^\star$. In the sequel, we will describe certain endomorphisms of V by specifying only some of their components, in matrix form: the unspecified $u_{i,j}$'s will always be given by Kronecker's symbol, namely $u_{i,j} = 0$ if $i \neq j$ and $u_{i,j} = \mathrm{Id} : V_i \to V_i$ if $i = j$.

We denote by $U(\alpha)$ the stabilizer of α in $U(V,\psi)$. Note that a given element $u \in U(V,\psi)$ fixes α if and only it also belongs to $\mathrm{End}_{\mathcal{N}}(V,\alpha)$ which amounts to requiring that all of its components $u_{j,i} : V_j \to V_i$ are actually morphisms $(V_j,\alpha_j) \to (V_i,\alpha_i)$ in the category $\mathcal{N}$. Indeed and more generally, an element $g \in O(V,\phi)$ fixes α if and only if it is compatible with the norm in the weak sense that $\alpha(gv) \le \alpha(v)$ for all $v \in V$, because then also

$$\alpha(v) = \sup \frac{|\phi(v,w)|}{\alpha(w)} = \sup \frac{|\phi(gv,gw)|}{\alpha(w)} \le \sup \frac{|\phi(gv,gw)|}{\alpha(gw)} = \alpha(gv).$$

This being said, we may construct some obvious families of elements in $U(\alpha)$:

- For $1 \le i \le n$ and $\gamma = \gamma_i : (V_i,\alpha_i) \to (V_i,\alpha_i)$ an isomorphism,

$$t(\gamma_i) = \begin{pmatrix} \gamma & \\ & \gamma^{\star -1} \end{pmatrix} \quad \text{in}\, U(V_i \oplus V_{-i}).$$

- For $1 \le i \ne j \le n$ and $x = x_{j,i} : (V_i,\alpha_i) \to (V_j,\alpha_j)$ a morphism,

$$u(x_{j,i}) = \begin{pmatrix} \mathrm{Id} & & & \\ x & \mathrm{Id} & & \\ & & \mathrm{Id} & -x^\star \\ & & & \mathrm{Id} \end{pmatrix} \quad \text{in } SU(V_i \oplus V_j \oplus V_{-i} \oplus V_{-j})$$

- For $1 \le i \ne j \le n$ and $x = x_{-j,i} : (V_i,\alpha_i) \to (V_{-j},\alpha_{-j})$ a morphism,

$$u(x_{-j,i}) = \begin{pmatrix} \mathrm{Id} & & & \\ & \mathrm{Id} & & \\ & -x^\star & \mathrm{Id} & \\ x & & & \mathrm{Id} \end{pmatrix} \quad \text{in } SU(V_i \oplus V_j \oplus V_{-i} \oplus V_{-j})$$

- For $1 \leq i \neq j \leq n$ and $x = x_{j,-i} : (V_{-i}, \alpha_{-i}) \to (V_j, \alpha_j)$ a morphism,

$$u(x_{-j,i}) = \begin{pmatrix} \mathrm{Id} & & & -x^\star \\ & \mathrm{Id} & x & \\ & & \mathrm{Id} & \\ & & & \mathrm{Id} \end{pmatrix} \quad \text{in } SU(V_i \oplus V_j \oplus V_{-i} \oplus V_{-j})$$

- For $1 \leq i \leq n$ and $x = x_{-i,i} : (V_i, \alpha_i) \to (V_{-i}, \alpha_{-i})$ a morphism such that $x^\star = -x$,

$$u(x_{-i,i}) = \begin{pmatrix} \mathrm{Id} & \\ x & \mathrm{Id} \end{pmatrix} \quad \text{in } SU(V_i \oplus V_{-i})$$

- For $1 \leq i \leq n$ and $x = x_{i,-i} : (V_{-i}, \alpha_{-i}) \to (V_i, \alpha_i)$ a morphism such that $x^\star = -x$,

$$u(x_{i,-i}) = \begin{pmatrix} \mathrm{Id} & x \\ & \mathrm{Id} \end{pmatrix} \quad \text{in } SU(V_i \oplus V_{-i})$$

- For $1 \leq i \neq j \leq n$ and $\theta = \theta_{j,i} : (V_i, \alpha_i) \to (V_j, \alpha_j)$ an isomorphism,

$$w(\theta_{j,i}) = \begin{pmatrix} & \theta^{-1} & & \\ \theta & & & \\ & & & \theta^\star \\ & & \theta^{\star -1} & \end{pmatrix} \quad \text{in } SU(V_i \oplus V_j \oplus V_{-i} \oplus V_{-j})$$

- For $1 \leq i \neq j \leq n$ and $\theta = \theta_{-j,i} : (V_i, \alpha_i) \to (V_{-j}, \alpha_j)$ an isomorphism,

$$w(\theta_{-j,i}) = \begin{pmatrix} & & & \theta^{-1} \\ & & \theta^{\star -1} & \\ & \theta^\star & & \\ \theta & & & \end{pmatrix} \quad \text{in } SU(V_i \oplus V_j \oplus V_{-i} \oplus V_{-j})$$

- For $1 \leq i \leq n$ and $\theta = \theta_{-i,i} : (V_i, \alpha_i) \to (V_{-i}, \alpha_{-i})$ an isomorphism,

$$w(\theta_{-i,i}) = \begin{pmatrix} & \theta^{\star -1} \\ \theta & \end{pmatrix} \quad \text{in } U(V_i \oplus V_{-i})$$

Lemma. *These elements span $U(\alpha)$.*

Proof. Without loss of generality, we may assume that $c_1 \geq c_2 \geq \cdots \geq c_n$. Let $g \in U(\alpha)$. By corollary 3.5 of [1], there exists $j \in \{\pm 1, \cdots, \pm n\}$ such that $g_{j,1} : (V_1, \alpha_1) \to (V_j, \alpha_j)$ is an isomorphism. Replacing g by $t(g_{1,1}^{-1}) \circ g$ (if $j = 1$) or $\omega(g_{j,1}) \circ g$ (if $j \neq 1$), we may assume that $j = 1$ and $g_{1,1} = \mathrm{Id} : V_1 \to V_1$. Replacing g by $u(-g_{\pm i,1}) \circ g$, we may also assume that $g_{\pm i,1} = 0$ for all $1 < i \leq m$. Then $g^\star_{-1,1} = -g_{-1,1}$, and thus replacing g by $u(-g_{-1,1}) \circ g$, we may finally also assume that $g_{-1,1} = 0$. Multiplying now g on the right by similar unipotent elements, we arrive at a situation where $g_{1,i} = g_{i,1} = 0$ for all $i \in \{-1, \pm 2, \cdots, \pm n\}$, with $g_{1,1} = \mathrm{Id}$. Then $g|V_1 \oplus V_{-1}$ is the identity, and $g' \in U(\alpha')$ where g' and α' are the restrictions of g and α to the orthogonal complement $V' = \perp_{i=2}^n (V_i \oplus V_{-i})$ of $V_1 \oplus V_{-1}$. We conclude by induction on n. □

We may now prove lemma 4.1.5. Of the above generators, only the first and last families contribute to the determinant. An isomorphism $\gamma_i : (V_i, \alpha_i) \to (V_i, \alpha_i)$ is the scalar multiplication by some element $\gamma_i \in \mathcal{O}_{r_i}^\times$ with $r_i = \lceil c_i \rceil$, and $\det t(\gamma_i)$ thus equals $\gamma_i / \overline{\gamma_i} \in U_{r_i}$. On the other hand, let $\theta_{-i,i} : (V_i, \alpha_i) \to (V_{-i}, \alpha_{-i})$ be an isomorphism. Then $\rho_{-i} \equiv \rho_i \bmod \mathbf{Z}$, i.e. $c_i - 2\rho_i = n_i \in \mathbf{Z}$ and $\theta_{-i,i}(e_i) = \pi^{n_i} \gamma_i e_{-i}$

for some $\gamma_i \in \mathcal{O}_{r_i}^\times$ (with $r_i = \lceil c_i \rceil$ as above). But then $\theta^\star_{-i,i}(e_i) = -\pi^{n_i}\overline{\gamma}_i e_{-i}$ and thus again $\det w(\theta_{-i,i}) = \gamma_i/\overline{\gamma}_i \in U_{r_i}$, QED.

6.2. **Counting Lemmas.** Recall from section 4.3.9 the functions

$$\begin{aligned}\pi(r) &= (q^r-1)(q^{r-1}-1)\cdots(q-1),\\ \sigma(r) &= (q^r+1)(q^{r-1}+1)\cdots(q+1),\\ \tau(r) &= (q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})\cdots(q+1).\end{aligned}$$

The following formulas are classical

$$|GL(r,\mathbf{F})| = q^{\frac{(r-1)r}{2}}\pi(r), \quad |U(r,\mathbf{F})| = q^{\frac{(r-1)r}{2}}\tau(r), \quad |Sp(2r,\mathbf{F})| = q^{r^2}\sigma(r)\pi(r).$$

Lemma 1. *Let (V,ψ) be a $\mathbf{E}$-hermitian space of $\mathbf{E}$-dimension $d = 2a+2b+2c+e$. The number of totally isotropic $\mathbf{F}$-subspaces H of V for which*

$$\dim_{\mathbf{E}}\begin{pmatrix} H\cap\eta H \\ H\cap\eta H^\perp + \eta H\cap H^\perp \\ H+\eta H\end{pmatrix} = \begin{pmatrix}1 & & \\ 1 & 1 & \\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

equals

$$q^{2c(b+e)+c(c-1)+\frac{b(b-1)}{2}}\cdot\frac{\tau(d)}{\pi(a)\pi(b)\pi(c)\sigma(a)\sigma(c)\tau(e)}$$

Proof. We consider the following diagram

$$\begin{array}{ccccccc} & & & H & & & \\ & & \nearrow & & \searrow & & \\ & H' & & & & \overline{H} & \\ \nearrow & & \searrow & & \nearrow & & \searrow \\ H_1 & & & H_2 & & & H_3 \\ \searrow & & \nearrow & & \searrow & & \nearrow \\ & \eta H' & & & & \eta\overline{H} & \\ & & \searrow & & \nearrow & & \\ & & & \eta\mathcal{H} & & & \end{array}$$

$$\begin{aligned} H_1 &= H\cap\eta H\\ H_2 &= H\cap\eta H^\perp + H^\perp\cap\eta H\\ H_3 &= H+\eta H\\ H' &= H\cap H_2\\ \overline{H} &= H+H_2\end{aligned}$$

We choose in this order (1) the totally isotropic $\mathbf{E}$-subspace H_1 of V, (2) the totally isotropic $\mathbf{E}$-subspace $\mathcal{H}_2 = H_2/H_1$ of $V_1 = H_1^\perp/H_1$, (3) the non-degenerate $\mathbf{E}$-subspace $V_3 = H_3/H_2$ of $V_2 = H_2^\perp/H_2$, (4) the totally isotropic $\mathbf{F}$-subspace $\mathcal{H}_4 = \overline{H}/H_2$ of V_3 such that $\mathcal{H}_4 \oplus \eta\mathcal{H}_4 = V_3$, (5) the $\mathbf{F}$-subspace $\mathcal{H}_5 = H'/H_1$ of $\mathcal{H}_2$ such that $\mathcal{H}_5 \oplus \eta\mathcal{H}_5 = \mathcal{H}_2$, and finally (6) the complement $\mathcal{H}_6 = H/H'$ of H_2/H' in the $\mathbf{F}$-space $\overline{H}/H'$. We now compute the number of possible choices at each step. For (1) and (2), [2, 9.4.1] gives respectively

$$\frac{\tau(d)}{\pi(a)\sigma(a)\tau(d-2a)} \quad \text{and} \quad \frac{\tau(d-2a)}{\pi(b)\sigma(b)\tau(d-2a-2b)}$$

For (3): the subspaces V_3 are all conjugated under $U(V_2)$, which gives

$$\frac{|U(d-2a-2b)|}{|U(2c)|\cdot|U(d-2a-2b-2c)|} = q^{2c(d-2a-2b-2c)}\frac{\tau(d-2a-2b)}{\tau(2c)\cdot\tau(d-2a-2b-2c)}$$

possible choices for V_3. For (4), we introduce the symplectic form $\theta = \mathrm{tr}_{\mathbf{E}/\mathbf{F}}\eta\psi$ on V_3. If $(e_{\pm i})$ is a Witt $\mathbf{E}$-basis of V_3 (with $\eta\psi(e_i, e_{-i}) = 1$), then $(e_{\pm i})$ spans over $\mathbf{F}$ a totally isotropic $\mathbf{F}$-subspace $\mathcal{H}_4$ of V_3 such that $\mathcal{H}_4 \oplus \eta\mathcal{H}_4 = V_3$, and $(e_{\pm i})$ is a Witt $\mathbf{F}$-basis for $(\mathcal{H}_4, \theta|\mathcal{H}_4)$. Conversely, for any totally isotropic $\mathbf{F}$-subspace $\mathcal{H}_4$ of

V_3 such that $\mathcal{H}_4 \oplus \eta\mathcal{H}_4 = V_3$, the restriction of θ to $\mathcal{H}_4$ is non-degenerate and for any Witt $\mathbf{F}$-basis $(e_{\pm i})$ of $(\mathcal{H}_4, \theta|\mathcal{H}_4)$, $(e_{\pm i})$ is also a Witt $\mathbf{E}$-basis of V_3. This gives

$$\frac{|U(2c)|}{|Sp(2c)|} = q^{c(c-1)} \cdot \frac{\tau(2c)}{\pi(c)\sigma(c)}$$

possible choices for $\mathcal{H}_4$. For (5), a similar argument gives

$$\frac{|GL(b,\mathbf{E})|}{|GL(b,\mathbf{F})|} = q^{\frac{(b-1)b}{2}} \cdot \sigma(b)$$

possible choices for $\mathcal{H}_5$. Finally for (6), there are q^{2bc} possible complements H/H' of H_2/H' in $\overline{H}/H'$. □

Lemma 2. *Let* V_1 *and* V_2 *be* $\mathbf{F}$*-vector spaces of dimension* $d = a + b + c$ *with a perfect pairing* $\langle \bullet, \bullet \rangle : V_1 \times V_2 \to \mathbf{F}$*. The number of pairs* (H_1, H_2) *of* $\mathbf{F}$*-subspaces* $H_1 \subset V_1$ *and* $H_2 \subset V_2$ *of* $\mathbf{F}$*-dimensions* a *and* b *such that* $\langle H_1, H_2 \rangle = 0$ *equals*

$$\frac{\pi(d)}{\pi(a)\pi(b)\pi(c)}.$$

Proof. We choose first H_1 in V_1 and then H_2 in $H_1^\perp \subset V_2$. There are

$$\frac{\pi(d)}{\pi(a)\cdot\pi(d-a)} \quad \text{and} \quad \frac{\pi(d-a)}{\pi(b)\pi(d-a-b)}$$

choices at each step. □

Lemma 3. *Let* (V, θ) *be a symplectic* $\mathbf{F}$*-vector space of* $\mathbf{F}$*-dimension* $2d$ *with*

$$d = a + b + c + b' + c' + e.$$

The number of pairs of $\mathbf{F}$*-subspaces* (B, C) *of* V *such that* $\theta(B, C) = 0$ *and*

$$\dim_{\mathbf{F}} \begin{pmatrix} B \cap C \\ B \cap B^\perp \\ C \cap C^\perp \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 0 & 2 & \\ 1 & 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ b' \\ c' \end{pmatrix}$$

equals

$$q^{2b'(e+c'+c)+2c'(b+e)+bc} \frac{\pi(d)\sigma(d)}{\pi(a)\pi(b)\pi(c)\pi(b')\pi(c')\pi(e)\sigma(b')\sigma(c')\sigma(e)}$$

Proof. We consider the following diagram

$$\begin{array}{ccccccc} & & & B & & & \\ & & \nearrow & & \searrow & & \\ & B' & & & & \overline{B} & \\ \nearrow & & \searrow & & \nearrow & & \searrow \\ A_1 & & & A_2 & & & A_3 \\ \searrow & & \nearrow & & \searrow & & \nearrow \\ & C' & & & & \overline{C} & \\ & & \searrow & & \nearrow & & \\ & & & C & & & \end{array} \qquad \begin{aligned} A_1 &= B \cap C \\ A_2 &= B \cap B^\perp + C \cap C^\perp \\ A_3 &= B + C \\ X' &= X \cap X^\perp \\ \overline{X} &= A_2 + X \end{aligned}$$

for $X \in \{B, C\}$. We choose in this order: (1) the totally isotropic subspace A_1 of V, (2) the totally isotropic subspace $\mathcal{A}_2 = A_2/A_1$ of $V_1 = A_1^\perp/A_1$, (3) the non-degenerate subspace $\mathcal{A}_3 = A_3/A_2$ of $V_2 = A_2^\perp/A_2$, (4) the orthogonal decomposition $\overline{\mathcal{B}} \perp \overline{\mathcal{C}} = \mathcal{A}_3$ where $\overline{\mathcal{X}} = \overline{X}/A_2$ for $X \in \{B, C\}$, (5) the decomposition $\mathcal{B}' \oplus \mathcal{C}' = \mathcal{A}_2$

where $\mathcal{X}' = X'/A_1$ for $X \in \{B, C\}$, and finally (6) the complement $\mathcal{X} = X/X'$ of A_2/X' in $\overline{X}/X'$ for $X \in \{B, C\}$. We now compute the number of possible choices at each step. For (1) and (2), [2, 9.4.1] gives respectively

$$\frac{\pi(d)\sigma(d)}{\pi(a)\pi(d-a)\sigma(d-a)} \quad \text{and} \quad \frac{\pi(d-a)\sigma(d-a)}{\pi(b+c)\pi(d-a-b-c)\sigma(d-a-b-c)}.$$

For (3) and (4) together, note that all pairs $(\overline{\mathcal{B}}, \overline{\mathcal{C}})$ are conjugated by $Sp(V_2)$. Thus

$$\frac{|Sp(V_2)|}{\left|Sp(\overline{\mathcal{B}})\right| \cdot \left|Sp(\overline{\mathcal{C}})\right| \cdot \left|Sp(\overline{\mathcal{D}})\right|} = q^{2(b'+c')e+2b'c'} \cdot \frac{\pi(d-a-b-c)\sigma(d-a-b-c)}{\pi(b')\sigma(b')\pi(c')\sigma(c')\pi(e)\sigma(e)}$$

where $\overline{\mathcal{D}} \perp \overline{\mathcal{B}} \perp \overline{\mathcal{C}} = V_2$. For (5), we obtain

$$\frac{|GL(b+c, \mathbf{F})|}{|GL(b, \mathbf{F})| \cdot |GL(c, \mathbf{F})|} = q^{bc} \cdot \frac{\pi(b+c)}{\pi(b) \cdot \pi(c)}$$

and finally for (6), $q^{2b'c+2bc'}$. □

Lemma 4. *Let (V, ϕ) be a symmetric $\mathbf{F}$-space. Suppose that we are given a sequence of totally isotropic subspaces $B \subset T^\perp \subset S \subset M$ of V, and let*

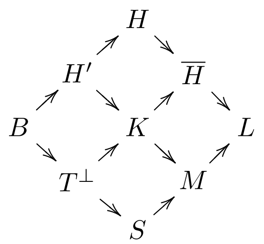
$$e = \dim_{\mathbf{F}} T^\perp/B, \quad t = \dim_{\mathbf{F}} S/T^\perp, \quad m = \dim_{\mathbf{F}} M/S \quad \text{and} \quad d = \dim_{\mathbf{F}} M^\perp/M.$$

The number of totally isotropic $\mathbf{F}$-subspaces H of V for which

$$H \cap S = B, \quad H \cap S^\perp + S = M \quad \text{and} \quad H + S^\perp = T$$

equals $q^{me+t(d+m+e+\frac{1}{2}(t-1))}$.

Proof. We consider the following diagram



$H' = H \cap S^\perp$

$\overline{H} = H + T^\perp$

$K = H' + T^\perp$

$L = H + S$

We choose in this order (1) the complement H'/B of S/B in M/B, which then gives $K = H' + T^\perp$, (2) the complement L/M of $M^\perp/M$ in $K^\perp/M$, which gives the non-degenerate subspace L/K of $K^\perp/K$, (3) the totally isotropic $\mathbf{F}$-subspace $\overline{H}/K$ of L/K such that $\overline{H}/K \oplus M/K = L/K$, and (4) the complement H/H' of K/H' in $\overline{H}/H'$. For (1), (2) and (4), there are respectively $q^{m(e+t)}$, q^{dt} and q^{et} possible choices. For (3), there are $q^{\frac{t(t-1)}{2}}$ choices by a simple case of [2, 9.4.2]. □

Lemma 5. *Let V be an $\mathbf{F}$-vector space of $\mathbf{F}$-dimension $d \geq 1 + a$ with an $\mathbf{F}$-line D and an $\mathbf{F}$-subspace $A \not\supset D$ of $\mathbf{F}$-dimension a. The number of $\mathbf{F}$-hyperplanes H of V such that $A \subset H$ and $D \oplus H = V$ equals q^{d-a-1}.*

Proof. Easy. □

Lemma 6. *Let (V, ϕ) be an odd-dimensional symmetric $\mathbf{F}$-space, D an anisotropic $\mathbf{F}$-line in V and S a totally isotropic $\mathbf{F}$-subspace of $W = D^\perp$. Put $a = \dim_{\mathbf{F}} S$ and $2d = \dim_{\mathbf{F}} W$. The number of $\mathbf{F}$-hyperplanes H of V such that*

$$H^\perp \subset H, \quad S \subset H \quad \text{and} \quad H \oplus D = V$$

equals $q^{d-1}(q^{d-a}-1)$ *if* V *is split and* $q^{d-1}(q^{d-a}+1)$ *otherwise.*

Proof. This is also the number of isotropic $\mathbf{F}$-lines $L = H^\perp$ in $S^\perp \oplus D$ such that $L \not\subset S^\perp$, where $S^\perp$ is the orthogonal complement of S in W. Let $\mathcal{L}$ be the image of L in $\mathcal{V} = \mathcal{W} \oplus D$ where $\mathcal{W} = S^\perp/S$. This is an isotropic $\mathbf{F}$-line of $\mathcal{V}$ which is not contained in $\mathcal{W}$. By [2, 9.4.1], the number of such lines equals $q^{d-a-1}(q^{d-a}-1)$ if V is split and $q^{d-a-1}(q^{d-a}+1)$ otherwise. For each $\mathcal{L} = L + S/S$, there are exactly q^a complements L of S in $L+S$. □

Lemma 7. *Let* (V, ϕ) *be a symmetric* $\mathbf{F}$*-space,* D *an isotropic* $\mathbf{F}$*-line in* V *and* S *a totally isotropic subspace of* V *such that* $D \not\subset S$ *but* $D \subset S^\perp$*. Write* $d = \dim_{\mathbf{F}} V$ *and* $a = \dim_{\mathbf{F}} S$*. The number of* $\mathbf{F}$*-hyperplanes* H *in* V *such that* $H^\perp \subset H$*,* $S \subset H$ *and* $H \oplus D = V$ *is equal to* q^{d-a-2}*.*

Proof. This is also the number of isotropic $\mathbf{F}$-lines $L = H^\perp$ in $S^\perp$ such that $L \not\subset D^\perp$. Let $\mathcal{L}$ and $\mathcal{D}$ be the images of L and D in $\mathcal{V} = S^\perp/S$. Then $\mathcal{D}$ is an isotropic $\mathbf{F}$-line in $\mathcal{V}$ and $\mathcal{L}$ is an isotropic $\mathbf{F}$-line in $\mathcal{V}$ such that $\mathcal{L} \not\subset \mathcal{D}^\perp$. By [2, 9.4.2] or lemma 4 above, the number of all such lines $\mathcal{L}$ equals q^{d-2a-2}. For each $\mathcal{L} = L + S/S$, there are q^a-choices for the complement L of S in $L+S$. □

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CNRS - Institut de Mathématiques de Jussieu. UMR 7586, Case 7012, Bâtiment Chevaleret, 75205 Paris CEDEX 13.

E-mail address: cornut@math.jussieu.fr