NOTE

Simultaneously Colouring the Edges and Faces of Plane Graphs

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In a simultaneous colouring of the edges and faces of a plane graph we colour edges and faces so that every two adjacent or incident pair of them receive different colours. In this paper we prove a conjecture of Mel’nikov which states that for this colouring every plane graph can be coloured with $\Delta + 3$ colours, where $\Delta$ is the maximum degree of a vertex in the graph.

1. INTRODUCTION

The elements of a plane graph $G$ are the edges, vertices, and faces of $G$. We say that two elements are neighbours in $G$ if they are incident with or are mutually adjacent with each other in $G$. The simultaneous colouring of distinct elements of a planar graph was first introduced by Ringel [12]. In his paper Ringel considered the problem of colouring the vertices and faces of a plane graph in such a way that every vertex and face receives a different colour from any of its neighbouring vertices and faces. Ringel’s conjecture that six colours always suffice for this colouring was proved by Borodin [2], this bound being best possible.

In this paper we consider the simultaneous colouring of edges and faces of a plane graph, where every edge and face receives a different colour from any of its neighbouring edges and faces. We define $\chi_{ef}(G)$ to be the least number of colours needed for such a colouring of the plane graph $G$. This problem was first studied by Jucovic [8] and Fiamcik [7] for 3- and 4-regular graphs. Mel’nikov [11] conjectured that $\chi_{ef}(G) \leq \Delta + 3$ for every plane graph $G$, where $\Delta$ is the maximum degree of a vertex in $G$. It is easy to see that this bound is attained by odd cycles. Borodin [3] proved this conjecture for the special case $\Delta \leq 3$ (this result was also proved independently by Lin et al. [9]). In [4, 5] Borodin gave further results on this
problem including that for a plane graph \( G \), \( \chi_{ef}(G) \leq A + 1 \) if \( A \geq 10 \) and \( \chi_{ef} \leq 11 \) if \( A \leq 10 \) (which implies the proof of the conjecture for \( A \geq 8 \)).

We give a short proof of the conjecture for all plane graphs using a technique due to McDiarmid and Sánchez-Arroyo [10] for obtaining total colourings of graphs, the problem of simultaneously colouring the edges and vertices of graphs. We adapt their technique to the colouring of edges and faces by using the concept of list colouring which was introduced independently by Vizing [15] and Erdős et al. [6].

It has been brought to the author’s attention that a proof of the conjecture is included in a paper which has been submitted by Sanders and Zhao [13].

2. PROOF OF THE CONJECTURE

We define \( \chi'(G) \) to be the chromatic index of \( G \), the smallest number of colours needed to properly colour the edges of \( G \). We use the following well known results.

**Theorem 2.1.** (Appel et al. [1]). The faces of every bridgeless plane graph can be properly coloured with 4 colours.

**Theorem 2.2.** (Vizing [14]). For any graph \( G \), \( \chi'(G) \leq A + 1 \).

A graph \( G \) is said to be \( k \)-choosable if however we assign lists of \( k \) colours to each vertex of \( G \) we can choose a proper vertex-colouring of \( G \) where each vertex receives a colour from its list. We say a graph \( G \) is \( k \)-edge-choosable if its line graph is \( k \)-choosable. Erdős, Rubin and Taylor [6] proved that paths and even cycles are 2-choosable. The line graph of a collection of disjoint paths and even cycles is again a collection of disjoint paths and even cycles and hence such a graph is 2-edge-choosable.

We are now ready to prove the main result.

**Theorem 2.3.** If \( G \) is a bridgeless plane graph then \( \chi_{ef}(G) \leq \chi'(G) + 2 \).

**Proof.** Let \( G \) be a bridgeless plane graph. Properly colour the faces with colours from the set \( \{1, 2, 3, 4\} \) and properly colour the edges with colours from the set \( \{3, 4, \ldots, \chi'(G) + 2\} \), which we can do by Theorem 2.1 and Theorem 2.2. We may now assume that some of the edges coloured 3 or 4 are incident with faces of the same colour.

Let \( S \) be the set of edges coloured 3 or 4. Note that the subgraph induced by \( S \) consists of the disjoint union of paths and even cycles as the edges of \( S \) are coloured with two colours in a proper edge-colouring.
of \( G \). Uncolour all edges in the set \( S \) and assign the list of colours \( L(e) = \{1, 2, 3, 4\} \) to each edge \( e \in S \). For each edge \( e \in S \), remove from the list \( L(e) \) the colours of the (at most) two faces incident with \( e \).

The problem of obtaining the required colouring becomes one of choosing an edge-colouring of the subgraph induced by \( S \) where each edge receives a colour from its list, which we can do by the fact that \(|L(e)| \geq 2 \forall e \in S\) and that the subgraph induced by \( S \) consists of disjoint paths and even cycles.

**Corollary 2.4.** For any bridgeless plane graph \( G \), \( \chi_{ef}(G) \leq \delta + 3 \).

**Proof.** Follows from Theorem 2.2.

**REFERENCES**