BOUNDS ON THE SIGNED 2-INDEPENDENCE NUMBER IN GRAPHS

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Abstract

Let \( G \) be a finite and simple graph with vertex set \( V(G) \), and let \( f : V(G) \to \{-1,1\} \) be a two-valued function. If \( \sum_{x \in N[v]} f(x) \leq 1 \) for each \( v \in V(G) \), where \( N[v] \) is the closed neighborhood of \( v \), then \( f \) is a signed 2-independence function on \( G \). The weight of a signed 2-independence function \( f \) is \( w(f) = \sum_{v \in V(G)} f(v) \). The maximum of weights \( w(f) \), taken over all signed 2-independence functions \( f \) on \( G \), is the signed 2-independence number \( \alpha_2^s(G) \) of \( G \).

In this work, we mainly present upper bounds on \( \alpha_2^s(G) \), as for example \( \alpha_2^s(G) \leq n - 2\lceil \Delta(G)/2 \rceil \), and we prove the Nordhaus-Gaddum type inequality \( \alpha_2^s(G) + \alpha_2^s(G) \leq n + 1 \), where \( n \) is the order and \( \Delta(G) \) is the maximum degree of the graph \( G \). Some of our theorems improve well-known results on the signed 2-independence number.

Keywords: bounds, signed 2-independence function, signed 2-independence number, Nordhaus-Gaddum type result.

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1. Introduction

Domination and independence in graphs are well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [2, 3].

All graphs considered are undirected, simple and finite. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \). The order \( n = n(G) \) and size \( q = q(G) \) of a graph \( G \) is the number of vertices and edges, respectively. The open neighborhood of \( v \in V(G) \) is \( N_G(v) = \{ u \in V(G) | uv \in E(G) \} \) and the closed neighborhood of \( v \) is \( N_G[v] = N_G(v) \cup \{ v \} \). The degree of \( v \) in \( G \),
denoted by $d_G(v)$, is the cardinality of $N_G(v)$. We write $\Delta(G)$ and $\delta(G)$ for the maximum and minimum degree of $G$. If the graph $G$ is clear from context, we simply use $N(v), N[v], d(v), \Delta$ and $\delta$ instead of $N_G(v), N_G[v], d_G(v), \Delta(G)$ and $\delta(G)$, respectively. For two disjoint subsets $A$ and $B$ of $V(G)$, let $e(A, B)$ denote the number of edges between $A$ and $B$. A graph $G$ is $r$-partite with vertex classes $V_1, V_2, \ldots, V_r$ if $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$, $V_i \cap V_j = \emptyset$ whenever $1 \leq i < j \leq r$, and no edge joins two vertices in the same class. The subgraph of $G$ induced by $A$ is denoted by $G[A]$. The complete graph of order $n$ is denoted by $K_n$. A graph is $K_{r+1}$-free if it does not contain the complete graph $K_{r+1}$ as a subgraph. The complement of a graph $G$ is denoted by $\overline{G}$.

For a two-valued function $f : V(G) \to \{-1, 1\}$, the weight of $f$ is $w(f) = \sum_{v \in V(G)} f(v)$. For a subset $A \subseteq V(G)$, we define $f(A) = \sum_{v \in A} f(v)$ and so $w(f) = f(V(G))$. For a vertex $v$ in $V(G)$, we denote $f(N[v])$ by $f[v]$ for notational convenience. The function $f$ is defined in [1] to be a signed dominating function of $G$ if $f[v] = f(N[v]) \geq 1$ for every $v \in V(G)$. The signed domination number of $G$ is the minimum weight of a signed dominating function on $G$.

The function $f : V(G) \to \{-1, 1\}$ is defined in [7] to be a signed 2-independence function on $G$ if $f[v] = f(N[v]) \leq 1$ for every $v \in V(G)$. The signed 2-independence number $\alpha_2^s(G)$ of $G$ is the maximum weight of a signed 2-independence function on $G$. Hence the signed 2-independence number is a certain dual to the signed domination number of a graph. Results on the signed 2-independence number can be found in [4, 5, 7].

In this paper we continue the investigations of the signed 2-independence number. We mainly present upper bounds on $\alpha_2^s(G)$ for general graphs and $K_{r+1}$-free graphs. In addition, we prove the Nordhaus-Gaddum type inequality $\alpha_2^s(G) + \alpha_2^s(\overline{G}) \leq n + 1$. Some of our results improve known bounds on the signed 2-independence numbers of graphs given by Henning [4] in 2002 and Shan, Sohn and Kang [5] in 2003.

Zelinka [7] determined the signed 2-independence number of complete graphs, and he established a sharp upper bound on $\alpha_2^s(G)$ for regular graphs $G$.

**Theorem 1** [7]. If $G$ is isomorphic to the complete graph $K_n$, then $\alpha_2^s(G) = 0$ when $n$ is even and $\alpha_2^s(G) = 1$ when $n$ is odd.

**Theorem 2** [7]. If $G$ is an $r$-regular graph of order $n$, then $\alpha_2^s(G) \leq n/(r + 1)$ when $r$ is even and $\alpha_2^s(G) \leq 0$ when $r$ is odd.

2. **Main Results**

**Theorem 3.** If $G$ is a graph of order $n$, then

$$2 - n \leq \alpha_2^s(G) \leq n - 2 \left\lfloor \frac{\Delta}{2} \right\rfloor.$$
Proof. Let \( w \in V(G) \) be a vertex of maximum degree \( d(w) = \Delta \), and let \( f \) be a signed 2-independence function on \( G \) for which \( f(V(G)) = \alpha_s^2(G) \). We define the two sets \( P = \{ v \in V(G) \mid f(v) = 1 \} \) and \( M = \{ v \in V(G) \mid f(v) = -1 \} \). If \( |P| = p \) and \( |M| = m \), then \( n = p + m \) and \( \alpha_s^2(G) = p - m = n - 2m \).

Assume first that \( f(w) = 1 \) and therefore \( w \in P \). The condition \( f[w] \leq 1 \) leads to the inequality \( |N(w) \cap P| - |N(w) \cap M| \leq 0 \), and since \( w \) is a vertex of maximum degree, we have \( |N(w) \cap P| + |N(w) \cap M| = \Delta \). Combining the last two inequalities, we deduce that \( m \geq |N(w) \cap M| \geq \lceil \Delta/2 \rceil \), and this yields to

\[
\alpha_s^2(G) = n - 2m \leq n - 2 \left\lceil \frac{\Delta}{2} \right\rceil .
\]

Assume second that \( f(w) = -1 \) and so \( w \in M \). As \( f[w] \leq 1 \) and \( d(w) = \Delta \), we obtain \( |N(w) \cap P| - |N(w) \cap M| \leq 2 \) and \( |N(w) \cap P| + |N(w) \cap M| = \Delta \). Combining these two inequalities, we conclude that

\[
m \geq |N(w) \cap M| + 1 = \frac{2|N(w) \cap M| + 2}{2} \geq \frac{\Delta}{2}
\]

and thus \( m \geq \lceil \Delta/2 \rceil \). This implies \( \alpha_s^2(G) = n - 2m \leq n - 2 \lceil \Delta/2 \rceil \) as above, and the upper bound on \( \alpha_s^2(G) \) is proved.

For the first inequality define \( f : V(G) \to \{-1, 1\} \) by \( f(v) = 1 \) for an arbitrary vertex \( v \in V(G) \) and \( f(x) = -1 \) for each vertex \( x \in V(G) \setminus \{v\} \). Obviously, \( f \) is a signed 2-independence function on \( G \) of weight \( 2 - n \) and thus \( \alpha_s^2(G) \geq 2 - n \).

If \( G \) is isomorphic to the star \( K_{1,\Delta} \), then

\[
\alpha_s^2(G) = n - 2 \left\lceil \frac{\Delta}{2} \right\rceil ,
\]

and therefore the upper bound on \( \alpha_s^2(G) \) in Theorem 3 is sharp.

**Corollary 4.** If \( G \) is a graph of order \( n \), then \( \alpha_s^2(G) = n \) if and only if \( G \) is the empty graph.

**Proof.** If \( G \) is the empty graph, then \( f : V(G) \to \{-1, 1\} \) with \( f(v) = 1 \) for each vertex \( v \in V(G) \) is a signed 2-independence function on \( G \) of weight \( n \) and thus \( \alpha_s^2(G) = n \).

Conversely, assume that \( \alpha_s^2(G) = n \). If we suppose that \( G \) is not the empty graph, then \( \Delta \geq 1 \), and Theorem 3 leads to the contradiction \( n = \alpha_s^2(G) \leq n - 2 \). Therefore \( G \) is the empty graph, and the proof is complete.

Obviously, \( \alpha_s^2(K_2) = 0 = n - 2 \), and therefore equality in the left inequality of Theorem 3 is achieved. However, if \( G \) is a graph of order \( n \geq 3 \), then the next result improves the lower bound in Theorem 3.
Theorem 5. If $G$ is a graph of order $n \geq 3$, then $\alpha^2_s(G) \geq 4 - n$.

Proof. If $G$ has two non-adjacent vertices $u$ and $v$, then $f : V(G) \to \{-1,1\}$ with $f(u) = f(v) = 1$ and $f(x) = -1$ for each $x \in V(G) - \{u,v\}$ is a signed 2-independence function on $G$ of weight $4 - n$ and thus $\alpha^2_s(G) \geq 4 - n$. Otherwise, $G$ is the complete graph. If $n \geq 4$, then it follows from Theorem 1 that $\alpha^2_s(G) \geq 0 \geq 4 - n$, and if $n = 3$, then Theorem 1 implies that $\alpha^2_s(G) = 1 = 4 - n$. ■

As an application of Theorems 1, 2, 3 and Corollary 4, we will prove the following Nordhaus-Gaddum type result.

Theorem 6. If $G$ is a graph of order $n$, then

$$\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq n + 1$$

with equality if and only if $n$ is odd and $G = K_n$ or $\overline{G} = K_n$.

Proof. Theorem 3 implies that

$$\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq n - \Delta(G) + n - \Delta(\overline{G})$$

(1) $$= n - \Delta(G) + n - (n - \delta(G) - 1)$$ $$= n + 1 - \Delta(G) + \delta(G),$$

and the desired bound follows, since $\delta(G) - \Delta(G) \leq 0$. If $n$ is odd and $G = K_n$ or $\overline{G} = K_n$, then we deduce from Theorem 1 and Corollary 4 that $\alpha^2_s(G) + \alpha^2_s(\overline{G}) = n + 1$.

If $\Delta(G) - \delta(G) \geq 1$, then the inequality chain (1) leads to $\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq n$.

Assume first that $n$ is even. If $\delta$ is even, then $\Delta(G) = n - \delta - 1$ is odd, and Theorem 2 implies that $\alpha^2_s(G) \leq 0$ and thus $\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq n$. If $\delta$ is odd, then Theorem 2 implies that $\alpha^2_s(G) \leq 0$ and thus $\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq n$.

Finally assume that $n$ is odd. The handshaking lemma implies that $\delta$ and $\Delta(G)$ are even. If $\delta = 0$ or $\Delta(G) = 0$, then $\overline{G} = K_n$ or $G = K_n$ and thus $\alpha^2_s(G) + \alpha^2_s(\overline{G}) = n + 1$. In the remaining case that $\delta \geq 2$ and $\Delta(G) \geq 2$, Theorem 2 shows that

$$\alpha^2_s(G) + \alpha^2_s(\overline{G}) \leq \frac{n}{\delta + 1} + \frac{n}{\Delta(G) + 1} \leq \frac{2n}{3} < n,$$

and the proof of Theorem 6 is complete. ■

The following upper bound on $\alpha^2_s(G)$ was obtained by Henning [4] in 2002.

Theorem 7 [4]. If $G$ is a connected graph of order $n \geq 2$ and size $q$, then

$$\alpha^2_s(G) \leq \frac{4q - n}{5}.$$
We now improve the bound in Theorem 7.

**Theorem 8.** If $G$ is a connected graph of order $n \geq 2$ and size $q$, then

$$\alpha_s^2(G) \leq \frac{4q + (2 - \delta - 2\lceil \delta/2 \rceil)n}{2 + \delta + 2\lceil \delta/2 \rceil}.$$  

**Proof.** Let $f$ be a signed 2-independence function on $G$ for which $f(V(G)) = \alpha_s^2(G)$, and let $P, M, p$ and $m$ be defined as in the proof of Theorem 3. Then $n = p + m$ and $\alpha_s^2(G) = p - m = 2p - n$. The condition $f[v] \leq 1$ implies that $|N(v) \cap P| \leq |N(v) \cap M|$ for $v \in P$ and $|N(v) \cap P| \leq |N(v) \cap M| + 2$ for $v \in M$. Thus we obtain

$$\delta \leq d(v) = |N(v) \cap P| + |N(v) \cap M| \leq 2|N(v) \cap M|$$

and so $|N(v) \cap M| \geq \lceil \frac{\delta}{2} \rceil$ for each $v \in P$. Hence we deduce that

$$e(P, M) = \sum_{v \in P} |N(v) \cap M| \geq p \left\lceil \frac{\delta}{2} \right\rceil = (n - m) \left\lceil \frac{\delta}{2} \right\rceil. \tag{2}$$

In addition, we have

$$e(P, M) = \sum_{v \in M} |N(v) \cap P| \leq \sum_{v \in M} (|N(v) \cap M| + 2) = 2|E(G[M])| + 2m. \tag{3}$$

Combining (2) and (3), we find that

$$2|E(G[M])| \geq p \left\lceil \frac{\delta}{2} \right\rceil - 2m. \tag{4}$$

Furthermore, we deduce from (2) that

$$e(P, M) + |E(G[P])| = \sum_{v \in P} |N(v) \cap M| + \frac{1}{2} \sum_{v \in P} |N(v) \cap P|$$

$$= \frac{1}{2} \sum_{v \in P} |N(v) \cap M| + \frac{1}{2} \sum_{v \in P} |N(v) \cap M| + \frac{1}{2} \sum_{v \in P} |N(v) \cap P|$$

$$\geq \frac{1}{2} p \delta + \frac{1}{2} p \left\lceil \frac{\delta}{2} \right\rceil. \tag{5}$$

According to (4) and (5), we have

$$2q = 2e(P, M) + 2|E(G[P])| + 2|E(G[M])| \geq p \delta + 2p \left\lceil \frac{\delta}{2} \right\rceil - 2m = p \left( 2 + \delta + 2 \left\lceil \frac{\delta}{2} \right\rceil \right) - 2n.$$
Hence
\[ p \leq \frac{2q + 2n}{2 + \delta + 2[\delta/2]} \]
and so we obtain the desired bound as follows
\[ \alpha_s^2(G) = 2p - n \leq \frac{4q + (2 - \delta - 2[\delta/2])n}{2 + \delta + 2[\delta/2]}. \]

Note that
\[ \frac{4q + (2 - \delta - 2[\delta/2])n}{2 + \delta + 2[\delta/2]} \leq \frac{4q - n}{5} \]
for \( \delta \geq 1 \), and therefore Theorem 8 is an improvement of Theorem 7.

For the next result, we use the famous theorem of Turán [6].

**Theorem 9** [6]. Let \( r \geq 1 \) be an integer. If \( G \) is a \( K_{r+1} \)-free graph of order \( n \), then
\[ |E(G)| \leq \frac{r - 1}{2r} n^2. \]

**Theorem 10.** If \( G \) is a \( K_{r+1} \)-free graph of order \( n \) with \( r \geq 2 \) and minimum degree \( \delta \geq 1 \), then
\[ \alpha_s^2(G) \leq n + \frac{r(2 + [\delta/2])}{r - 1} \sqrt{\left( \frac{r(2 + [\delta/2])}{r - 1} \right)^2 + \frac{4rn[\delta/2]}{r - 1}}. \]

**Proof.** Let \( f, P, M, p \) and \( m \) be defined as in the proof of Theorem 3. By (2) and (3), we obtain
\[ (n - m) \left[ \frac{\delta}{2} \right] \leq e(P, M) \leq 2|E(G[M])| + 2m. \]

Since \( G \) is \( K_{r+1} \)-free, the induced subgraph \( G[M] \) is also \( K_{r+1} \)-free, and hence it follows from Theorem 9 that \( |E(G[M])| \leq (r - 1)m^2/2r \). Using (6), we obtain
\[ (n - m) \left[ \frac{\delta}{2} \right] \leq e(P, M) \leq \frac{r - 1}{r} m^2 + 2m \]
and so
\[ m^2 + \frac{r}{r - 1}(2 + [\delta/2])m - \frac{r}{r - 1}n[\delta/2] \geq 0. \]

This yields
\[ m \geq -\frac{r}{2(r - 1)(2 + [\delta/2])} + \sqrt{\left( \frac{r}{2(r - 1)(2 + [\delta/2])} \right)^2 + \frac{r}{r - 1}n[\delta/2]}, \]
and we obtain the desired bound as follows

$$\alpha_2^s(G) = n - 2m \leq n + \frac{r(2 + \lceil \delta/2 \rceil)}{r - 1} - \sqrt{\left(\frac{r(2 + \lceil \delta/2 \rceil)}{r - 1}\right)^2 + \frac{4rn\lceil \delta/2 \rceil}{r - 1}}.$$ 

Since

$$\frac{r(2 + \lceil \delta/2 \rceil)}{r - 1} - \sqrt{\left(\frac{r(2 + \lceil \delta/2 \rceil)}{r - 1}\right)^2 + \frac{4rn\lceil \delta/2 \rceil}{r - 1}} \leq \frac{3r}{r - 1} - \sqrt{\left(\frac{3r}{r - 1}\right)^2 + \frac{4rn}{r - 1}}$$

for $\delta \geq 1$, the next known result is an immediate consequence of Theorem 10.

**Corollary 11** [5]. If $G$ is an $r$-partite graph of order $n$ with $r \geq 2$, then

$$\alpha_2^s(G) \leq n + \frac{3r}{r - 1} - \sqrt{\left(\frac{3r}{r - 1}\right)^2 + \frac{4rn}{r - 1}}.$$

**References**


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