General solution of certain matrix equations arising in filter design applications

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A B S T R A C T

In this work we present the explicit expression of all rectangular Toeplitz matrices $\mathbf{B}, \mathbf{C}$ which verify the equation $\mathbf{B}\mathbf{B}^H + \mathbf{C}\mathbf{C}^H = a\mathbf{I}$ for some $a > 0$. This matrix equation arises in some signal processing problems. For instance, it appears when designing the even and odd components of paraunitary filters, which are widely used for signal compression and denoising purposes. We also point out the relationship between the above matrix equation and the polynomial Bézout equation $|B(z)|^2 + |C(z)|^2 = a > 0$ for $|z| = 1$. By exploiting this fact, our results also yield a constructive method for the parameterization of all solutions $B(z), C(z)$. The main advantage of our approach is that B and C are built without need of spectral factorization. Besides these theoretical advances, in order to illustrate the effectiveness of our approach, some examples of paraunitary filters design are finally given.

Keywords:
Toeplitz matrices
Spectral factorization
Paraunitary filters
Filter design

1. Introduction

In this paper we will solve the following matrix equation problem:

Problem 1. Given $L \in \mathbb{N}$ and $a > 0$, find all rectangular complex Toeplitz matrices $\mathbf{B}, \mathbf{C} \in \mathbb{C}^{L\times(2L-1)}$ of the type

$$\mathbf{B} = \begin{pmatrix} b_0 & b_1 & \cdots & b_{L-1} & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{L-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_{L-1} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{L-1} & 0 & \cdots & 0 \\ 0 & c_0 & c_1 & \cdots & c_{L-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_0 & c_1 & \cdots & c_{L-1} \end{pmatrix}$$

such that

$$\mathbf{B}\mathbf{B}^H + \mathbf{C}\mathbf{C}^H = a\mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the identity matrix of order L.

The parameter L is the maximum number of nonzero row components of $\mathbf{B}$ or $\mathbf{C}$. Note that Problem 1 can be also generalized for Toeplitz matrices $\mathbf{B}, \mathbf{C}$ with the same structure but with a greater number of rows $p \geqslant L$ (in that case, $\mathbf{B}, \mathbf{C} \in \mathbb{C}^{p \times p+L-1}$ and $\mathbf{I}$ is the identity matrix of order p), but then we would obtain solutions with the same rows $(b_0, \ldots, b_{L-1})$ and $(c_0, \ldots, c_{L-1})$ as in the case $p = L$ of Problem 1. The reason is that both $\mathbf{B}\mathbf{B}^H$ and $\mathbf{C}\mathbf{C}^H$ are Toeplitz matrices of size p, and their entries are the scalar products of the rows of $\mathbf{B}$ and $\mathbf{C}$), respectively. Due to their structure, we only need to consider its L central diagonals, since the rest are null.

In other words, we can just consider Problem 1, because its solutions $\mathbf{B}, \mathbf{C}$ have the same rows than the corresponding solutions of the generalized problem defined for matrices with more than L rows. Therefore, the results here obtained can be eventually applied to a great variety of cases.

Easily, Problem 1 can be reformulated as

Problem 2. Given $L \in \mathbb{N}$ and $a > 0$, find all complex vectors $\mathbf{b} = (b_0, \ldots, b_{L-1})$ and $\mathbf{c} = (c_0, \ldots, c_{L-1})$ such that

$$\sum_{j=0}^{L-1-k} b_j \overline{b_{j+k}} + \sum_{j=0}^{L-1-k} c_j \overline{c_{j+k}} = \begin{cases} a & \text{if } k = 0, \\ 0 & \text{if } k = 1, \ldots, L-1, \end{cases} \tag{2}$$

where the superscript $^-$ denotes complex conjugation.

Thus, Problem 2 is equivalent to the matrix Problem 1, and it is stated in terms of the unknown vectors $\mathbf{b}$ and $\mathbf{c}$. Notice that the left hand side of Eq. (2) involves the sum of the autocorrelation of vector $\mathbf{b}$ and the autocorrelation of vector $\mathbf{c}$.

To solve any of these problems, our key idea is to relate them with an apparently very different approach: the theory of complex paraunitary filterbanks. Filterbanks are widely used in all signal processing areas; in particular, paraunitary filters are required for signal compression applications, since they yield unitary transforms. For this reason,

However, most results focus on real paraunitary filters, but not in complex ones. Complex filters also play an important role in Signal Processing (moreover, in Image Processing, since their real and imaginary parts yield bidimensional filters for digital images). Hence, the complex case is interesting not only from the theoretical point of view, but also for its applications.

In this work we will provide a new procedure for the design of paraunitary complex filters. Additionally, it will constitute the desired general solution of the initial matrix Problems 1 and 2.

The paper is organized as follows: in Section 2, the problem will be restated as a polynomial equation, denoted as Problem 3. In Section 3, we will introduce orthogonal paraunitary filters and their most important properties, which are necessary to follow the development of our work. Moreover, we will prove that the filter approach is equivalent to any one of such problems. In Sections 4 and 5 we will provide our new results on the general explicit expressions of both complex orthogonal and paraunitary filters, respectively. It will also be shown how to obtain the solutions of Problem 1 or 2 or 3 by means of the already designed complex paraunitary filters. By exploiting this equivalence, the original problem is finally solved in Section 6. Simple illustrative examples are also given in Section 7, before the conclusions of Section 8.

2. Formulation as a polynomial equation

Let us give a third equivalent statement of Problems 1 and 2:

Problem 3. Given $L \in \mathbb{N}$ and $a > 0$, find all complex polynomials B, C of degree at most $L - 1$:

$$B(z) = \sum_{k=0}^{L-1} b_k z^k, \quad C(z) = \sum_{k=0}^{L-1} c_k z^k,$$

which verify the polynomial equation on the unit torus:

$$|B(z)|^2 + |C(z)|^2 = a \quad \forall |z| = 1. \tag{3}$$

Here, $B(z)$ and $C(z)$ denote the polynomials whose coefficients are, respectively, the solutions $\mathbf{b} = (b_0, \dots, b_{L-1})$ and $\mathbf{c} = (c_0, \dots, c_{L-1})$ of Problem 2. In fact, the equivalence between Eqs. (3) and (2) follows from the fact that, for $|z| = 1$ we have that $\bar{z} = z^{-1}$, and

$$\begin{aligned} |B(z)|^2 &= \left(\sum_{j=0}^{L-1} b_j z^j\right)\left(\sum_{n=0}^{L-1} \overline{b_n} z^{-n}\right) = \sum_{n=0}^{L-1}\sum_{j=0}^{L-1} b_j \overline{b_n} z^{j-n} \\ &= \sum_{k=0}^{L-1}\left(\sum_{j=0}^{L-1-k} b_j \overline{b_{j+k}}\right) z^{-k} + \sum_{k=1}^{L-1}\left(\sum_{j=0}^{L-1-k} b_{n+j} \overline{b_j}\right) z^k. \end{aligned}$$

Hence $|B(z)|^2 + |C(z)|^2$ turns out to be

$$\sum_{k=0}^{L-1}\left(\sum_{j=0}^{L-1-k} b_j \overline{b_{j+k}} + c_j \overline{c_{j+k}}\right) z^{-k} + \sum_{k=1}^{L-1}\left(\sum_{j=0}^{L-1-k} b_{n+j} \overline{b_j} + c_{n+j} \overline{c_j}\right) z^k. \tag{4}$$

Note that this polynomial can be written as $\sum_{k=1}^{L-1} p_k z^{-k} + \sum_{k=1}^{L-1} \overline{p_k} z^k$ up to an additive constant. Hence, it is a constant polynomial if all its coefficients p_k are zero. In other words, the polynomial of Eq. (4) is identically equal to a if and only if (2) is fulfilled.

Eq. (3) can be considered a Bézout polynomial equation since it can be rewritten in the following way:

Proposition 1. *If $B(z) = \sum_{k=0}^{L-1} b_k z^k$, $C(z) = \sum_{k=0}^{L-1} c_k z^k$ satisfy Eq. (3), then the Bézout identity*

$$B(z)\,\widetilde{B}(z) + C(z)\,\widetilde{C}(z) = a z^{L-1} \quad \forall |z| = 1$$

is reached by the reciprocal polynomials of B, C, defined as:

$$\begin{aligned} \widetilde{B}(z) &= z^{L-1}\overline{B}(z^{-1}) = \sum_{k=0}^{L-1} \overline{b_{L-1-k}} z^k, \\ \widetilde{C}(z) &= z^{L-1}\overline{C}(z^{-1}) = \sum_{k=0}^{L-1} \overline{c_{L-1-k}} z^k. \end{aligned}$$

Proof. In effect, if B, C are the solutions of Eq. (3) then

$$B(z)\,\overline{B(z)} + C(z)\,\overline{C(z)} = a$$

and we can write, for $|z| = 1$,

$$\overline{B(z)} = \overline{B}(z^{-1}) = \sum_{n=0}^{L-1} \overline{b_n} z^{-n} = \sum_{k=0}^{L-1} \overline{b_{L-1-k}} z^{1-L+k} = z^{1-L}\widetilde{B}(z).$$

Operating analogously with $\overline{C(z)}$ and multiplicating by z^{L-1}, we get the result.

In Section 6 we will give the general explicit expression for the polynomial solutions of Problem 3; to this end, we will make use of the filter approach introduced in the next Section.

3. Paraunitary filter approach

Let us start with the definition of orthogonal and paraunitary complex filters:

Definition 1. An **orthogonal complex filter** of length M is a complex vector $\mathbf{h} = (h_1, h_2, \ldots, h_M)$ with $h_1 h_M \neq 0$, which is orthogonal to its even shifts:

$$\forall k = 1, \ldots, \lfloor M/2 \rfloor - 1 \quad \sum_{n=1}^{M-2k} h_n \overline{h_{n+2k}} = 0. \tag{5}$$

From the definition, the length M of the orthogonal filter must be even: if M is odd, then for $k = (M-1)/2$ the inner product of $\mathbf{h}$ and its $2k$th shift would be $h_1 \overline{h_M} = 0$, so either h_1 or h_M would be zero, and it contradicts the definition. Hence, the length M is an even number: from now on, let $M = 2L$.

Definition 2. A **paraunitary complex filter** of length $2L$ is an orthogonal complex filter $\mathbf{h} = (h_1, h_2, \ldots, h_{2L})$ with unit Euclidean norm:

$$\forall k = 0, \ldots, L-1 \quad \sum_{n=1}^{2L-2k} h_n \overline{h_{n+2k}} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0. \end{cases}$$

Let us mention some equivalences for paraunitary filters that will be used in this work:

Lemma 1. *The following statements are equivalent:*

- *The vector* $\mathbf{h} = (h_1, h_2, \ldots, h_{2L})$ *of length* $2L$ *is a paraunitary complex filter.*
- *The polynomial* $H(z) = \sum_{n=1}^{2L} h_n z^{n-1}$ *satisfies, for every* $|z| = 1$,

  $$|H(z)|^2 + |H(-z)|^2 = 2.$$
- *Polynomials* H_{even} *and* H_{odd}, *whose coefficients contain the even and odd components of* $\mathbf{h}$:

  $$H_{even}(z) = \sum_{n=1}^{L} h_{2n} z^{n-1} \quad H_{odd}(z) = \sum_{n=1}^{L} h_{2n-1} z^{n-1}$$

  verify the polynomial equation on the unit torus

  $$|H_{even}(z)|^2 + |H_{odd}(z)|^2 = 1 \quad \forall |z| = 1. \tag{6}$$

Proofs of these equivalences are easily derived in an analogous way to the proof given in Section 2. Finally, our idea is to compare the solutions of Problem 3 (Eq. (3)) and the solution of Eq. (6):

Corollary 1. *Polynomials* $B(z), C(z)$ *of degree less than* L *are solutions of the Eq.* (3) *if and only if*

$$B(z) = \sqrt{a} H_{even}(z), \quad C(z) = \sqrt{a} H_{odd}(z),$$

where H_{even}, H_{odd} *are the polynomials whose coefficients are the even and odd components of any complex paraunitary filter* $\mathbf{h}$ *of length* $2L$.

This is one of the main results of this work: it suffices to build all such complex paraunitary filters in an explicit way. This will be done in the next Section.

4. General explicit expression for complex orthogonal filters

In this Section we will first obtain a general explicit parameterization of all complex orthogonal filters, similar to the one obtained in for real filters. We begin by writing the orthogonality condition of Eq. (5) of a filter of length $M = 2L$, for any $k = 1, \dots, L-1$, as:

$$\sum_{n\text{odd}} h_n \overline{h_{n+2k}} = - \sum_{n\text{even}} h_n \overline{h_{n+2k}}.$$

For instance, if $k = L-1$ we have that $h_1 \overline{h_{2L-1}} = -h_2 \overline{h_{2L}}$; since $h_1 \neq 0$, $h_{2L} \neq 0$ (otherwise, the length of the filter is less than $2L$), there is be a complex number a_1 such that

$$a_1 = -\frac{\overline{h_{2L-1}}}{\overline{h_{2L}}} = \frac{h_2}{h_1}.$$

In other words, h_2, h_{2L-1} can be derived from h_{2L}, h_1:

$$h_2 = a_1 h_1 \quad h_{2L-1} = -\overline{a_1} h_{2L}.$$

Now the key question arises: can we always write the even components of the filter by means of the odd ones, and viceversa? Our first result proves that the answer is **yes**; moreover, it provides our first characterization for orthogonal complex filters. To this aim, we will use the following notation: for any set of complex numbers $(a_1, \dots, a_m)$, let us denote by $T(a_1, \dots, a_m)$ the Toeplitz lower triangular matrix of order m which contains these numbers in its first column, that is:

$$T(a_1, \dots, a_m) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \ddots & \vdots \\ a_3 & a_2 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_m & \cdots & a_3 & a_2 & a_1 \end{pmatrix}.$$

In particular, we will denote $\mathbf{A} = T(a_1, \dots, a_{L-1})$ for such Toeplitz lower triangular matrix of order $L-1$.

Now we are in conditions to provide our first main result:

Theorem 1. *The complex vector* $\mathbf{h} = (h_1, h_2, \dots, h_{2L})$ *is an orthogonal complex filter of length* $2L$ *if and only if there exist* $L-1$ *complex numbers* $a_1, \dots, a_{L-1}$ *such that, for any* $k = 1, \dots, L-1$:

$$h_{2k} = \sum_{j=1}^{k} h_{2k+1-2j} a_j,$$

$$h_{2L+1-2k} = -\sum_{j=1}^{k} h_{2L-2k+2j} \overline{a_j}.$$

Or, in an equivalent matricial way:

$$\begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{2L-2} \end{pmatrix} = \mathbf{A} \begin{pmatrix} h_1 \\ h_3 \\ \vdots \\ h_{2L-3} \end{pmatrix}, \quad \begin{pmatrix} h_3 \\ \vdots \\ h_{2L-3} \\ h_{2L-1} \end{pmatrix} = -\mathbf{A}^H \begin{pmatrix} h_4 \\ \vdots \\ h_{2L-2} \\ h_{2L} \end{pmatrix}. \tag{7}$$

Proof. Eq. (5) may be easily rewritten matricially as

$$-T\left(h_1,h_3,\dots,h_{2L-3}\right)\begin{pmatrix}\overline{h_{2L-1}}\\ \overline{h_{2L-3}}\\ \vdots\\ \overline{h_3}\end{pmatrix}=\overline{T\left(h_{2L},h_{2L-2},\dots,h_4\right)}\begin{pmatrix}h_2\\ h_4\\ \vdots\\ h_{2L-2}\end{pmatrix},$$

where we have used our notation for lower triangular Toeplitz matrices. As $h_1\cdot h_{2L}\neq 0$, both matrices are nonsingular; besides, their inverses are also lower triangular Toeplitz matrices; finally, such matrices always commute, so we can state that

$$-\left(\overline{T\left(h_{2L},h_{2L-2},\dots,h_4\right)}\right)^{-1}\begin{pmatrix}\overline{h_{2L-1}}\\ \overline{h_{2L-3}}\\ \vdots\\ \overline{h_3}\end{pmatrix}$$

$$=\left(T\left(h_1,h_3,\dots,h_{2L-3}\right)\right)^{-1}\begin{pmatrix}h_2\\ h_4\\ \vdots\\ h_{2L-2}\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_{L-1}\end{pmatrix},$$

where $(a_1,\dots,a_{L-1})$ denotes any of these identical products. Note that the first coefficient a_1 verifies $a_1=-\overline{h_{2L-1}}/\overline{h_{2L}}=h_2/h_1$. Let us show that the whole vector verify Eq. (7): we now have

$$-T\left(h_{2L},h_{2L-2},\dots,h_4\right)\begin{pmatrix}\overline{a_1}\\ \vdots\\ \overline{a_{L-1}}\end{pmatrix}=\begin{pmatrix}h_{2L-1}\\ h_{2L-3}\\ \vdots\\ h_3\end{pmatrix}, \tag{8}$$

$$T\left(h_1,h_3,\dots,h_{2L-3}\right)\begin{pmatrix}a_1\\ \vdots\\ a_{L-1}\end{pmatrix}=\begin{pmatrix}h_2\\ h_4\\ \vdots\\ h_{2L-2}\end{pmatrix}. \tag{9}$$

So we have shown that it is possible to express each odd coefficient of the filter by means of its following even coefficients, and each even coefficient by means of its former odd coefficients. Moreover, Eqs. (8), (9) can be rewritten as

$$-\overline{T\left(a_1,\dots,a_{L-1}\right)}\begin{pmatrix}h_{2L}\\ h_{2L-2}\\ \vdots\\ h_4\end{pmatrix}=\begin{pmatrix}h_{2L-1}\\ h_{2L-3}\\ \vdots\\ h_3\end{pmatrix},$$

$$T\left(a_1,\dots,a_{L-1}\right)\begin{pmatrix}h_1\\ h_3\\ \vdots\\ h_{2L-3}\end{pmatrix}=\begin{pmatrix}h_2\\ h_4\\ \vdots\\ h_{2L-2}\end{pmatrix}.$$

To finish, in the top identity it suffices to reverse the order of the equations, and reverse the components of the vector $(h_{2L},\dots,h_4)$; this is achieved by multiplication of the antidiagonal permutation matrix $\mathbf{P}$. As $\mathbf{A}=T(a_1,\dots,a_{L-1})$ is a Toeplitz matrix, then it is easy to deduce that

$$\mathbf{P}\overline{\mathbf{A}}\mathbf{P} = \overline{\mathbf{A}}^t = \mathbf{A}^H,$$

which concludes the proof.

Corollary 2. $\mathbf{h} = (h_1, h_2, \ldots, h_{2L})$ *is an orthogonal complex filter if and only if there exist L complex numbers* $a_1, \ldots, a_L$ *such that*

$$h_{2L+1-2k} = -\sum_{j=1}^{k} h_{2L-2k+2j}\overline{a_j} \quad \forall k = 1, \ldots, L-1, \tag{10}$$

$$h_{2k} = \sum_{j=1}^{k} h_{2k+1-2j} a_j \quad \forall k = 1, \ldots, L. \tag{11}$$

Proof. The first $L-1$ equations of (11) are just the first set of equations of Theorem 1. For the last one (the one with $k = L$) set

$$a_L = \frac{h_{2L} - a_{L-1}h_3 - \cdots - a_1 h_{2L-1}}{h_1}$$

(which is well defined because $h_1 \neq 0$) and, then

$$a_L h_1 + a_{L-1}h_3 + \cdots + a_1 h_{2L-1} = h_{2L}.$$

which is precisely Eq. (11) for $k = L$. □

4.1. Design procedure of orthogonal complex filters

Now we will obtain a new explicit expression for all orthogonal complex filters. Notice that the two identities of Corollary 2 present some kind of redundancy: the coefficient h_{2L} appears as a parameter in Eq. (10) and as an unknown in (11). By exploiting this redundancy, we will generate $\mathbf{h}$ using only external independent parameters, such as $a_1, \ldots, a_L$. This will then lead us to find the desired design method for all orthogonal complex filters.

To this end, by means of the parameters $(a_1, \ldots, a_L)$, we just build:

- the lower triangular Toeplitz matrix $\mathbf{A}$ already defined in this Section:

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \ddots & \vdots \\ a_3 & a_2 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{L-1} & \cdots & a_3 & a_2 & a_1 \end{pmatrix}.$$

- and three column vectors of length $L-1$:

$$\left.\begin{array}{l} \mathbf{a} = (a_2, a_3, \ldots, a_L)^t \\ \mathbf{d} = \left(\mathbf{I} + \mathbf{A}\mathbf{A}^H\right)^{-1}\mathbf{a} \\ \mathbf{e} = -\mathbf{A}^H\mathbf{d} \end{array}\right\}, \tag{12}$$

  where $\mathbf{d}$ is well defined because $\mathbf{I} + \mathbf{A}\mathbf{A}^H$ is always an invertible (positive definite) matrix.

For the sake of simplicity, from now on we will denote the vectors of length $L-1$:

$$\mathbf{h}_{even} = (h_4, h_6, \ldots, h_{2L-2}, h_{2L})^t,$$
$$\mathbf{h}_{odd} = (h_3, h_5, \ldots, h_{2L-3}, h_{2L-1})^t,$$

which contain the even and odd indexed coefficients of $\mathbf{h}$ *except for* h_1, h_2.

Now we are finally ready to express all the components of the filter by means of h_1 and the L parameters. This is one of the main results of this paper, which constitutes the first parameterization of all orthogonal complex filters:

Theorem 2. $\mathbf{h}$ *is an orthogonal complex filter of length* $2L$ *if and only if their components are of the form*:

$$\begin{pmatrix} h_1 \\ \mathbf{h}_{odd} \\ h_2 \\ \mathbf{h}_{even} \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ \mathbf{e} \\ a_1 \\ \mathbf{d} \end{pmatrix}, \tag{13}$$

where the vectors $\mathbf{d}, \mathbf{e}$ *are built from the parameters* $a_1, \ldots, a_L$ *by means of Eq.* (12).

Proof. We start by writing Eq. (11) as:

$$\begin{pmatrix} h_2 \\ \mathbf{h}_{even} \end{pmatrix} = \begin{pmatrix} h_1 a_1 \\ h_1 \mathbf{a} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{A}\mathbf{h}_{odd} \end{pmatrix} \Leftrightarrow \begin{cases} h_2 = a_1 h_1 \\ \mathbf{h}_{even} = h_1 \mathbf{a} + \mathbf{A}\mathbf{h}_{odd} \end{cases} \tag{14}$$

and Eq. (10) as

$$\mathbf{h}_{odd} = -\mathbf{A}^H \mathbf{h}_{even} \tag{15}$$

so it suffices to substitute (15) into (14):

$$\begin{aligned} &\mathbf{h}_{even} = h_1 \mathbf{a} - \mathbf{A}\mathbf{A}^H \mathbf{h}_{even} \\ \Leftrightarrow &\left(\mathbf{I} + \mathbf{A}\mathbf{A}^H\right) \mathbf{h}_{even} = h_1 \mathbf{a} \\ \Leftrightarrow &\,\mathbf{h}_{even} = h_1 \left(\mathbf{I} + \mathbf{A}\mathbf{A}^H\right)^{-1} \mathbf{a} = h_1 \mathbf{d}. \end{aligned}$$

We just finally have $\mathbf{h}_{odd} = -\mathbf{A}^H \mathbf{h}_{even} = -h_1 \mathbf{A}^H \mathbf{d} = h_1 \mathbf{e}$. □

Remark 1. We have derived that, by choosing L arbitrary complex parameters $(a_1, \ldots, a_L)$ and one arbitrary nonzero number h_1, the whole set of orthogonal complex filters $\mathbf{h}$ of length $2L$ can be parameterized. In other words, these filters are characterized by means of just $L+1$ parameters. And this representation is *unique*: different sets of parameters always yield different filters, so there is no redundancy in this parameterization.

Recall from Section 2 that the even and odd components of orthogonal filters are associated to the polynomials

$$\left.\begin{aligned} H_{even}(z) &= \sum_{k=0}^{L-1} h_{2k+2} z^k = h_1 B(z) \\ H_{odd}(z) &= \sum_{k=0}^{L-1} h_{2k+1} z^k = h_1 C(z) \end{aligned}\right\},$$

where the polynomials B and C are

$$B(z) = a_1 + \sum_{k=1}^{L-1} d_k z^k, \quad C(z) = 1 + \sum_{k=1}^{L-1} e_k z^k. \tag{16}$$

Remark 2. Hence, we have just provided a procedure to design the solutions of the polynomial equation

$$|B(z)|^2 + |C(z)|^2 = \frac{1}{|h_1|^2} \quad \forall |z| = 1,$$

which is the same as Eq. (3). Moreover, we conclude that the general solution of Eq. (3) is the one given in Eq. (16), up to a factor.

5. General expression for paraunitary complex filters

In this Section we will impose the orthogonal complex filter to be unitary. This way, we obtain the first general parameterization of paraunitary complex filters:

Theorem 3. **h** *is a paraunitary filter if and only if its components can be written as*

$$\begin{pmatrix} h_1 \\ \mathbf{h}_{odd} \\ h_2 \\ \mathbf{h}_{even} \end{pmatrix} = \frac{e^{i\alpha}}{\sqrt{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}} \begin{pmatrix} 1 \\ \mathbf{e} \\ a_1 \\ \mathbf{d} \end{pmatrix}. \tag{17}$$

where $e^{i\alpha}$ *is an arbitrary unitary complex number.*

Proof. It suffices to impose the vector **h** of Eq. (13) to have unit norm:

$$1 = |h_1|^2 \left(1+|a_1|^2+\|\mathbf{e}\|^2+\|\mathbf{d}\|^2\right)$$

and so, this condition determines h_1 (up to an unitary complex number). By using the relationship between **d** and **e** we can simplify this expression:

$$\begin{aligned} \|\mathbf{e}\|^2+\|\mathbf{d}\|^2 &= \mathbf{e}^H\mathbf{e}+\mathbf{d}^H\mathbf{d} = \mathbf{d}^H\mathbf{A}\mathbf{A}^H\mathbf{d}+\mathbf{d}^H\mathbf{d} \\ &= \mathbf{d}^H\left(\mathbf{A}\mathbf{A}^H+\mathbf{I}\right)\mathbf{d} = \mathbf{d}^H\mathbf{a} \geqslant 0 \end{aligned}$$

so the unitary condition is

$$|h_1|^2 = \frac{1}{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}. \quad \square$$

Notice that the paraunitary complex filters are expressed directly via L parameters $a_1, \ldots, a_L$ and one complex unitary number $e^{i\alpha}$. Unlike other approaches our procedure requires no iteration process: hence, Eq. (17) directly provides the **simplest general expression** for paraunitary complex filters. This constitutes one of the main results of this paper.

6. Solution of the original polynomial equation

The above results lead us to the desired explicit solution of the polynomial Problem 3. Recall that this problem has already been solved by means of Remark 2, up to a normalizing factor. Let us give the general solution in this Section; we solve it first for $a = 1$:

Theorem 4. *Two polynomials of complex coefficients* $B_0(z)$ *and* $C_0(z)$ *of degree at most* $L-1$ *satisfy the polynomial equation on the unit torus*

$$|B_0(z)|^2 + |C_0(z)|^2 = 1 \tag{18}$$

if and only if they can be written as

$$B_0(z) = H_{even}(z) = \frac{e^{i\alpha}}{\sqrt{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}}\left(a_1 + \sum_{k=1}^{L-1} d_k z^k\right),$$

$$C_0(z) = H_{odd}(z) = \frac{e^{i\alpha}}{\sqrt{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}}\left(1 + \sum_{k=1}^{L-1} e_k z^k\right),$$

where **a**, **d**, **e** *are defined in Eq. (12), by means of L independent parameters.*

Proof. Lemma 1 assures that the solutions of the polynomial equation (18) are given by the polynomials H_{even} and H_{odd} associated to a paraunitary filter. By exploiting the general expression of paraunitary complex filters provided by Eq. (17), we directly derive this result. □

Now we provide the expression of the general solution:

Corollary 3. *For $a > 0$, all the polynomials B, C of degree $< L$ which verify the polynomial equation*

$$|B(z)|^2 + |C(z)|^2 = a \quad \forall |z| = 1$$

are explicitly given by

$$B(z) = \sqrt{a} B_0(z) = \frac{\sqrt{a} e^{i\alpha}}{\sqrt{1 + |a_1|^2 + \mathbf{d}^H \mathbf{a}}} \left(a_1 + \sum_{k=1}^{L-1} d_k z^k \right)$$

$$C(z) = \sqrt{a} C_0(z) = \frac{\sqrt{a} e^{i\alpha}}{\sqrt{1 + |a_1|^2 + \mathbf{d}^H \mathbf{a}}} \left(1 + \sum_{k=1}^{L-1} e_k z^k \right),$$

where $\mathbf{a}, \mathbf{d}, \mathbf{e}$ are vectors defined by Eq. (12), and can be explicitly parameterized by choosing L arbitrary complex parameters $a_1, \ldots, a_L$ via our design method.

Remark 3. The main advantage of our approach is that $B(z)$ are $C(z)$ are built **directly**, without need of spectral factorization or root finding procedure, unlike most procedures in the literature [2,5,6].

7. Design example

Let us build all the orthogonal complex filters of length $2L = 4$: according to our results, it just suffices to choose $L = 2$ complex parameters a_1, a_2. In this case, $\mathbf{A} = a_1$, $\mathbf{a} = a_2$ and

$$\mathbf{d} = \frac{a_2}{1 + |a_1|^2}, \quad \mathbf{e} = -\frac{\overline{a_1} a_2}{1 + |a_1|^2}$$

are also complex numbers, not vectors. Hence, the explicit expression of such filters are:

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ a_1 \\ \mathbf{e} \\ \mathbf{d} \end{pmatrix} = \frac{h_1}{1 + |a_1|^2} \begin{pmatrix} 1 + |a_1|^2 \\ a_1 \left(1 + |a_1|^2\right) \\ -\overline{a_1} a_2 \\ a_2 \end{pmatrix}.$$

Notice that this general expression for orthogonal complex filters of length 4 also includes the set of filters of minor length: in fact, when choosing $a_2 = 0$ we would obtain all orthogonal filters of length 2. Our procedure yields the parameterization of all orthogonal complex filters of length **at most** $2L$. For the paraunitary case, just an additional normalization step would be required.

7.1. Application to the solution of a polynomial equation

Let us find the whole set of polynomials B, C of degree $\leqslant 1$ which verify the equation

$$|B(z)|^2 + |C(z)|^2 = a > 0 \quad \forall |z| = 1.$$

By using our proposed approach, as B, C contain $L = 2$ coefficients, we associate them with the even and odd components of a paraunitary complex filter $\mathbf{h}$ of length $2L = 4$. Thus, polynomials $B(z), C(z)$ are explicitly given by Corollary 3:

$$B(z)=\frac{\sqrt{a}e^{i\alpha}}{\sqrt{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}}\,(a_1+\mathbf{d}z)\,,$$
$$C(z)=\frac{\sqrt{a}e^{i\alpha}}{\sqrt{1+|a_1|^2+\mathbf{d}^H\mathbf{a}}}\,(1+\mathbf{e}z)\,.$$

In this Section we have obtained their final expression:

$$B\,(z)=\beta\left(a_1\left(1+|a_1|^2\right)+a_2z\right),$$

$$C\,(z)=\beta\left(1+|a_1|^2-\overline{a_1}a_2z\right),$$

where β is a normalizing factor. Therefore, they are uniquely determined via a_1, a_2; the multiplicative factor β is also computed by means of them, in order to match the quantity a in the original polynomial Equation (3).

7.2. Application to the rectangular Toeplitz solutions of a matrix equation

The previous example also yields all the rectangular complex Toeplitz matrices $\mathbf{B}, \mathbf{C}$ which are **bidiagonal**, and fulfill the matrix equation

$$\mathbf{B}\mathbf{B}^H+\mathbf{C}\mathbf{C}^H=a\mathbf{I}.$$

Indeed, we obtain that such matrices are, up to a factor β, the ones whose rows are of the form:

$$\text{Rows of }\mathbf{B}:\left(\ldots,0,0,a_1\left(1+|a_1|^2\right),a_2,0,0,\ldots\right),$$
$$\text{Rows of }\mathbf{C}:\left(\ldots,0,0,1+|a_1|^2,-\overline{a_1}a_2,0,0,\ldots\right)$$

for any set of complex numbers a_1, a_2.

8. Conclusions

Some matrix equations can be investigated by using approaches from other disciplines. In this paper, filterbank theory has helped us to solve certain Toeplitz matrix equations. To this end, we have presented new results on complex orthogonal and paraunitary filters. Moreover, we have provided a novel parameterization of all complex orthogonal and paraunitary filters. This characterization not only yields the explicit general solution of the previous matrix equations, but also solves their corresponding Bézout polynomial identities. Our proposed procedure is direct, simple, non redundant, non iterative, and does not require any spectral factorization technique.

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