A Stochastic Ellipsoid Approach to Repeatability Modelisation of Industrial Manipulator Robots

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Abstract

Most industrial manipulator robots are used in assembly tasks. Their manufacturers use repeatability parameters to show their effectiveness. ISO9283 standard details the process of measuring repeatability. As loads, speed and other various factors affect repeatability, a high number of experimental trials are needed to obtain significant values. Moreover, repeatability is also affected by the location of the robot’s endpoint in its workspace. In this paper, we construct a stochastic model to evaluate repeatability in the whole workspace of the robot. For this, we only need to determine the robot geometry and sensor sensitivity. Consequently, it is possible with only a few experimental measures to map robot repeatability and obtain relevant information about the spatial error distribution around the points’ mean position. This method has been applied to a SCARA robot and has led to specific mapping. Experimental results have been compared with the results of the stochastic ellipsoid model.

1. Introduction

Assembly tasks require high accuracy from industrial manipulator robots. But driving a robot very accurately is difficult because numerous factors affect robot accuracy. A classification of these errors enables us to understand robot behaviour. For instance, some errors stem from our lack of knowledge of robot geometry. Some others appear when we compute the robot’s forward or inverse kinematics function for control purpose. Others are a consequence of the robot’s imperfect mechanisms (loose speed reducer, non-linearity…). To sum up, error sources can be classified in three main categories: robot design (geometry and mechanisms), control computation and environment (temperature, forces or torques on the robot created by gravity or contacts). As improving robot accuracy is an important challenge, a lot of research has been done in this field. For example, research in the calibration field [1,2,3] has reduced robot geometric model inaccuracy. The influence of gravity has been diminished with the use of special mechanisms, like springs. Consequently, some of the robot position errors factors have been diminished.

Anyway, the search for better accuracy must be thought in order to avoid the use of too complex models because they will not suit the adaptability needed for an industrial application. If the robot is controlled using a dynamic model, then it is very difficult to achieve correct behaviour with changing positions and unknown loads.

In fact, industrial manipulator robots are often used for pick and place tasks and the main criterion necessary to succeed is good repeatability. Moreover, the pose accuracy obtained with visual based control methods should be close to the pose repeatability obtained with joint PID feedback control. So, robot manufacturers often emphasize the pose repeatability parameters of their robots and do not give much information about unidirectional pose accuracy or multidirectional pose accuracy variation as specified in the ISO 9283 standard [4].

The repeatability of a robot measures its precision when its endpoint achieves a particular pose under repeated commands to the same set of joint angles. It is defined as:

\[ R_p = \bar{L} + 3S_f \]

where \( L_f = \sqrt{(X_f - \bar{X})^2 + (Y_f - \bar{Y})^2 + (Z_f - \bar{Z})^2} \)

\( (X_f, Y_f, Z_f) \) coordinates of the i-th measurement

\( \bar{X}, \bar{Y}, \bar{Z}, \bar{L} \) average values

\( S_f \) standard deviation of \( L \)

The procedure detailed in the standard ISO9283 to compute repeatability is extremely long. Five positions must be attained thirty times. The positions are precisely distributed in the volume of a cube chosen to be the largest possible in the workspace. So, the manufacturer is supposed to give five different estimations of the robot’s pose repeatability. In practice, only one figure is
given and the user does not know where it has been measured.
Of course, if standard ISO9283 requires robot manufacturers to carry out five different trials for repeatability, it is because repeatability varies within the robot workspace [5]. This phenomenon is well-known [6] and has been verified in laboratories. Riemer and Edan [7,8] have proposed a simplified model to estimate repeatability in different locations of the workspace.
In our paper, we propose a stochastic approach to repeatability. This model has several advantages:
- It describes the spatial distribution of the workspace errors around the mean position
- Results are easily computed and lead to characteristic numbers
- Results can be visualized graphically with stochastic ellipsoids
- Few experimental measures are needed to construct the model
- The stochastic model can be used for robot design optimisation or assembly task layout.
In the first part of the paper, we set out the hypothesis necessary to construct the mathematical model. Then we explain the concept of stochastic ellipsoids. In the third part, we apply the model to a SCARA robot. Repeatability maps are drawn to visualise results.

2. Hypothesis and Modelisation
The forward kinematics function of a robot transforms joint coordinates \( \theta \) into workspace coordinates \( X \):
\[
X = F(\theta)
\]
(1)
The Jacobian function maps the joint velocity vector to the Cartesian velocity vector:
\[
dX = J(\theta)d\theta
\]
(2)
Let the joint coordinates of the robot be fixed at \( \theta_j \), the joint error \( d\theta \) is transformed into a workspace error \( dX \) by the former linear transformation.
The variations of the random variable \( d\theta \) around the mean position are responsible for the pose repeatability phenomenon.
Let the target location, the speed, the load, the approach direction of the robot arm be constant, then the final position of the robot arm will be in the same interval of the angular encoders. In the case where the robot is internally controlled by a PID feedback device, the difference between the desired and the actual final position of the robot will be nil, after a sufficient time lapse, in the discrete space of the angular sensor. The angular sensor encoder gives the same discrete value as long as the random variable \( d\theta \) stays in the same interval of one bit width.
Of course, it is possible that the difference between the desired and the measured positions for the joint actuator is not nil. One reason can be insufficient influence of the integral factor in the control. But as conditions for every trial are the same, control must be the same and we will assume that the final discrete position in the angular sensor encoder space will be the same. This is our first hypothesis, illustrated in figure 1. But it is not as restricted as it seems. For example, instead of limiting the width of the final interval to one bit, it is possible to widen it to a higher number of bits or to link it with the sensor sensitivity which would have been experimentally measured. This does not discredit the modelisation.

![Figure 1: Final positions of angular coordinates](image1)

We can estimate the upper limit \( \Delta X \) for the workspace coordinates knowing the width of the bit interval of each joint actuator sensor \( \Delta \theta \) using the Jacobian matrix
\[
\Delta X = J(\theta_j)\Delta \theta
\]
Nevertheless, this calculation is limited because it considers the worst case, which we do not know how often it occurs. Another disadvantage of this calculation is the difficulty to link it with the experimental measurement of repeatability, simply because we do not consider the workspace error from a probabilistic point of view, so that it is impossible to find a mathematical relation between \( R_s \) and \( \Delta X \).
For this reason, it is necessary to have more precision about the probability density of the random variable \( dX \). Knowing more about this probability law would allow us to link repeatability to the robot’s sensor sensitivity.
Experiments on a SCARA robot with precise measurement instruments show that the probability density of the robot joint angular position was a Gaussian law. Standard variation was function of the bit width. So, in our second hypothesis, we assume that the probability density of the joint angular error \( d\theta \) around the mean is a Gaussian law (figure 2).

![Figure 2: angular error distribution](image2)
3. Stochastic modelisation

Let $d\theta$ be the random variable describing angular variations around the angular mean position. Assuming the hypothesis explained in the preceding paragraph, the vector $d\theta$ is an independent Gaussian vector whose covariance matrix is obtained through the calculation of each independent angular random variable.

Using probability theory, the random vector $dX$ is a Gaussian vector because it is the image of a Gaussian vector by a linear application. Moreover, the covariance matrix $C$ of $dX$ is given by calculating:

$$C = JD J^T$$

The forward kinematics function of the SCARA robot is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

The Jacobian function is obtained through differentiation:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = J \begin{bmatrix} d\theta \end{bmatrix} = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_1 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_1 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

Assuming that angular error follows Gaussian law, let the covariance matrix be given by $Cov\theta$.

Thus, the workspace error is a Gaussian vector whose covariance matrix is:

$$CovX = J Cov\theta J^T$$

Stochastic ellipsoids

The density of the Gaussian vector $dX$ is illustrated in figure 4 and given by:

$$g(dx, dy) = \beta \exp \left( -\frac{1}{2} \begin{bmatrix} dx \\ dy \end{bmatrix} \left( CovX \right)^{-1} \begin{bmatrix} dx \\ dy \end{bmatrix} \right)$$

The density contour lines are ellipses drawn in figure 5.
The main characteristics of the ellipses can easily be determined from a reference stochastic ellipse generated as follows:

- The lengths of the semiaxes of the ellipses are the square root of the eigenvalues of the covariance matrix $\text{Cov}_X$.
- The directions of the principal axes are the eigenvectors of the covariance matrix $\text{Cov}_X$.

It is possible to obtain the stochastic ellipse of risk $\alpha$ from a reference stochastic ellipse using a central homothety. The centres of the ellipses are the same and the ratio of the homothety is calculated by the following process. First we link the risk $\alpha$ to the ellipse $E$ with the integral:

$$
1 - \alpha = P(dX \in E) = \int \beta \exp\left(-\frac{1}{2}dX^T \left[\text{Cov}_X\right]^{-1}dX\right) d(dX)
$$

In the transformation (2), Cartesian stochastic ellipses are associated with joint stochastic ellipses according to the relation:

$$
K = dX^T \left[\text{Cov}_X\right]^{-1}dX = d\theta^T \left[\text{Cov}\theta\right]^{-1}d\theta
$$

Thus the reference Cartesian stochastic ellipse is associated with a reference joint stochastic ellipse. Hence the ratio $r(\alpha)$ of the central homothety which transforms the Cartesian stochastic ellipse of risk $\alpha$ into the Cartesian stochastic ellipse of reference can be used to find the joint stochastic ellipse of risk $\alpha$.

As it is easier to compute the integral in the joint space, we use equation (2) and find the ratio $r(\alpha)$:

$$
1 - \alpha = P(d\theta \in E') = \int \beta' \exp\left(-\frac{1}{2}d\theta^T \left[\text{Cov}\theta\right]^{-1}d\theta\right) d\theta
$$

Normalizing the integral leads to:

$$
\beta' = \frac{1}{2\pi} \frac{1}{\sigma_1 \sigma_2}
$$

Then we use the following elliptic coordinates:

$$
d\theta_1 = \sigma_1 r \cos \omega
$$
$$
d\theta_2 = \sigma_2 r \sin \omega
$$

the integral is now changed into:

$$
1 - \alpha = \frac{1}{2\pi} \frac{1}{\sigma_1 \sigma_2} \int \int \exp\left(-\frac{1}{2}\frac{\omega^2}{\sigma_1^2} - \frac{\omega^2}{\sigma_2^2}\right) d\sigma_1 d\sigma_2 d\omega
$$

Ultimately we find the value of $r(\alpha)$:

$$
r(\alpha) = \sqrt{\ln \left(\frac{1}{\alpha\sigma_1^2}\right)}
$$

In figure 6, we plot the ratio $r(\alpha)$ versus the risk.

So, we have defined how to draw precisely the stochastic ellipses of risk $\alpha$. To sum up, the lengths of the semiaxes of the stochastic ellipse are given by:

$$
L_m = r(\alpha) \sqrt{\lambda_m} \quad \text{and} \quad L_M = r(\alpha) \sqrt{\lambda_M}
$$

where $\lambda_m$ and $\lambda_M$ are the eigenvalues of $\text{Cov}_X$. And the directions of the principal axes of the ellipse are the eigenvectors of $\text{Cov}_X$.

What is the variation of the stochastic ellipse surface if the risk decreases, for example, from 10% to 1%? The ellipse surface is proportional to the square of $r(\alpha)$, so the surface ratio will be:

$$
q = \left[ \frac{r(0.01)}{r(0.10)} \right]^2 \approx 2
$$

The size of the stochastic ellipses does not grow as fast as one could expect because of the Gaussian distribution. Figure 7 illustrates the size of different stochastic ellipses with different risks.

When looking for simple curves to visualize the evolution of repeatability for robots, we find that we have to plot the variation of the lengths of the stochastic reference ellipses in the workspace. For the SCARA robot, we began by plotting these lengths versus the
θ₁ angle because they are independent from θ₁ due to the central symmetry of the robot’s first joint.

Maximax and minimax curves

We name the ellipse semi-axes curves:
- “Maximax error curve” for the major semi-axis of the stochastic ellipse. For a risk α, it corresponds to the upper limit of the error in a direction which maximises this upper limit.
- “Minimax error curve” for the minor semi-axis of the stochastic ellipse. For a risk α, it corresponds to the upper limit of the error in a direction which minimises this upper limit.

Figure 8 shows such curves for a SCARA robot with the following characteristics:

\[ L_1 = L_2 = 1; \sigma_1 = \sigma_2 = 2; \alpha = 0.01 \]

Some interesting conclusions can be drawn on examining the curves:
- The maximax error has high variation in the workspace. Its value is doubled between the two positions corresponding to \( \theta_1 = \pi \) and \( \theta_1 = 0 \);
- The minimax error is not very high throughout the interval. So in some special directions, repeatability is quite good. That is an important result that does not appear in the classical model of repeatability.

The main result is that our model allows a better understanding of the distribution of error around the mean. The classical approach is not a stochastic one and consequently, it doesn’t take into account the anisotropy of error distribution. Because in assembly tasks, tolerances are often anisotropic, this stochastic model can therefore be of great interest for positioning the task in the workspace.

What’s more, it is possible to know if a given assembly task can be successfully achieved by the robot and in which area. For instance, if a peg-in-hole process is to be carried out, the radial tolerance can be compared to the maximax error and we can then deduce whether the assembly task can be achieved and its rate of success.

SCARA Robot Repeatability Map

Even if workspace conditions can be read on the maximax error curves, it is more convenient to locate the corresponding area in the robot’s workspace. To improve visualisation, we have drawn a colour map of the maximax error in the robot’s workspace. Colours are indexed according to the intensity of the maximax errors. Figure 9 is an illustration of this map.

Maximax and minimax errors can be drawn with the same colour scale and it is then easy to identify the best area according to the task specifications. Another criterion can be used, like for instance the ellipse surface. Figure 10 gives an illustration of these last three maps.

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Nevertheless, one more useful indication can be added to help in visualising the axes of the ellipse. As the main axes are orthogonal, it is sufficient to be able to plot the longer one. For this purpose, we have drawn the field lines for the major axis. It is then possible to draw the major axis for one point anywhere in the workspace. Once the point is given, we carry out a rotation of the field line around the centre of the workspace and stop when an intersection point appears. Then the tangent vector to the field line is the direction of the major axis of the ellipse. Figure 11 illustrates the construction process.
Experimental results

It is possible to construct the model with few experimental measures. To obtain the standard variations necessary to write the covariance matrix of angular errors, the process was the following. First we locked the robot two arms together mechanically and we operated with the first joint. The \( \theta \) angle was set to 0. The error density was a Gaussian law and it was easy to estimate the standard deviation. Taking into account the lever arm, we estimated the standard deviation for angle \( \theta_1 \). A similar process gave standard deviation for angle \( \theta_2 \) once the first joint was firmly locked.

To verify the effectiveness of the model, we chose another location in the workspace and measured 2D error. The distribution was then compared to the stochastic ellipses at this point. Experimental and theoretical results match. Figure 12 shows some experimental results and two stochastic ellipsoids are plotted with \( \alpha = 0.05 \) and \( \alpha = 0.001 \). In the first case 8 trials out of 150 fall out of the ellipse corresponding to 5.33% and in the second case no trial among the 150 is found outside the ellipse. The theoretical model fits well to reality.

5. Conclusions

In this paper we have proposed a stochastic model for repeatability. This model is interesting because Cartesian error distribution is described as a Gaussian random variable. This model has several advantages: first, we do not need to perform long experiments to determine repeatability; we only need a few experimental measures to construct the model and then we are able to compute repeatability in the whole workspace. Secondly, we have more information about robot ability. The modelisation takes into account more parameters. For instance, error distribution is well described. So, we can determine directions where the upper limit of error is at its maximum or minimum. These indications are very appropriate in an assembly process. Thirdly, we have proved that, for a given position, the size of the stochastic ellipsoids is a function of risk.

Finally, we have applied the method to a SCARA robot. In this case, we have drawn maps to allow easy visualisation of stochastic ellipsoids. Experimental and theoretical results have been compared and hence the general model of stochastic ellipsoids has been validated.

6. References