Note

Independent branchings in acyclic digraphs

Andreas Huck *

Institute of Mathematics, University of Hannover, Welfengarten 1, 30167 Hannover, Germany

Received 21 May 1996; revised 21 May 1997; accepted 31 August 1998

Abstract

Let \( D \) be a finite directed acyclic multigraph and \( t \) be a vertex of \( D \) such that for each other vertex \( x \) of \( D \), there are \( n \) pairwise openly disjoint paths in \( D \) from \( x \) to \( t \). It is proved that there exist \( n \) spanning trees \( B_1, \ldots, B_n \) in \( D \) directed toward \( t \) such that for each vertex \( x \neq t \) of \( D \), the \( n \) paths from \( x \) to \( t \) in \( B_1, \ldots, B_n \) are pairwise openly disjoint. © 1999 Elsevier Science B.V. All rights reserved

AMS classification: 05C05; 05C40

Keywords: Branchings

Throughout this paper, each digraph is finite and may have multiple edges but no loops. Digraphs without multiple edges directed equally are called simple. Let \( D \) be a digraph. \( V(D) \) and \( E(D) \) denote the set of the vertices of \( D \) and the set of the edges of \( D \), respectively. For any distinct \( x, y \in V(D) \), we denote by \( \delta(D; x, y) \) the number of edges directed from \( x \) toward \( y \). Moreover, we let \( \delta^+(D; x) \) and \( \delta^-(D; x) \) denote the number of edges directed from \( x \) and directed toward \( x \), respectively.

Paths and cycles in \( D \) are always directed and are not allowed to use a vertex more than once. A path from a vertex \( x \) to a vertex \( y \) is also called an \( x, y \)-path. As usual, digraphs without cycles are called acyclic.

Let \( P_1 \) and \( P_2 \) be two edge-disjoint paths in \( D \). For each \( i \in \{1, 2\} \), let \( x_i \) be the start vertex and \( y_i \) be the end vertex of \( P_i \). Then \( P_1 \) and \( P_2 \) are called disjoint if \( V(P_1) \cap V(P_2) = \emptyset \), nearly disjoint if \( V(P_1) \cap V(P_2) \subseteq \{x_1\} \cap \{x_2\} \), and openly disjoint if \( V(P_1) \cap V(P_2) \subseteq \{x_1, y_1\} \cap \{x_2, y_2\} \). As usual, we call \( D \) \( n \)-(edge)-connected if for any two vertices \( x \) and \( y \) of \( D \), there are at least \( n \) openly disjoint (edge-disjoint) \( x, y \)-paths in \( D \). Moreover, if \( t \) is a vertex of \( D \) such that for all \( x \in V(D) - t \), there are \( n \) pairwise openly disjoint (edge-disjoint) \( x, t \)-paths in \( D \), then \( D \) is called \( (t, n) \)-(edge)-connected.

* E-mail: huck@math.uni-hannover.de.

0012-365X/99/$ - see front matter © 1999 Elsevier Science B.V. All rights reserved

PII: S0012-365X(98)00338-0
An acyclic digraph $B$ is called a branching if there exists $t \in V(B)$ with $\delta^+(B; t) = 0$ and $\delta^+(B; x) = 1$ for each $x \in V(B) - t$. $t$ is called the root of $B$ and $B$ is also called a $t$-branching. For each $x \in V(B)$, $xBt$ denotes the unique $x,t$-path in $B$. Now let $B_1$ and $B_2$ be two branchings and for each $i \in \{1, 2\}$, let $t_i$ be the root of $B_i$. Then $B_1$ and $B_2$ are called (weakly) independent if for each $x \in V(B_1) \cap V(B_2)$, the paths $xB_1t_1$ and $xB_2t_2$ are openly disjoint (edge-disjoint).

A subdigraph $D'$ of a digraph $D$ is called spanning if $V(D') = V(D)$. If $D$ is a digraph and $t \in V(D)$, then an obviously necessary condition for the existence of $n$ pairwise (weakly) independent spanning $t$-branchings in $D$ is that $D$ is $(t,n)$-(edge)-connected. Ref. [2] proved that $(t,n)$-edge-connectivity is even sufficient for the existence of $n$ pairwise weakly independent spanning $t$-branchings. So it is near to examine the following conjecture for each integer $n \geq 0$.

**Conjecture 1.** Let $D$ be a $(t,n)$-connected digraph with $t \in V(D)$. Then there exist $n$ pairwise independent spanning $t$-branchings in $D$.

The following variation is due to A. Frank and appeared in [6].

**Conjecture 1'.** Let $D$ be an $n$-connected digraph and $t \in V(D)$. Then there exist $n$ pairwise independent spanning $t$-branchings in $D$.

Clearly, for each integer $n \geq 0$, Conjecture 1 implies Conjecture 1'. The converse also holds: Assume that $D$ is a $(t,n)$-connected digraph with $t \in V(D)$. Clearly, we may assume that $\delta^+(D; t) = 0$. Let $\delta := \delta^-(D; t)$ and let $f_1, \ldots, f_\delta$ be the edges of $D$ directed toward $t$. Moreover, take any $n$-connected digraph $H$ with $|V(H)| \geq \delta$ and any pairwise distinct vertices $x_1, \ldots, x_\delta$ of $H$. Finally, let the digraph $D'$ be obtained from $D$ by replacing $t$ by $H$ such that $f_i$ is directed toward $x_i$ in $D'$ for each $i \leq \delta$ and by adding an edge directed from $x$ toward $y$ for each $x \in V(H)$ and $y \in V(D) - t$. It is standard to check that $D'$ is $n$-connected. Take any $t' \in V(H)$. Then it is easy to see that any $n$ pairwise independent spanning $t'$-branchings in $D'$ yield $n$ pairwise independent spanning $t$-branchings in $D$.

Clearly, Conjecture 1 and Conjecture 1' are valid for $n = 1$ (and trivial for $n = 0$). [7] verified these conjectures for $n = 2$ (a short proof is given in [3]) and [4] constructed counterexamples for each $n \geq 3$. Analogous statements for undirected graphs were proved for each $n \leq 3$ by [1,5] and are open for each $n \geq 4$.

With regard to the counterexamples of Conjectures 1 and 1', it is natural to look for additional conditions that ensure the existence of $n$ pairwise independent spanning $t$-branchings. We will show that acyclicity is such a condition, i.e. we will prove the following theorem.

**Theorem 1.** Let $D$ be an acyclic $(t,n)$-connected digraph with $t \in V(D)$ and $n \geq 0$. Then there exist $n$ pairwise independent spanning $t$-branchings in $D$. 
Note that there are non-trivial acyclic \((t,n)\)-connected digraphs since we allow multiple edges (openly disjoint paths toward \(t\) may consist of parallel edges directed toward \(t\)).

It turned out that it is more convenient to deal with modifications of Conjecture 1 and Theorem 1 where we have \(n\) pairwise distinct vertices \(t_1, \ldots, t_n\) instead of a single vertex \(t\). Let \(D\) be a digraph and \(t_1, \ldots, t_n \in V(D)\) be pairwise distinct. If \(x \in V(D)\) and \(P_i\) is an \(x, t_i\)-path in \(D\) for each \(i \leq n\) such that \(P_1, \ldots, P_n\) are pairwise nearly disjoint, then the \(n\)-tuple \((P_{1:n}) := (P_1, \ldots, P_n)\) is called an \(x, (t_i)_{n}\)-linkage. If there exists an \(x, (t_i)_{n}\)-linkage for each \(x \in V(D) - \{t_1, \ldots, t_n\}\), then \(D\) is called \((t_i)_{n}\)-connected. An \(n\)-tuple \((B_i)_{n}\) is called an independent \((t_i)_{n}\)-system if for each \(i \leq n\), \(B_i\) is a \(t_i\)-branching in \(D\) with \(V(B_i) = V(D) - \{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}\) and if \(B_1, \ldots, B_n\) are pairwise independent, i.e. for each \(x \in V(D) - \{t_1, \ldots, t_n\}\), \((xB_i, t_i)\) is an \(x, (t_i)_{n}\)-linkage.

Clearly, a necessary condition for the existence of an independent \((t_i)_{n}\)-system in \(D\) is that \(D\) is \((t_i)_{n}\)-connected. So it is natural to examine the validity of the following conjecture for each integer \(n \geq 0\).

**Conjecture 2.** Let \(D\) be a \((t_i)_{n}\)-connected digraph with pairwise distinct \(t_1, \ldots, t_n \in V(D)\). Then there exists an independent \((t_i)_{n}\)-system in \(D\).

It is not difficult to prove that for each integer \(n \geq 0\), Conjecture 1 and Conjecture 2 are equivalent (see also [3,4]). Particularly, Conjecture 2 is valid for all \(n \leq 2\) and false for all \(n \geq 3\). But we will show that this conjecture becomes valid for all \(n \geq 0\) if we restrict ourselves to acyclic digraphs. Obviously, to prove this restricted version, we only have to consider simple digraphs and therefore it suffices to prove the following theorem.

**Theorem 2.** Let \(D\) be a simple acyclic digraph and \(t_1, \ldots, t_n \in V(D)\) be pairwise distinct with \(n \geq 0\). Moreover, assume that \(\delta^+(D; x) \geq n\) for each \(x \in V(D) - \{t_1, \ldots, t_n\}\). Then there exists an independent \((t_i)_{n}\)-system in \(D\).

Theorem 2 also implies Theorem 1: Assume that \(D\) is an acyclic \((t,n)\)-connected digraph with \(t \in V(D)\) and \(n \geq 0\). Obviously, we may assume that \(D - t\) is simple. Let \(D'\) be obtained from \(D - t\) by adding pairwise distinct new vertices \(t_1, \ldots, t_n\) and an edge directed from \(x\) toward \(t_i\) for each \(x \in V(D) - t\) and each \(i \in \{1, 2, \ldots, \min(n, \delta(D; x))\}\). Clearly, \(D'\) is simple and acyclic. Moreover, since \(D\) is \((t,n)\)-connected, it is easy to see that \(\delta^+(D'; x) \geq n\) for each \(x \in V(D) - t\). Thus by Theorem 2, there exists an independent \((t_i)_{n}\)-system in \(D'\). Using this system, it is easy to construct \(n\) pairwise independent spanning \(t\)-branchings in \(D\).

Let us prepare the proof of Theorem 2. It is well-known and easy to see that for each acyclic digraph \(D\), there exists a numbering \(\{x_1, \ldots, x_r\}\) of \(V(D)\) such that \(i > j\) for all \(i, j \leq r\) with \(\delta(D; x_i, x_j) > 0\). Such a numbering is called *topological*. Now let \(D\) be a digraph and \(t_1, \ldots, t_n\) be pairwise distinct vertices of \(D\) with \(n \geq 0\). Then \(D\) is
called \((t_i)_n\)-admissible if the premises of Theorem 2 are satisfied, i.e. if \(D\) is acyclic and simple and if \(\delta^+(D; x) \geq n\) for all \(x \in V(D) - \{t_1, \ldots, t_n\}\).

**Lemma 1.** Let \(D\) be a \((t_i)_n\)-admissible digraph with pairwise distinct \(t_1, \ldots, t_n \in V(D)\) and \(n \geq 1\). Then there exists a spanning \(t_n\)-branching \(B\) in \(D - \{t_1, \ldots, t_{n-1}\}\) and a topological numbering \(\{x_1, \ldots, x_r\}\) of \(V(B) - t_n\) such that \(D - E(B) - t_n\) is \((t_i)_{n-1}\)-admissible and \(\{x_r, x_{r-1}, \ldots, x_1\}\) is a topological numbering of \(D - E(B) - \{t_1, \ldots, t_n\}\).

**Proof.** We prove this lemma by induction on \(|V(D)|\). Clearly, we may assume that \(V(D) - \{t_1, \ldots, t_n\} \neq \emptyset\). Take any topological numbering \(\{y_1, \ldots, y_r\}\) of \(D - \{t_1, \ldots, t_n\}\). Then clearly, also \(D* := D - y_r\) is \((t_i)_n\)-admissible. Thus by the induction-hypothesis, there exists a spanning \(t_n\)-branching \(B^*\) of \(D* - \{t_1, \ldots, t_{n-1}\}\) and a topological numbering \(\{x_1, \ldots, x_{r-1}\}\) of \(B^* - t_n\) such that \(D* - E(B^*) - t_n\) is \((t_i)_{n-1}\)-admissible and \(\{x_{r-1}, \ldots, x_1\}\) is a topological numbering of \(D* - E(B^*) - \{t_1, \ldots, t_n\}\). Define \(x_0 := t_n\). Then since \(\delta^+(D; y_r) \geq n\), we find a minimal \(j \in \{0, \ldots, r - 1\}\) with \(\delta(D; y_r, x_j) > 0\). Let \(B\) be obtained from \(B^*\) by adding \(y_r\) and the edge of \(D\) directed from \(y_r\) toward \(x_j\). Clearly, \(B\) is a spanning \(t_n\)-branching in \(D - \{t_1, \ldots, t_{n-1}\}\). Moreover, by construction, \(\{x_1, \ldots, x_j, y_r, x_{j+1}, \ldots, x_{r-1}\}\) is a topological numbering of \(D - B\) and \(\{x_{r-1}, \ldots, x_{j+1}, y_r, x_j, x_1\}\) is a topological numbering of \(D - E(B) - \{t_1, \ldots, t_n\}\). Finally, it is easy to see that \(D - E(B) - t_n\) is \((t_i)_{n-1}\)-admissible (note that the unique edge of \(B\) directed from \(y_r\) is directed toward \(t_n\) if \(\delta(D; y_r, t_n) > 0\)).

Now we are able to prove Theorem 2. Let \(D\) be a \((t_i)_n\)-admissible digraph with pairwise distinct \(t_1, \ldots, t_n \in V(D)\). We prove by induction on \(n\) that \(D\) contains an independent \((t_i)_n\)-system. Clearly, we may assume that \(n \geq 1\). Take a spanning \(t_n\)-branching \(B_n\) of \(D - \{t_1, \ldots, t_{n-1}\}\) and a topological numbering \(\{x_1, \ldots, x_r\}\) of \(B_n - t_n\) according to Lemma 1. By the induction-hypothesis, there exists an independent \((t_i)_{n-1}\)-system \((B_i)_{n-1}\) in \(D - E(B_n) - t_n\). Let \(j \in \{1, \ldots, r\}\). Then by construction, we have \(V(x_jB, t_n) \subseteq \{t_n, x_1, \ldots, x_r\}\) and \(V(x_jB, t_i) \subseteq \{t_i, x_1, \ldots, x_j\}\) for all \(i \leq n - 1\). Thus \((x_jB, t_i, t_n)\) is a \((t_i)_n\)-linkage and hence \((B_i)_n\) is an independent \((t_i)_n\)-system.

This completes the proof of Theorem 2.

Let us finally consider some algorithmic aspects. Following the proof of Lemma 1, we can develop an algorithm of complexity \(O(|E(D)|)\) which constructs a \(t_n\)-branching \(B\) and a topological numbering of \(B - t_n\) as described in Lemma 1 for each \((t_i)_n\)-admissible digraph \(D\). Take any topological numbering \(\{y_1, \ldots, y_r\}\) of \(D - \{t_1, \ldots, t_n\}\). Start with \(V(B) := \{t_n\}, E(B) := \emptyset\), and the empty topological numbering of \(B - t_n\). Then add successively \(y_1, \ldots, y_r\) to \(B\) and to the topological numbering of \(B - t_n\) according to the proof of Lemma 1. Therefore, by the proof of Theorem 2, an independent \((t_i)_n\)-system in a \((t_i)_n\)-admissible graph \(D\) can be constructed in \(O(n|E(D)|)\) steps. Now by the derivation of Theorem 1 from Theorem 2, we see that
n pairwise independent spanning t-branchings in a (t,n)-connected acyclic digraph D can also be constructed in O(n|E(D)|) steps.

References