Projectivity and unification in the varieties of locally finite monadic $MV$-algebras

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Abstract

A description of finitely generated free monadic $MV$-algebras and a characterization of projective monadic $MV$-algebras in locally finite varieties is given. It is shown that unification type of locally finite varieties is unitary.

1 Introduction

Monadic $MV$-algebras (monadic Chang algebras by Rutledge’s terminology) were introduced and studied by Rutledge in [5] as an algebraic model for the predicate calculus $qL$ of Lukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos’ study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [5], showing the completeness of the monadic predicate calculus, has been a great interest. Adapting for the propositional case the axiomatization of monadic $MV$-algebras given by Rutledge in [5], we can define modal Lukasiewicz propositional calculus $MLPC$ as a logic which contains Lukasiewicz propositional calculus $Luk$, the formulas as the axioms schemes:

\[
\begin{align*}
\alpha \rightarrow \exists \alpha, & \exists (\alpha \lor \beta) \equiv \exists \alpha \lor \exists \beta, \exists (\neg \exists \alpha) \equiv \neg \exists \alpha, \exists (\exists \alpha + \exists \beta) \equiv \exists \alpha + \exists \beta, \\
\end{align*}
\]

and closed under modus ponens and necessitation (if $\alpha$, then $\forall \alpha$, where $\forall \alpha = \neg \exists \neg \alpha$).

Let $L$ denote a first-order language based on $\cdot, +, \rightarrow, \neg, \exists$ and $L_m$ denotes monadic propositional language based on $\cdot, +, \rightarrow, \neg, \exists$ and $Form(L)$ and $Form(L_m)$ - the set of all formulas of $L$ and $L_m$, respectively. We fix a variable $x$ in $L$, associate with each propositional letter $p$ in $L_m$ a unique monadic predicate $p^*(x)$ in $L$ and define by induction a translation $\Psi : Form(L_m) \rightarrow Form(L)$ by putting: $\Psi(p) = p^*(x)$ if $p$ is propositional variable, $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \rightarrow$, $\Psi(\exists \alpha) = \exists \Psi(\alpha)$.

Through this translation $\Psi$, we can identify the formulas of $L_m$ with monadic formulas of $L$ containing the variable $x$. Moreover, it is routine to check that $\Psi(MLPC) \subseteq QL$.

2 Monadic $MV$-algebras

The characterization of monadic $MV$-algebras as pair of $MV$-algebras, where one of them is a special kind of subalgebra, are given in [3, 2]. $MV$-algebras were introduced by Chang in [1] as an algebraic model for infinitely valued Lukasiewicz logic.

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An MV-algebra is an algebra $A = (A, \oplus, \circ, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1, x^{**} = x, 0^* = 1, x \oplus x^* = 1, (x^* \oplus y)^* \oplus y = (x^* \oplus y^*) \oplus x, x \circ y = (x^* \oplus y^*)^*$.

An algebra $A = (A, \oplus, \circ, ^*, \exists, 0, 1)$ is said to be monadic MV-algebra (for short MMV-algebra) if $A = (A, \oplus, \circ, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities: $x \leq 3x, 3(x \vee y) = 3x \vee 3y, 3(3x)^* = (3x)^*, 3(3x \vee 3y) = 3x \vee 3y, 3(x \circ y) = 3x \circ 3y, 3(x \circ y) = 3 \circ 3y, \exists(x \circ y) = 3 \circ 3y$.

We shall denote a monadic MV-algebra $A = (A, \oplus, \circ, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. Let $\exists A = \{x \in A : x = 3x\}$.

A subalgebra $A_0$ of an MV-algebra $A$ is said to be relatively complete if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has the least element.

A subalgebra $A_0$ of an MV-algebra $A$ is said to be $m$-relatively complete, if $A_0$ is relatively complete and two additional conditions hold:

$$(\#) (\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \circ v \leq x),$$

$$(\#\#)(\forall a \in A)(\forall x \in A)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \circ v \leq x).$$

**Proposition 1.** [3] Let $(A, \oplus, \circ, ^*, \exists, 0, 1)$ be a monadic MV-algebra. Then the MV-subalgebra $\exists A$ of $A$ of monadic MV-algebra $(A, \oplus, \circ, ^*, \exists, 0, 1)$ is $m$-relatively complete.

**Proposition 2.** [4] There exists a one-to-one correspondence between:

1. monadic MV-algebras $(A, \exists)$;
2. the pairs $(A, A_0)$, where $A_0$ is $m$-relatively complete subalgebra of $A$.

### 3 Projective monadic MV-algebras

From the variety of monadic MV-algebras MMV select the subvariety $K_n$ for $1 \leq n \neq \omega$, which is defined by the following equation [3]: $(K_n) x^n = x^{n+1}$, that is $K_n = MMV + (K_n)$.

**Proposition 3.** [3] If $(A, \exists)$ is a totally ordered monadic MV-algebra, then $A = \exists A$.

**Proposition 4.** [3] If $(A, \exists)$ is a finite monadic MV-algebra with totally ordered $A$, then $MV$-algebra $A$ is isomorphic to a product of totally ordered $MV$-algebras $A_i$, $i \in I$, $A_i \cong \exists A$ and $\exists A$ is isomorphic to the diagonal subalgebra of the product.

It is defined a unique monadic operator $\exists$ on $S^k_n$, where $S_n = (S_n; \oplus, \circ, ^*, 0, 1)$ and $S_n = \{0, 1, n, \ldots, n^{-1}, 1\}$, which corresponds to $m$-relatively complete linearly ordered $MV$-algebra $S^k_n$. This subalgebra coincides with the greatest diagonal subalgebra, i.e. $d(S^k_n) = \{(x, \ldots, x) \in S^k_n : x \in S_n\}$. Denote this monadic $MV$-algebra by $(S^k_n, \exists_d)$. In this case the monadic operator $\exists_d$ is defined as follows: $\exists_d(x_1, \ldots, x_k) = (x_j, \ldots, x_j)$, where $x_j = \max(x_1, \ldots, x_k)$. The operator $\forall_d$ is defined dually: $\forall_d(x_1, \ldots, x_k) = (x_i, \ldots, x_i)$, where $x_i = \min(x_1, \ldots, x_k)$.

Notice that $K_n$ is generated by $(S^k_p, \exists_d)$, $p = 1, \ldots, n$ and $k \in \omega$. Moreover, $K_n$ is locally finite and there exists maximal $k \in \omega$, depending on $n$, such that $(S^k_n, \exists_d)$ is $m$-generated. The maximal $k$ we denote by $t(n)$. There exists also a positive number $r(k, n)$ depending on $k$ and $n$ such that $(S^k_n, \exists_d)^{r(k,n)}$ is $m$-generated. So,

**Theorem 5.**

$$\prod_{p=1}^{n} \prod_{k=1}^{t(p)} (S^k_p, \exists_d)^{r(k,p)}$$

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is a free $m$-generated algebra $F_{K_n}(m)$ in the variety $K_n$.

Let us notice, that exact description of one-generated free $MMV$-algebra in the variety $K_n$ is given in [3].

**Theorem 6.** The $m$-generated $MMV$-algebra $A$ from $K_n$ is projective iff $A$ is isomorphic to $(S^1_1, \exists_d) \times A'$.

**Theorem 7.** Any subalgebra of the free $m$-generated algebra $F_{K_n}(m)$ is projective.

Let $V_n$ be the variety generated by $\{S_1, \ldots, S_n\}$. Let us observe that

$$
\prod_{p=1}^n (S^1_{p}, \exists)^{(1,p)}
$$

is an algebra with trivial monadic operator $\exists$ (i.e. $\exists x = x$) which is isomorphic as an $MV$-algebra to the $m$-generated free $MMV$-algebra $F_{V_n}(m)$. Denote this algebra as $(F_{V_n}(m), \exists)$. It holds

**Theorem 8.** The $MMV$-algebra $(F_{V_n}(m), \exists)$ is a retract of the algebra of the free $m$-generated algebra $F_{K_n}(m)$. So, $(F_{V_n}(m), \exists)$ is projective.

### 4 Monadic operators on finite $MV$-algebras

Suppose that $A$ is a finite $MV$-algebra. Then $A \cong S_{n_1} \times S_{n_2} \times \ldots S_{n_k}$ where the $n_i \geq 1$. Let $\Pi = \{K_1, K_2, \ldots, K_m\}$ be a partition of $\{1, 2, \ldots, k\}$. We shall say that $\Pi$ is homogeneous if $i, j \in K_l$ implies $S_{n_i} = S_{n_j}$. Given such a $\Pi$, each $K_i$ has associated a unique $S_{n_i}$ which we shall denote by $A_i$. We clearly have $A \cong A_1^{K_1} \times \ldots \times A_m^{K_m}$. Since each $K_i$ is finite, there is a monadic operator $\exists_i$ defined on $A_i^{K_i}$ such that $(A_i^{K_i}, \exists_i)$ is an $MMV$-algebra with $\exists_i(A_i^{K_i}) = A_i$. Setting $\exists = \exists_1 \times \ldots \times \exists_m$ and acting pointwise, we obtain a monadic operator $\exists$ on $A$, that is, $(A, \exists)$ is an $MMV$-algebra. If a $K_i \in \Pi$ has at least two members, then determined the monadic operator will not be trivial, that is will not be the identity operator.

**Proposition 9.** Suppose that $A$ is a finite $MV$-algebra, say $A = S_{n_1} \times S_{n_2} \times \ldots S_{n_k}$.

(i) For each homogeneous partition $\{K_1, K_2, \ldots, K_m\}$ of $\{1, 2, \ldots, k\}$, there is a monadic operator defined on $A$. Conversely, each monadic operator defined on $A$ is determined by some homogeneous partition of $\{K_1, K_2, \ldots, K_m\}$.

(ii) If $A = S_n^k$, then any partition on $\{1, 2, \ldots, k\}$ determines a monadic operator on $A$ and conversely, each monadic operator on $A$ comes from some partition of $\{1, 2, \ldots, k\}$.

### 5 Unification problem

Let $V$ be a variety of algebras and $F_V(m)$ $m$-generated free algebra over the variety $V$. Recall that an algebra $A$ of $V$ is finitely presented if it is a quotient of the form $A = F_V(m)/\theta$, with $\theta$ a finitely generated congruence. Following [1], by an algebraic unification problem we mean a finitely presented algebra $A$ of $V$. An algebraic unifier for $A$ is a homomorphism $u : A \rightarrow P$ with $P$ a $m$-generated projective algebra in $V$ and $A$ is algebraically unifiable if such an algebraic unifier exists. Given another algebraic unifier $u : A \rightarrow Q$, we say that $u$ is more general than $w$, written $w \preceq u$, if there is a homomorphism $g : P \rightarrow Q$ such that $w = gu$. The algebraic unification type of an algebraically unifiable finitely presented algebra $A$ in the variety $V$ is
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now defined exactly as in the symbolic case, using the partially order $\leq$ induced by the quasi-order $\preceq$. Let $U_V(P)$ be the set of unifiers $\sigma : F_V(m) \rightarrow F_V(m)$ for the unification problem $P(x_1, \ldots, x_m)$; it is a quasi-ordered set. The problem $P(x_1, \ldots, x_m)$ is solvable iff $U_V(P) \neq \emptyset$. Let $(\Sigma, \leq)$ be a poset, where $\leq$ is the ordering induced by the quasi-ordering identifying the equivalence classes with its elements. $\text{Max} \Sigma$ is said to be basis of unifiers for $P$.

We say that an equational theory $E$ has:

1. Unification type 1 iff for every solvable unification problem $P$, $\text{Card}(\text{Max} \Sigma) = 1$.
2. Unification type $\omega$ iff for every solvable unification problem $P$, $\text{Card}(\text{Max} \Sigma) = n \neq 1$, $n \in \omega$.
3. Unification type $\infty$ iff for every solvable unification problem $P$, $\text{Card}(\text{Max} \Sigma)$ is infinite.
4. Unification type nullary, if none of the preceding cases applies.

We say that $V$ has finitary unification type iff it has type 1 or $\omega$.

**Theorem 10.** The unification type of the equational class $K_n$ is 1, i.e. unitary.

**References**