ABSTRACT

In this paper, the static balancing of existing spatial and planar parallel manipulators by the addition of balancing elements is addressed. Static balancing is defined here as the set of conditions on manipulator dimensional and inertial parameters which, when satisfied, ensure that the weight of the links does not produce any force (or torque) at the actuators for any configuration of the manipulator, under static conditions. These conditions are derived here for spatial six-degree-of-freedom parallel manipulators and it is shown that planar three-degree-of-freedom parallel manipulators can be treated as a particular case of the spatial 6-dof mechanisms. The static balancing conditions associated with planar mechanisms can therefore easily be found, but are not given here because of space limitations. A brief geometric interpretation of the balancing conditions which are associated with statically balanced spatial mechanisms is then carried out. It is shown that balancing is generally possible even when the dimensional parameters are imposed, which is a useful property since dimensional parameters are usually obtained from kinematic design or optimization. Finally, examples of balanced planar and spatial parallel manipulators are given. Static balancing leads to considerable reduction in the actuator forces (or torques), which in turn leads to less powerful actuators and more efficient designs. Moreover, the possibility of balancing existing systems by introducing additional elements, as demonstrated here, is of interest for retrofitting existing parallel mechanisms.

1 INTRODUCTION

Parallel mechanical architectures have been first introduced in tire-testing and motion simulation applications (Stewart 1965) and later on in robotics (see for instance (Hunt 1983), (Fichter 1986)). Parallel mechanisms are characterized by the fact that the end-effector is connected to the base via multiple kinematic chains and that all the actuators can be located on or close to the base. This leads to high stiffness and load-carrying capacity and to very good dynamic properties since the inertia of the moving parts is considerably reduced. However, current parallel mechanisms typically require powerful actuators to overcome the weight of the links. In commercial flight simulators, for instance, maintaining the platform in static equilibrium requires considerable power because of the large forces induced in the actuators which lead to hydraulic fluid losses. Hence, the kinetostatic design of parallel mechanisms is an important issue. Indeed, the reduction or elimination of the static forces at the actuators due to the weight of the links would lead to designs requiring less powerful actuators and would also improve the efficiency.

In this paper, the static balancing of spatial and planar parallel manipulators is addressed. Static balancing is first defined and the formulation allowing the derivation of the conditions associated with balanced mechanisms is presented. These conditions are then obtained for spatial six-degree-of-freedom parallel manipulators. It is also shown that planar three-degree-of-freedom parallel manipulators are in fact a particular case of the spatial 6-dof mechanisms. The static balancing conditions associated
with planar mechanisms can therefore easily be found, but are not given here because of space limitations. Before going any further, it is very important to point out that the objective of this work is to balance mechanisms that already exist and on which one can only add new elements. The motivation for this approach is that there are several parallel manipulators currently in use in the industry and since these costly mechanisms usually have a long life expectancy they cannot be easily replaced by newly designed mechanisms — featuring static balancing for instance — or have their intrinsic structure modified. Hence, because of this restriction, the static balancing of these mechanisms can be achieved only by adding new parts to the existing systems.

After obtaining the balancing conditions themselves, a brief geometric interpretation of these conditions is carried out. It is shown that balancing is generally possible even when the dimensional parameters are imposed, which is a useful property since dimensional parameters are usually obtained from kinematic design or optimization. Finally, examples of balanced planar and spatial parallel manipulators are given.

This paper focuses on spatial mechanisms. Moreover, the planar manipulators that are studied are assumed to move in a vertical plane since static balancing is irrelevant for planar manipulators moving in a horizontal plane. Applications of this work include flight simulators, positioning devices, manual manipulators, and high-performance robotic manipulators.

2 STATIC BALANCING OF MECHANISMS AND MANIPULATORS

The balancing of mechanisms has been an important research topic for several decades (Lowen et al. 1983; Bagci 1982). Mechanisms are said to be force-balanced when the total force applied by the mechanism on the fixed base is constant for any motion of the mechanism. In other words, a mechanism is force-balanced when its global center of mass remains fixed, for any arbitrary motion of the mechanism. This condition is very important in machinery since unbalanced forces on the base will lead to vibrations, wear and other undesirable side effects. For robotic manipulators or motion simulation devices, however, the forces on the base are usually not critical and designers are mostly concerned with the forces (or torques) which are required at the actuators to maintain the manipulator or mechanism in static equilibrium. Hence, in this context, manipulators or mechanisms are said to be statically balanced, when the weight of the links does not produce any force (or torque) at the actuators under static conditions, for any configuration of the manipulator or mechanism. This condition is also referred to as gravity compensation. Gravity-compensated serial manipulators have been designed in (Ulrich and Kumar 1991) and (Herve 1986) using cams, pulleys and/or springs. Also, gravity-compensation has been performed for planar 1-, 2- and 3-dof parallel manipulators in (Jean and Gosselin 1996). When springs are used, static balancing can be defined as the set of conditions for which the total potential energy in the mechanism (including gravitational energy and the elastic energy stored in the springs) is constant for any configuration of the mechanism. When no springs (or other means of storing elastic energy) are used, then static balancing conditions imply that the center of mass of the mechanism does not move in the direction of the gravity vector, for any motion of the mechanism.

The latter conditions will be used here in order to define static balancing and to apply it to spatial and planar parallel manipulators. Expressions for the coordinates of the center of mass of the mechanisms as functions of the dimensional and inertial properties, as well as the actuated joint coordinates, will first be derived. Then, the expression obtained for the coordinate associated with gravity ($z$ coordinate here) will be examined and the coefficients multiplying the functions of the joint coordinates will be set to zero, in order to lead to a constant value and hence, to the conditions for static balancing.

3 STATIC BALANCING OF SPATIAL 6-DOF PARALLEL MECHANISMS WITH PRISMATIC ACTUATORS

3.1 Gough-Stewart platform

A spatial six-degree-of-freedom parallel mechanism with prismatic actuators is illustrated in Fig. 1. This mechanism is known as the Gough-Stewart platform. It consists of six identical legs connecting the base to the platform. Each of these legs has

![Gough-Stewart platform](image-url)
the following design: a passive Hooke joint attached to the base, a first moving link, an actuated prismatic joint, a second moving link and a passive spherical joint attached to the platform. The coordinate frame of the base, designated as the \(O-x,y,z\) frame is fixed to the base with its \(Z\)-axis pointing vertically upward. Similarly, the moving coordinate frame \(O'-x',y',z'\) is attached to the platform.

The Cartesian coordinates of the platform are given by the position of point \(O'\) with respect to the fixed frame, noted \(p\) and the orientation of the platform (orientation of frame \(O' - x',y',z'\) with respect to the fixed frame), represented by matrix \(Q\)

\[
p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \quad Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}
\] (1)

where the entries can be expressed as functions of linear invariants, quadratic invariants, Euler angles or any other representation.

Additionally, the coordinates of points \(A_i\) and \(B_i\) relative to the fixed coordinate frame of the base are noted

\[
a_i = \begin{bmatrix} a_{ix} \\ a_{iy} \\ a_{iz} \end{bmatrix}, \quad b_i = \begin{bmatrix} b_{ix} \\ b_{iy} \\ b_{iz} \end{bmatrix}, \quad i = 1, \ldots, 6
\] (2)

Also, the coordinates of points \(B_i\) relative to the moving coordinate frame of the platform are noted

\[
b'_i = \begin{bmatrix} b'_{ix} \\ b'_{iy} \\ b'_{iz} \end{bmatrix}, \quad i = 1, \ldots, 6
\] (3)

Finally, let \(\rho_i\) be the length of the \(i\)th leg. This scalar is defined by taking the Euclidian norm of the vector \((b_i - a_i)\), which leads to

\[
\rho_i = \left\| b_i - a_i \right\|
\] (4)

As stated in the introduction, static balancing will be performed here on mechanisms that already exist and on which one can only add new elements. The first step is to verify whether or not the Gough-Stewart platform itself can be statically balanced. This is accomplished by finding \(r = [r_x, r_y, r_z]^T\), the position vector of the global center of mass of the mechanism, which in fact depends on the position of the center of mass of each and every link composing the mechanism, including the platform. As shown in Fig. 1 (See also Fig. 6), each leg has two parts: a cylinder (the lower part) and a piston (the upper part). The global center of mass of the mechanism can be written as

\[
r = m_{pf}c_{pf} + \sum_{i=1}^{6} \left( m_{li}c_{li} + m_{ui}c_{ui} \right) \frac{Q}{M}
\] (5)

where \(M\) is the total mass of all moving links of the mechanism, \(m_{pf}\), \(m_{li}\) and \(m_{ui}\) are respectively the masses of the platform, the lower link and the upper link of the \(i\)th leg, and

\[
M = m_{pf} + \sum_{i=1}^{6} (m_{li} + m_{ui})
\] (6)

while \(c_{pf}, c_{li}\) and \(c_{ui}\) are respectively the position vectors of the center of mass of the platform and of the lower and upper links of the \(i\)th leg, which are assumed to be located on the \(A_iB_i\) segment, for the sake of simplicity. Thus,

\[
c_{pf} = p + Qc'_{pf}
\] (7)
\[
c_{li} = a_i + \alpha_i(b_i - a_i), \quad i = 1, \ldots, 6
\] (8)
\[
c_{ui} = a_i + \beta_i(b_i - a_i), \quad i = 1, \ldots, 6
\] (9)

with

\[
b_i = p + Qb'_i, \quad i = 1, \ldots, 6
\] (10)

where \(c'_{pf}\) is the position vector of the center of mass of the platform expressed in its local reference frame, and whose components are given as

\[
c'_{pf} = \begin{bmatrix} c'_{pfx} \\ c'_{pfy} \\ c'_{pfz} \end{bmatrix}
\] (11)

Also, the dimensionless parameters \(\alpha_i\) and \(\beta_i\) are given as

\[
\alpha_i = \frac{d_{li}}{\rho_i}, \quad i = 1, \ldots, 6
\] (12)
\[
\beta_i = \frac{\rho_i - d_{ui}}{\rho_i}, \quad i = 1, \ldots, 6
\] (13)

where \(d_{li}\) is the position of the center of mass of the lower link relative to point \(A_i\), \(d_{ui}\) the position of the center of mass of the upper link relative to point \(B_i\) and \(\rho_i\) the variable representing the movement of the \(i\)th leg.
Substituting eqs. (7), (8) and (9) into eq. (5), one then obtains

\[
\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}
\] (14)

where

\[
r_x = \sum_{i=1}^{6} \left( \frac{D_i d_{xi}}{\rho_i} \right) - \rho_x \sum_{i=1}^{6} \left( \frac{D_i}{\rho_i} \right) - q_{11} \sum_{i=1}^{6} \left( \frac{D_i h_{xi}}{\rho_i} \right) - q_{12} \sum_{i=1}^{6} \left( \frac{D_i h_{yi}}{\rho_i} \right) - q_{13} \sum_{i=1}^{6} \left( \frac{D_i h_{zi}}{\rho_i} \right) \]

\[
r_y = \sum_{i=1}^{6} \left( \frac{D_i d_{yi}}{\rho_i} \right) - \rho_y \sum_{i=1}^{6} \left( \frac{D_i}{\rho_i} \right) - q_{21} \sum_{i=1}^{6} \left( \frac{D_i h_{xi}}{\rho_i} \right) - q_{22} \sum_{i=1}^{6} \left( \frac{D_i h_{yi}}{\rho_i} \right) - q_{23} \sum_{i=1}^{6} \left( \frac{D_i h_{zi}}{\rho_i} \right) \]

\[
r_z = \sum_{i=1}^{6} \left( \frac{D_i d_{zi}}{\rho_i} \right) - \rho_z \sum_{i=1}^{6} \left( \frac{D_i}{\rho_i} \right) - q_{31} \sum_{i=1}^{6} \left( \frac{D_i h_{xi}}{\rho_i} \right) - q_{32} \sum_{i=1}^{6} \left( \frac{D_i h_{yi}}{\rho_i} \right) - q_{33} \sum_{i=1}^{6} \left( \frac{D_i h_{zi}}{\rho_i} \right) \]

and where the coefficients are expressed as

\[
D_i = m_{ui} d_{ui} - m_{ui} d_{ui}, \quad i = 1, \ldots, 6
\] (15)

\[
E = m_{pf} + \sum_{i=1}^{6} m_{ui}
\] (16)

\[
\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m_{pf} \mathbf{c}_{pf} + \sum_{i=1}^{6} (m_{ui} \mathbf{b}_i)
\] (17)

\[
\mathbf{G} = \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \sum_{i=1}^{6} (m_{ui} \mathbf{a}_i)
\] (18)

\[G_x, G_y \text{ and } G_z \text{ being constant coefficients.}\]

Since no elastic components (springs) are used for the static balancing of this mechanism, it will be statically balanced provided its global center of mass does not move in the direction of the gravity vector, the Z-direction, for any motion of the mechanism. Sufficient conditions for \(r_z\) to be constant can then be written as

\[
D_i = E = 0, \quad i = 1, \ldots, 6
\] (19)

\[
\mathbf{F} = \mathbf{0}
\] (20)

However, from eq. (16), it is clear that the only way to satisfy eq. (19) is by forcing all the masses to be zero, which is a trivial solution. Therefore, it is impossible to statically balance the Gough-Stewart platform without adding other mechanical elements.

Since the mechanism has to be balanced, new links are added to the structure. However, these new parts must be placed in such a way that the mechanism then fulfills the following imperative condition: To statically balance a mechanism there must be at least one kinematic chain in which there are no prismatic joints between the base and the end-effector (in this case the platform). See (Ye and Smith 1994). This means that new legs with revolute joints have to be added.

The simplest way to statically balance the Gough-Stewart platform is to add an unactuated 6-dof leg having the following architecture: a passive revolute joint attached to the base, a first moving link, a passive Hooke joint, a second moving link and a passive spherical joint attached to the platform (Wang and Gosselin 1998). Thus, any parallel mechanism can become statically balanced. However, this often leads to a too restrictive set of conditions. One way to compensate that restriction is to add more than one leg. However, the more moving components one adds to a mechanism the larger the inertia, which is a drawback, from a dynamical point of view. Since one does not want to increase too much the inertia of the mechanism in achieving gravity compensation, springs can also be used. These components have low masses and thus, low inertia, and for many applications they are considered to have a negligible weight. Just like the added links, the springs must not be attached in a random way. For the given added 2-link leg that was mentioned above the springs proved to be useful only when attached from ground to the first (lower) moving link. Conversely, when springs are attached from ground to the second (upper) moving link or to the platform, the set of conditions leads to a zero-value of the spring stiffness constants. This is mainly because some elastic energy terms among the total potential energy equation cannot combine with the gravitational energy terms. One way to avoid that is to replace the above-mentioned balancing leg by a different one: a special leg with a parallelogram (Sreit and Shin 1990; Wang and Gosselin 1998) makes it possible to use more springs. The most important characteristic of this device is the fact that the attachment distances of the springs become constant relative to the pivot (see Fig. 2). Consequently, all the elastic energy terms combine with the gravitational energy terms and thereby, extra terms leading to zero-value spring stiffness constants are no longer present. In the following, four different architectures of legs based on the
parallelom principle are introduced for the static balancing of the Gough-Stewart platform. The same nomenclature as in the Gough-Stewart platform holds, except for the added parts.

3.2 One-stage parallelogram special leg

As represented in Fig. 2, each added leg is a planar mechanism with a parallelogram four-bar linkage $C_jD_jH_jG_j$ — used instead of the first link of the $j$th extra leg mentioned in the previous section — a distal link $D_jE_j$ and a spherical joint at point $E_j$. Having distances $G_jH_j$ and $C_jD_j$ identical and the same for distances $C_jG_j$ and $D_jH_j$, the parallelogram enables the attachment of a spring to the upper (distal) link of the leg and to a support that is maintained vertically. A spring is also attached to the parallelogram. In addition, each added leg is completely unactuated and mounted on a revolute joint having a vertical axis of rotation passing through point $F_j$.

Since new links are being added, new reference frames, as well as new matrices and vectors, have to be defined. A reference frame noted $O_{1j} = x_{1j}, y_{1j}, z_{1j}$ is attached to the parallelogram at point $F_j$, whose coordinates in the base coordinate frame are $(f_{jx}, f_{jy}, f_{jz})$, where $j = 1, ..., n$ and $n$ is the number of special legs added. The $Y_1j$-axis points in the $F_jC_j$ direction while the $Z_1j$-axis points vertically upward.

In addition, let $\gamma_j$ be the angle (joint variable) between the positive direction of the $X$-axis of the base coordinate frame and the $X_1j$-axis, where it is assumed that the latter axis is contained in the $XY$ plane of the fixed reference frame. One can then write the rotation matrix giving the orientation of frame $O_{1j} = x_{1j}, y_{1j}, z_{1j}$ with respect to the reference frame attached to the base as

\[
Q_{1j} = \begin{bmatrix}
\cos \gamma_j - \sin \gamma_j & 0 & 0 \\
\sin \gamma_j & \cos \gamma_j & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad j = 1, ..., n
\]

Similarly, a second reference frame noted $O_{2j} = x_{2j}, y_{2j}, z_{2j}$ is attached to the parallelogram, but this time at point $C_j$, whose coordinates in the base coordinate frame are $(c_{jx}, c_{jy}, c_{jz})$. The $Z_2j$-axis points from $C_j$ towards $D_j$ while the $X_2j$-axis is defined along coincident with the direction of the revolute joint axis at point $C_j$.

Furthermore, let $\theta_j$ be the joint variable of the revolute joint at point $C_j$ of the $j$th leg. One can then write the rotation matrix giving the orientation of frame $O_{2j} = x_{2j}, y_{2j}, z_{2j}$ with respect to the $O_{1j} = x_{1j}, y_{1j}, z_{1j}$ moving frame as

\[
Q_{2j} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_j - \sin \theta_j & 0 \\
0 & \sin \theta_j & \cos \theta_j
\end{bmatrix}, \quad j = 1, ..., n
\]

The center of mass of each of the five links of the special leg can now be defined. Vectors $c_{p0j}$, $c_{p1j}$, $c_{p2j}$, $c_{p3j}$ and $c_{p4j}$ represent respectively the position vector of the center of mass of links $G_jC_jI_j$, $C_jD_j$, $D_jE_j$, $G_jH_j$ and $H_jD_j$. With respect to the reference frame attached to the base, namely

\[
c_{p0j} = f_j + Q_{1j} (l_{FC} + r_{p0j}), \quad j = 1, ..., n
\]

\[
c_{p1j} = f_j + Q_{1j} (l_{FC} + Q_{2j} r_{p1j}), \quad j = 1, ..., n
\]

\[
c_{p2j} = d_j + Q_{2j} (e_j - d_j), \quad j = 1, ..., n
\]

\[
c_{p3j} = f_j + Q_{1j} (l_{FC} + l_{DH}), \quad j = 1, ..., n
\]

\[
c_{p4j} = f_j + Q_{1j} (l_{FC} + r_{p4j}), \quad j = 1, ..., n
\]

with

\[
d_j = f_j + Q_{1j} l_{FC} + Q_{1j} Q_{2j} r_{p1j}, \quad j = 1, ..., n
\]

\[
e_j = p + Qe_j', \quad j = 1, ..., n
\]

where $l_{FC}$, $r_{p0j}$, $l_{DH}$ and $r_{p4j}$ are respectively the vectors representing distances $F_jC_j$ and $D_jH_j$ and the center of mass of links $G_jC_jI_j$ and $H_jD_jJ_j$ with respect to their local reference frames — having the same orientation as the $O_{1j} = x_{1j}, y_{1j}, z_{1j}$ moving frame — located respectively at points $F_j$, $C_j$ and $D_j$; $l_{p1j}$, $r_{p1j}$ and $r_{p3j}$ are respectively the vectors representing the length of link $C_jD_j$ and the center of mass of links $C_jD_j$ and $G_jH_j$ with respect to their local reference frames — having the same orientation as the $O_{2j} = x_{2j}, y_{2j}, z_{2j}$ moving frame — located respectively at points $C_j$, and $G_j$; $d_j$, $e_j$, $f_j$ are the vectors representing respectively the position of points $D_j$, $E_j$ and $F_j$. 

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with respect to the fixed reference frame attached to the base and $\mathbf{e}_j$ is a vector representing point $E_j$ on the platform with respect to the moving reference frame attached to the latter element. As with the prismatic actuators, one can impose, for the sake of simplicity, that the center of mass of link $D_jE_j$ be located on the main axis of the link. Hence, scalars $r_{pj}$ and $l_{pj}$ represent respectively the position of the center of mass of link $D_jE_j$ relative to point $D_j$ and the length of link $D_jE_j$.

Eq.(5) then becomes

$$\mathbf{r} = \frac{m_{pf}\mathbf{e}_{pf} + \sum_{i=1}^{6}(m_{li}\mathbf{e}_{li} + m_{ui}\mathbf{e}_{ui}) + \sum_{j=1}^{n}\sum_{k=0}^{4}m_{pkj}\mathbf{e}_{pkj}}{M}$$

(31)

with

$$M = m_{pf} + \sum_{i=1}^{6}(m_{li} + m_{ui}) + \sum_{j=1}^{n}\sum_{k=0}^{4}m_{pkj}$$

(32)

where $m_{pkj}$ are the masses of the added links.

Finally, the gravitational potential energy $V_g$ is given by

$$V_g = Mgr_z$$

(33)

where $r_z$ is the component of vector $\mathbf{r}$ in the direction of the gravity field and $g$ is the gravitational acceleration.

However, since springs are also used for the static balancing, their contribution must be taken into account. Thereby, the total potential energy of the mechanism is in fact

$$V = V_g + V_e$$

(34)

where $V_e$, the elastic potential energy, can be written, for this mechanism, as

$$V_e = \frac{1}{2}\sum_{j=1}^{n}(k_i e_{ij}^2 + k_u e_{uj}^2)$$

(35)

where $k_{ij}$ is the stiffness of the lower spring of the $j$th leg and $e_{ij}$ its length, $k_{uj}$ is the stiffness of the upper spring of the $j$th leg and $e_{uj}$ its length. It is assumed here that the undeformed length of the springs is equal to zero in order to obtain complete balancing (Sreit and Shin 1990). As shown in the latter reference, this condition can easily be met in a practical design. Using the law of cosines, the lengths of the springs can be written as

$$e_{ij} = \sqrt{h_{ij}^2 + l_{ij}^2 - 2h_{ij}l_{ij}\cos \theta_j}, \quad j = 1, \ldots, n$$

(36)

$$e_{uj} = \sqrt{h_{uj}^2 + l_{uj}^2 - 2h_{uj}l_{uj}\cos \beta_j}, \quad j = 1, \ldots, n$$

(37)

where $h_{ij}$ and $l_{ij}$ are the distances from the revolute joint located at point $C_j$ to the attachment points of the lower spring (Fig. 2) while $h_{uj}$ and $l_{uj}$ are the distances from the revolute joint located at point $D_j$ to the attachment points of the upper spring. Finally, $\cos \beta_j$ can be expressed as a function of angle $\theta_j$, as well as the position and orientation of the platform, i.e.,

$$\cos \beta_j = \frac{e_{uj} - d_{uj}}{l_{pj}}$$

(38)

Substituting eqs.(33), (35), (36) and (37) into eq.(34), one obtains

$$V = p_z\left[\sum_{j=1}^{n}(\frac{D_j}{\rho_j}) \right] + \sum_{j=1}^{n}\left[\left(\frac{m_{pf}\mathbf{e}_{pf} + \sum_{i=1}^{6}(m_{li}\mathbf{e}_{li} + m_{ui}\mathbf{e}_{ui}) + \sum_{k=0}^{4}m_{pkj}\mathbf{e}_{pkj}}{M}\right)\right] +$$

$$\sum_{j=1}^{n}\left[\left(\frac{D_{d\text{ij}}}{\rho_i}\right) + \sum_{j=1}^{n}(P_j \sin \theta_j) + \sum_{k=0}^{4}(R_j \cos \theta_j) + S_z\right]$$

(39)

with the coefficients

$$D_j = m_{ui}d_{ui} - m_{ui}d_{ui}, \quad i = 1, \ldots, 6$$

(40)

$$L = \left(\frac{m_{pf} + \sum_{i=1}^{6}m_{ui}}{g}\right) + \sum_{j=1}^{n}\left(\frac{m_{pj}\mathbf{e}_{pf} + \sum_{i=1}^{6}(m_{li}\mathbf{e}_{li} + m_{ui}\mathbf{e}_{ui})}{l_{pj}}\right)$$

(41)

$$N = \left[\begin{array}{c} N_x \\ N_y \\ N_z \end{array}\right] = \left(\frac{m_{pf}\mathbf{e}_{pf} + \sum_{i=1}^{6}(m_{li}\mathbf{e}_{li} + m_{ui}\mathbf{e}_{ui})}{l_{pj}}\right) + \sum_{j=1}^{n}\left(\frac{m_{pj}\mathbf{e}_{pf}}{l_{pj}}\right)$$

(42)

$$P_j = m_{p1j}r_{p1j} + m_{p3j}r_{p3j}, \quad j = 1, \ldots, n$$

(43)

$$R_j = \left(\frac{m_{pj} + m_{pkj}}{l_{pj}}\right) + \left(\frac{m_{pj} + m_{pkj}}{l_{pj}}\right)$$

(44)

Coefficient $S_z$ is a constant term, regardless of the configuration of the mechanism.

As defined above, a mechanism is statically balanced only when its total potential energy, given by eq.(34), is constant. For this mechanism, sufficient conditions for $V$ to be constant can be written as

$$D_j = L = 0, \quad i = 1, \ldots, 6$$

(45)

$$N = 0$$

(46)

$$P_j = R_j = 0, \quad j = 1, \ldots, n$$

(47)

thus leading to

$$V = S_z = \text{cte}$$

(48)
By comparing this mechanism with the Gough-Stewart platform alone, one can see that the set of conditions is similar to the one obtained in the previous case. However, in the present case the springs make it possible to statically balance the platform. Moreover, the constraint given by eq.(16), which lead to a trivial solution, has now been replaced by eq.(41), the latter condition allowing the mechanism to be balanced. Similarly, all of the other constraint equations can be satisfied. Nevertheless, these constraints can sometimes be inadequate according to the targeted goals and the analysis of different leg designs then becomes of great interest. In the next subsection, a slightly modified version of the one-stage parallelogram special leg is introduced.

### 3.3 Two-stage parallelogram special leg

For the balancing of the mechanism, a leg composed of two parallelograms — one above the other — is now used. The reason for adding an extra parallelogram to the leg is to take advantage of an extra spring, the latter being attached from the upper parallelogram to the platform itself (Fig. 3). The purpose of this modification is to determine whether the conditions will become less restrictive and thus, allow more flexibility for the balancing of the mechanism.

Once again, all quantities defined in the previous case remain valid, with the only difference that more elements are now being added. Keeping the same reference frames, one can now define the center of mass for the two additional links on the special leg. Vectors \( \mathbf{c}_{p5j} \) and \( \mathbf{c}_{p6j} \), represent respectively the position vector of the center of mass of links \( K_jL_j \) and \( L_jE_jM_j \) with respect to the reference frame attached to the base, namely

\[
\mathbf{c}_{p5j} = \mathbf{d}_j + \frac{r_{p5j}}{p_{2j}} (\mathbf{e}_j - \mathbf{d}_j) + \mathbf{Q}_j \mathbf{L}_j \mathbf{E}_j, \quad j = 1, \ldots, n
\]

\[
\mathbf{c}_{p6j} = \mathbf{p} + \mathbf{Q}_j \mathbf{e}_j + \mathbf{Q}_j \mathbf{r}_{p6j}, \quad j = 1, \ldots, n
\]

where \( \mathbf{L}_j \mathbf{E}_j \) and \( \mathbf{r}_{p6j} \) are respectively vectors representing the distance \( L_jE_j \) and the position of the center of mass of link \( L_jE_jM_j \) with respect to their local reference frame — having the same orientation as the \( O_{1j} - x_{1j}, y_{1j}, z_{1j} \) moving frame — located at point \( E_j \). As in the one-parallelogram case, one can impose, for the sake of simplicity, that the center of mass of link \( K_jL_j \) be located on its main axis. The position of this center of mass relative to point \( K_j \) is given by scalar \( r_{p5j} \).

Eq.(31) then becomes

\[
\mathbf{r} = \frac{m_{pf} \mathbf{c}_{pf} + \sum_{i=1}^{6} (m_{li} \mathbf{e}_i + m_{ui} \mathbf{c}_i) + \sum_{j=1}^{n} \sum_{k=0}^{6} (m_{pkj} \mathbf{c}_{pkj})}{M}
\]

with

\[
M = m_{pf} + \sum_{i=1}^{6} (m_{li} + m_{ui}) + \sum_{j=1}^{n} \sum_{k=0}^{6} m_{pkj}
\]

where \( m_{pkj} \) are the masses of the added links.

The expression of the elastic potential energy for this mechanism then becomes

\[
V_e = \frac{1}{2} \sum_{j=1}^{n} \left( k_{ej} e_{ej}^2 + k_{uj} e_{uj}^2 + k_{ej} e_{ej}^2 \right)
\]

where \( k_{ej} \) is the stiffness of the spring connecting the platform to the upper parallelogram of the \( j \)th leg and \( e_{ej} \) its length. Using the law of cosines, the length of this third spring can be written as

\[
e_{ej} = \sqrt{h_{ej}^2 + l_{ej}^2 - 2h_{ej}l_{ej} \cos \epsilon_j}, \quad j = 1, \ldots, n
\]

where \( h_{ej} \) and \( l_{ej} \) are the distances from the revolute joint located at point \( E_j \) to the attachment points of the third spring on the platform (Fig. 3). Moreover, \( \cos \epsilon_j \) can be expressed as a function of the position and orientation of the platform, i.e.,

\[
\cos \epsilon_j = \frac{r_{zj} - e_{zj}}{l_{ej}}, \quad j = 1, \ldots, n
\]

where \( r_{zj} \) is the Z-component of vector \( \mathbf{r}_j \), the latter representing the position of point \( R_j \) — the location on the platform where the
spring is attached — with respect to the fixed reference frame attached to the base.

As in the one-parallelogram case, the expression of the total potential energy is given by eq.(39), but with different coefficients, i.e.

\[ D_i = m_i d_i - m_i d_{di}, \quad i = 1, \ldots, 6 \]  
\[ L = \left( m_{pf} + \sum_{i=1}^{6} m_{ai} \right) g + \]
\[ \sum_{j=1}^{n} \left( \frac{m_p f \beta_j + m_p s \gamma_j}{l_p j} - \frac{k_a \delta_j l_j}{l_p j} \right) \]  
\[ N = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} = \left( m_{pf} x_{pf} + \sum_{i=1}^{6} m_{ai} b_{ai} \right) g + \]
\[ \sum_{j=1}^{n} \left( \frac{m_p f \beta_j + m_p s \gamma_j}{l_p j} - \frac{k_a \delta_j l_j}{l_p j} \right) \]  
\[ P_j = m_p f \theta_j b_{pf} + m_p s \gamma_j, \quad j = 1, \ldots, n \]  
\[ R_j = \left( \frac{m_p f \beta_j + m_p s \gamma_j}{l_p j} \right) g - \frac{k_a \delta_j l_j}{l_p j} \]  
\[ \delta_j = 1, \ldots, n \]  

The newly obtained coefficient \( S_z \) is a constant term, regardless of the configuration of the mechanism.

Finally, for the set of conditions, the same equations as the ones resulting from the previous case, i.e., eqs.(45), (46) and (47), are obtained while \( V \) is given by eq.(48).

### 3.4 Other designs of legs using the parallelogram principle

In addition to the previously proposed architectures for the balancing legs (Figs. 2 and 3), it is possible to statically balance parallel mechanisms with the alternative architectures shown in Figs. 4 and 5.

The leg shown in Fig. 4 is very similar to the one shown in Fig. 2. The main differences are in the way the joints are arranged. Unlike the leg of Fig. 2, the alternative leg presented here cannot rotate around a Z-axis (through \( F_j \) and in the direction of gravity) and the revolute joint at point \( D_j \) has been replaced by a Hooke joint. It is also interesting to point out that both legs result in the same set of balancing conditions.

As for the architecture of Fig. 5, it still uses the parallelogram principle, but somewhat differently (Sreit and Shin 1990). Links 1 and 3 are not rigidly attached to each other, thus allowing them to move independently. The main advantage of this architecture is the fact that the springs are closer to the base and attached to a link that does not move vertically. Finally, as in the previous cases, a similar set of balancing condition is obtained.
4 STATIC BALANCING OF PLANAR 3-DOF PARALLEL MECHANISMS WITH PRISMATIC ACTUATORS

For the static balancing of planar parallel mechanisms with prismatic actuators, the same equations as those obtained for the spatial mechanisms are used. Indeed, the analysis of 3-dof planar mechanisms is simply a special case of what was previously developed for 6-dof spatial mechanisms. Since the mechanisms are constrained to have three dofs and to move in a 2-D space, a few parameters must be changed.

First, the legs themselves must reduce to three dofs. For the legs with prismatic actuators, each of them has the following design: a passive revolute joint attached to the base, a first moving link, an actuated prismatic joint, a second moving link and a passive revolute joint attached to the platform (Gosselin and Angeles 1988). Also, the number of these legs must be reduced from six to three. Thereby, the equations become valid only with  \(i = 1, \ldots, 3\).

Similarly, the added special legs composed of parallelograms reduce to three dofs. Thus, for a given leg, the Hooke joint connecting the leg to the base must be replaced by a single revolute joint. The same applies to the spherical joint connecting the leg to the platform.

Moreover, one component of all of the 3-dimensional vectors must be simply ignored. If it is assumed that this component is the \(X\)-component, then all the vectors become 2-dimensional vectors. They are all contained in a same \(YZ\)-plane and the \(X\)-component is no longer taken into account.

From a mathematical point of view, this amounts to forcing all the \(X\)-terms in the equations for spatial mechanisms to be zero. Also, by forcing angle \(\gamma_j\) to be zero, the associated rotation matrix \(Q_{ij}\) becomes equal to the identity matrix.

Hence, with the slight modifications stated above, the conditions found earlier can easily apply to the 3-dof planar case and this case will not be further discussed here.

5 GEOMETRIC INTERPRETATION OF THE BALANCING CONDITIONS

5.1 Design parameters

For the parallel mechanisms introduced above, the \(p\)-dimensional space defined by the design parameters (variables \(m_{li}, m_{ui}, d_{li}\) and \(d_{ui}\) for \(i = 1, \ldots, m\); variables \(m_{pkj}, r_{pkj}, l_{pkj}, \alpha_{ij}, \beta_{ij}, \beta_{uj}, \beta_{uj}, \alpha_{ij}\) and \(l_{uj}\), and vectors \(e_j^l\) and \(e_j^r\) for \(j = 1, \ldots, n\) and \(k = 1, \ldots, 6\) can be thought of as the design space, in which each point defines a mechanism. In this multi-dimensional space, the balancing conditions define a subspace which can be defined as the set of balanced mechanisms. Since the number of parameters is relatively large and the balancing conditions do not introduce many constraints, the set of balanced mechanisms is also a multi-dimensional space. In other words, an infinity of solutions can be obtained. However, it has to be kept in mind that balancing is performed here only by adding new links. Nothing can be removed from original structure of the existing Gough-Stewart platform and this has to be taken into account in the design parameters.

5.2 Prismatic actuators

Now, referring to the mechanisms of the above sections, a closer look at eqs.(15), (40) and (56) leads to the conclusion that for all of these mechanisms the same conditions hold for the prismatic actuators, i.e.,

\[
m_{li}d_{li} = m_{ui}d_{ui}, \quad i = 1, \ldots, 6
\]

From that equation, two important consequences follow. First, since the equation does not include any parameter from the platform and/or the added legs, the prismatic actuators must then be statically balanced themselves, regardless of the other elements composing the system. The second consequence is a little less obvious. As shown in Fig. 6, for the \(i\)th prismatic actuator, one can express \(z_i\) — the vertical (direction of gravity) coordinate of its global center of mass — as a function of \(\rho_i\) — its total length — and \(\alpha_i\) — the angle it makes with the \(XY\)-plane of the fixed reference frame of the base. It is given by

\[
z_i = \frac{m_{li}z_{li} + m_{ui}z_{ui}}{m_{li} + m_{ui}}, \quad i = 1, \ldots, 6
\]

where

\[
z_{li} = d_{li}\sin\alpha_i, \quad i = 1, \ldots, 6
\]
\[
z_{ui} = (\rho_i - d_{ui})\sin\alpha_i, \quad i = 1, \ldots, 6
\]
Substituting eq.(61) into eq.(62) one then obtains
\[ z_i = \frac{m_{ui}}{m_{ii} + m_{ui}} p_i \sin \alpha_i, \quad i = 1, \ldots, 6 \]  
(63)

Now, since \( p_i \sin \alpha_i \) is in fact the Z-coordinate \( b_i \) of point \( B_i \), eq.(63) is then written as
\[ z_i = \frac{m_{ui}}{m_{ii} + m_{ui}} (p_z + [\mathbf{Q}]_{ii} b_i), \quad i = 1, \ldots, 6 \]  
(64)

with \( z_i \) expressed in the fixed reference frame of the base. From eq.(64), it is clear that the global center of mass of the \( n \)th leg depends only on the height of the platform at point \( B_i \), regardless of the orientation of the leg. In other words, for any motion of the platform for which point \( B_i \) is kept at a constant height, the height of the center of mass of the associated prismatic actuator will also be constant.

### 5.3 Particular geometries

Another point of interest is the effect of symmetry on the studied mechanisms. Symmetries can be introduced in order to simplify the equations representing the balancing conditions, which leads to an easier physical understanding of these equations, without the use of numerical data. Consider the Gough-Stewart platform with the one-parallelogram special legs (Fig. 2). As a simplification, it is first assumed that the \( n \) special legs that might be added are all identical. The same applies to the \( m \) prismatic actuators (here, \( m = 6 \)). Thereafter, substituting eq.(41) into eqs.(42) and (44), equating them to zero and rearranging, the set of conditions becomes
\[ m_{ad} d_a = m_1 d_1 \]  
(65)
\[ k_1 w_d l_a = \left( \frac{m_{pf} + m_{ad}}{n} \right) l_{p 2} + m_{p2} r_{p2} \]  
(66)
\[ m_{pf} \left( \frac{1}{n} \sum_{j=1}^{n} e_j - \frac{e_{pf}}{n} \right) = m_u \left( \sum_{j=1}^{n} b_j - m \sum_{j=1}^{n} e_j \right) \]  
(67)
\[ m_{p1} r_{p1y} = -m_{p3} r_{p3y} \]  
(68)
\[ k_1 h l_1 = \left( \frac{m_{pf} + m_{ad}}{n} + m_{p2} + m_{p4} \right) l_{p1} + m_{p1} r_{p1z} + m_{p3} r_{p3z} \]  
(69)

Now, symmetrical conditions are imposed. Although there are actually numerous ways in which symmetry can simplify the equations, only one option is treated here. Hence, assuming that all of the \( m \) legs with prismatic actuators are placed symmetrically with respect to the center of mass of the platform, i.e.,
\[ \sum_{i=1}^{m} (b_i' - \frac{e_{pf}}{n}) = 0 \]  
(70)
eq(67) reduces to
\[ \sum_{j=1}^{n} e_j = n e_{pf} \]  
(71)

When \( n = 1 \), eq.(71) then simplifies to
\[ e_1 = e_{pf} \]  
(72)
and thus, it is clear that the leg with the parallelogram must be attached to the center of mass of the platform. However, this location might not always be accessible for the attachment of new parts because of the design constraints. That is why the use of more than one balancing leg is considered here. When \( n = 2 \), eq.(71) becomes
\[ \frac{e_1 + e_2}{2} = e_{pf} \]  
(73)
The latter equation shows that the center of mass of the platform must be located exactly at the middle point of the line segment formed by points \( E_1 \) and \( E_2 \). Finally, when \( n = 3 \), eq.(71) can be written as
\[ \frac{e_1 + e_2 + e_3}{3} = e_{pf} \]  
(74)

By analogy to the previous case, the center of mass of the platform must now be located at the centroid of the triangle formed by points \( E_1 \), \( E_2 \) and \( E_3 \). However, the above result is valid only when \( e_1 \), \( e_2 \) and \( e_3 \) are coplanar. For \( n > 3 \), the same kind of analysis can be performed.

Consider now the Gough-Stewart platform with the two-parallelogram special legs (Fig. 3). Making the same assumptions and performing the same substitutions as those made in the one-parallelogram leg case, similar equations to eqs.(65)-(69) are found. However, for the geometric interpretation being done here, only one equation is important. The latter equation is in fact the only one where \( k_e \) appears and is given by
\[ k_e h \sum_{j=1}^{n} (e_j' - r_{pf}) = m_{pf} \left( \frac{1}{n} \sum_{j=1}^{n} e_j' - \frac{e_{pf}}{n} \right) g + m_u \left( \frac{m}{n} \sum_{j=1}^{n} e_j - \frac{n}{n} \sum_{j=1}^{n} b_j' \right) g \]  
(75)

Similarly to the one-parallelogram case, symmetry is imposed. It is assumed that all of the \( m \) legs with prismatic actuators and all of the \( n \) special legs are placed symmetrically with respect to the center of mass of the platform. These two conditions are given by
\[ \sum_{i=1}^{m} (b_i' - \frac{e_{pf}}{n}) = 0 \]  
(76)
\[ \sum_{j=1}^{n} (e'_{j} - e'_{pf}) = 0 \]  
(77)

Eq.(75) then reduces to

\[ \sum_{i=1}^{n} r'_{i} = ne'_{pf} \]  
(78)

Variable \( k_{e} \) does not appear in the latter equation. This means that the springs connecting the legs to the platform are in this case useless. In fact, the symmetry pattern made by the \( n \) special legs already makes possible the static balancing, as it has been demonstrated for the mechanism with one-parallelogram legs. Therefore, it becomes obvious that these springs become useful only when there is no symmetry of the added legs with respect to the center of mass of the platform.

In conclusion to that section, it is pointed out that only a partial geometric analysis has been accomplished here and that there are many other possibilities that will not be discussed in this paper.

6 EXAMPLES OF STATICALLY BALANCED MANIPULATORS

Using the balancing conditions developed above, examples of statically balanced manipulators are easily found. Following the approach presented in the preceding section, some constraints can be imposed (link lengths, masses, etc.) and the remaining families of balanced mechanisms can be investigated. Optimization of additional criteria could also be performed.

In Figs. 7 and 8, simple examples of planar 3-dof and spatial 6-dof statically balanced mechanisms are illustrated. No torque is produced at the actuators by the weight of the links, for any configuration of these mechanisms. Hence, the mechanisms can be left at rest in any configuration.

7 CONCLUSION

In this paper, the static balancing of spatial and planar parallel manipulators has been addressed. Static balancing has been defined here as the set of conditions on manipulator dimensional and inertial parameters which, when satisfied, ensure that the weight of the links does not produce any force (or torque) at the actuators for any configuration of the manipulator, under static conditions. Conditions for the static balancing of spatial 6-dof parallel manipulators have been obtained. It has been shown that since the static balancing of planar 3-dof parallel manipulators is in fact a particular case of the balancing of the spatial 6-dof mechanisms, the associated conditions, which are very similar, can easily be found. Because the number of constraints introduced by the static balancing is not very large, balancing is generally possible even when the dimensional parameters are imposed, which is a useful property since dimensional parameters are usually obtained from kinematic design or optimization. Also, it is important to remember that the balanc-
ing has, in this case, been achieved for existing mechanisms on which only the addition of new parts is possible. A brief geometric interpretation of these static constraints which are associated with statically balanced mechanisms has been carried out. Finally, simple examples of balanced planar and spatial parallel manipulators have been given. Static balancing leads to a considerable reduction in the actuator forces (or torques), which in turn leads to less powerful actuators and more efficient designs. This is of great interest in the design of manipulators and motion simulation systems. Future work includes the optimization of the design of the balancing legs that are to be added to existing planar 3-dof and spatial 6-dof parallel manipulators and the fabrication of balanced prototypes.

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