A SUMMATION METHOD DUE TO CARR: PART 1

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Abstract. We present a summation method extending an idea found in Carr's synopsis.

1. Introduction

The collection of formulas by Carr [3] was made famous by being one of the few texts available to S. Ramanujan in India. A complete discussion of these texts has been given by Berndt and Rankin in [1]. In statement 2708 of [3] we find an identity between a series and an integral, which we state as our first theorem.

Theorem 1.1. Let $f_k(x)$ be a sequence of functions and assume the expansion of φ in terms of $\{f_k\}$

$$\varphi(x) = \sum_{k=0}^{\infty} A_k f_k(x) \tag{1.1}$$

converges uniformly. For any function $X(x)$, normalized by

$$\int_a^b X(x) f_0(x)\, dx = 1, \tag{1.2}$$

define the coefficients C_k by the relation

$$C_k = \int_a^b X(x) f_k(x) dx. \tag{1.3}$$

Then

$$\sum_{k=0}^{\infty} A_k C_k = \int_a^b X(x)\varphi(x) dx. \tag{1.4}$$

Proof. Evaluate the integral by replacing the expansion (1.1) in (1.4). □

Note. Observe that the identity (1.4) stays the same without the normalization (1.2). This relation expresses an identity between a series and a definite integral.

We first state three corollaries of Theorem 1.1 that will be used later. The proofs are direct.

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Corollary 1.2. Let $f_k(x) = x^k,\ k \geq 0$ and $X(x) = x^{\beta-1}e^{-x},\ \beta > 0$. Suppose

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^k.$$

Then

$$\sum_{k=0}^{\infty} A_k \Gamma(k+\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} \varphi(x)\, dx. \tag{1.5}$$

In particular, for $\beta = 1$, we obtain

$$\sum_{k=0}^{\infty} k!\, A_k = \int_0^{\infty} e^{-x} \varphi(x)\, dx. \tag{1.6}$$

Corollary 1.3. Let $p > 0$ be fixed, $f_{k,p}(x) = x^{k/p},\ k \geq 0$, and $X(x) = x^{\beta-1}e^{-x},\ \beta > 0$. Suppose

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^{k/p}.$$

Then

$$\sum_{k=0}^{\infty} A_k \Gamma(k/p+\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} \varphi(x)\, dx. \tag{1.7}$$

Corollary 1.4. Let $p > 0$ be fixed, $f_k(x) = x^k,\ k \geq 0$, and $X(x) = x^{\beta-1}e^{-x^p},\ \beta > 0$. Suppose

$$\varphi(x) = \sum_{k=0}^{\infty} A_k x^k.$$

Then

$$\sum_{k=0}^{\infty} A_k \Gamma\left(\frac{\beta+k}{p}\right) = p \int_0^{\infty} x^{\beta-1} e^{-x^p} \varphi(x)\, dx. \tag{1.8}$$

In this paper we provide examples of evaluations of series and definite integrals that result from Theorem 1.1 and its corollaries. The first few sections contain examples obtained by choosing the function $\varphi(x)$ appropriately. Section 2 considers the exponential function, Section 3 hyperbolic functions, Section 4 the incomplete gamma function $\Gamma(\alpha, x)$, and Section 5 evaluates definite integrals involving Bessel functions. In Section 6 we obtain identities by prescribing the coefficients of the Taylor expansion of $\varphi(x)$. Section 7 presents an example in which the basic family $\{f_k\}$ is modified.

2. The exponential

Example 2.1. Let $a < 1$ and apply Corollary 1.2 to

$$\varphi(x) = e^{ax} = \sum_{k=0}^{\infty} \frac{a^k}{k!} x^k \tag{2.1}$$

to obtain

$$\sum_{k=0}^{\infty} \frac{a^k}{k!}\Gamma(\beta+k) \quad = \quad \int_0^\infty x^{\beta-1} e^{-(1-a)x} dx = \frac{\Gamma(\beta)}{(1-a)^\beta}. \tag{2.2}$$

Note. Using the Pochhammer symbol

$$(\beta)_k \quad := \quad \frac{\Gamma(\beta+k)}{\Gamma(\beta)} = \beta(\beta+1)(\beta+2)\cdots(\beta+k-1), \tag{2.3}$$

the identity (2.2) becomes the binomial expansion

$$\sum_{k=0}^{\infty} a^k \binom{\beta+k-1}{k} = \sum_{k=0}^{\infty} \frac{a^k(\beta)_k}{k!} \quad = \quad \frac{1}{(1-a)^\beta}. \tag{2.4}$$

The presence of a free parameter in Example 2.1 permits differentiation of (2.1) with respect to β. This results in the next example.

Example 2.2. Let $a < 1$. Then

$$\sum_{k=0}^{\infty} \frac{a^k}{k!}\Gamma'(\beta+k) \quad = \quad \frac{\Gamma(\beta)}{(1-a)^\beta} \times \left[\frac{\Gamma'(\beta)}{\Gamma(\beta)} - \ln(1-a) \right]. \tag{2.5}$$

Note. Introduce the function $\psi(x) = \Gamma'(x)/\Gamma(x)$ (the logarithmic derivative of $\Gamma(x)$) to write (2.5) as

$$\sum_{k=0}^{\infty} \frac{a^k}{k!}\psi(\beta+k)\Gamma(\beta+k) \quad = \quad \frac{\Gamma(\beta)}{(1-a)^\beta} \left[\psi(\beta) - \ln(1-a) \right]. \tag{2.6}$$

Now let $\beta \to 1$ and recall that $\gamma = -\psi(1)$ is Euler's constant to obtain

$$\sum_{k=1}^{\infty} \psi(k) a^k \quad = \quad \frac{a}{a-1} \times (\gamma + \ln(1-a)). \tag{2.7}$$

Corollary 2.1. The function ψ satisfies $\psi(k) = -\gamma + H_{k-1}$, where $H_0 = 0$ and $H_{k-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}$ is the $(k-1)-$th harmonic number.

Proof. Compare the coefficients of a^k on both sides of (2.7). □

Example 2.3. Apply Corollary 1.4 to the expansion

$$\varphi(x) \quad = \quad e^{ax} = \sum_{k=0}^{\infty} \frac{a^k}{k!} x^k \tag{2.8}$$

to produce

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \Gamma\left(\frac{k+\beta}{p}\right) \quad = \quad p \int_0^\infty x^{\beta-1} e^{ax-x^p}\, dx. \tag{2.9}$$

Special case: For $\beta = 1$ and $p = 2$ we obtain

$$\int_0^\infty e^{-x^2+ax}\, dx \quad = \quad \frac{1}{2} \sum_{k=0}^{\infty} \Gamma\left(\frac{k+1}{2}\right) \frac{a^k}{k!}. \tag{2.10}$$

The integral can be evaluated in terms of the *error function*

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt, \tag{2.11}$$

yielding

$$\sum_{k=0}^{\infty} \Gamma\left(\frac{k+1}{2}\right) \frac{a^k}{k!} = \sqrt{\pi} e^{a^2/4} \left(1 + \text{erf}\left(\frac{a}{2}\right)\right). \tag{2.12}$$

Note. The right hand side is

$$\sqrt{\pi} \left(\sum_{j=0}^{\infty} \frac{a^{2j}}{2^{2j} j!} \right) \times \left(1 + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{a^{2k+1}}{2^{2k+1}(2k+1)} \right) =$$

$$\sqrt{\pi} \sum_{j=0}^{\infty} \frac{a^{2j}}{2^{2j} j!} + 2 \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!(2k+1)} \right\} \frac{a^{2n+1}}{2^{2n}},$$

where we have separated the odd and even powers. Then (2.12) follows from the identities

$$\Gamma(k+1/2) = \frac{\sqrt{\pi}(2k)!}{2^{2k} k!} \quad \text{and} \quad \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!(2k+1)} = \frac{2^{2n} n!}{(2n+1)!}.$$

Special case: $\beta = p$ gives

$$\sum_{k=1}^{\infty} \frac{a^k}{(k-1)!} \Gamma(k/p) = p^2 \int_0^{\infty} t^{p-1} e^{at-t^p}\,dt \tag{2.13}$$

$$= p \int_0^{\infty} e^{av^{1/p}-v}\,dv.$$

We now express the series in (2.13) as a finite sum of hypergeometric terms. For $p \in \mathbb{N}$ we write $k = np + j$ with $n = 0, 1, \cdots$ and $1 \leq j \leq p-1$, so that

$$\sum_{j=1}^{p} a^j \Gamma\left(\frac{j}{p}\right) \sum_{n=0}^{\infty} \frac{a^{np}}{(np+j-1)!} \left(\frac{j}{p}\right)_n = p \int_0^{\infty} e^{av^{1/p}-v}\,dv. \tag{2.14}$$

Now define

$$\mathbf{P} := \left\{ \frac{1+j}{p}, \frac{2+j}{p}, \cdots, \frac{p-1+j}{p} \right\}. \tag{2.15}$$

The inner sum in (2.14) can be identified as a hypergeometric series and we obtain

$$\int_0^{\infty} e^{av^{1/p}-v}\,dv = \frac{1}{p} \sum_{j=1}^{p} \frac{a^j}{(j-1)!} \Gamma(j/p)\ {}_1F_{p-1}\left[\begin{matrix} 1 \\ \mathbf{P} \end{matrix}; \left(\frac{a}{p}\right)^p \right]. \tag{2.16}$$

Example 2.4. Applying Corollary 1.2 to the function

$$e^{-x^p} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{kp} \tag{2.17}$$

yields

$$\int_0^{\infty} x^{\beta-1} e^{-x-x^p}\,dx = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma(\beta + jp). \tag{2.18}$$

Special case: $\beta = 1$ gives

$$\int_0^\infty e^{-x-x^p}dx \quad = \quad \sum_{j=1}^\infty \frac{(-1)^j}{j!}(jp)! \tag{2.19}$$

for $0 < p < 1$. The specific case $p = 1/2$ results in

$$\sum_{j=1}^\infty \frac{(-1)^j}{j!}(p/2)! \quad = \quad 1 - \frac{1}{2}e^{1/4}\sqrt{\pi} + \frac{1}{2}e^{1/4}\sqrt{\pi}\ \text{erf}\,(1/2). \tag{2.20}$$

3. Hyperbolic functions

In this section we explore the consequences of Theorem 1.1 on the expansions

$$\sinh bx \quad = \quad \sum_{j=0}^\infty \frac{b^{2j+1}}{(2j+1)!}x^{2j+1} \tag{3.1}$$

and

$$\cosh bx \quad = \quad \sum_{j=0}^\infty \frac{b^{2j}}{(2j)!}x^{2j}. \tag{3.2}$$

Example 3.1. Corollary 1.2 applied to the expansion (3.1) yields

$$\int_0^\infty x^{\beta-1}e^{-x}\sinh bx\,dx \quad = \quad \sum_{j=0}^\infty \frac{b^{2j+1}}{(2j+1)!}\Gamma(2j+1+\beta). \tag{3.3}$$

This also follows directly from (3.1).

Special cases: $\beta = 1$ gives

$$\int_0^\infty e^{-x}\sinh bx\,dx \quad = \quad \sum_{j=0}^\infty b^{2j+1} = \frac{b}{1-b^2}, \tag{3.4}$$

a result from elementary calculus.

• $\beta = 1/2$ gives

$$\int_0^\infty e^{-x}x^{-1/2}\sinh bx\,dx \quad = \quad \sum_{j=0}^\infty \frac{b^{2j+1}}{(2j+1)!}\Gamma(2j+3/2). \tag{3.5}$$

Evaluating the integral and letting $b = 4\sqrt{c}$ yields the classical evaluation

$$\sum_{j=0}^\infty \binom{4j+1}{2j+1}c^j \quad = \quad \frac{2}{\sqrt{c}}\left[\frac{1}{\sqrt{1-4\sqrt{c}}} - \frac{1}{\sqrt{1+4\sqrt{c}}}\right]. \tag{3.6}$$

Example 3.2. Apply Corollary 1.2 to the function

$$\frac{\sinh b\sqrt{x}}{b\sqrt{x}} \quad = \quad \sum_{k=0}^\infty \frac{b^{2j}\,x^j}{(2j+1)!} \tag{3.7}$$

to obtain

$$\int_0^\infty x^{\beta-1}e^{-x}\frac{\sinh b\sqrt{x}}{b\sqrt{x}}\,dx \quad = \quad \sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!}b^{2j}, \tag{3.8}$$

so that

$$\int_0^\infty t^{2\beta-2} e^{-t^2/b^2} \sinh t \, dt = \frac{1}{2} \sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!} b^{2(j+\beta)}. \tag{3.9}$$

Special case: $\beta = 1$ produces

$$\int_0^\infty e^{-t^2/b^2} \sinh t \, dt = \frac{1}{2} \sum_{j=0}^\infty \frac{j!}{(2j+1)!} b^{2(j+1)}. \tag{3.10}$$

Evaluating the integral and replacing $b/2$ by b gives

$$\sum_{j=0}^\infty \frac{j! \, b^{2j+1}}{(2j+1)! 2^{2j+2}} = 2\sqrt{\pi} e^{b^2} \operatorname{erf}(b). \tag{3.11}$$

Example 3.3. Apply Corollary 1.2 to

$$\varphi(x) = \sqrt{x} \sinh \sqrt{x} = \sum_{k=1}^\infty \frac{1}{\Gamma(2k)} x^k \tag{3.12}$$

to produce

$$\int_0^\infty x^{\beta-1/2} e^{-x} \sinh \sqrt{x} \, dx = \sum_{k=1}^\infty \frac{\Gamma(\beta+k)}{\Gamma(2k)}. \tag{3.13}$$

Mathematica evaluates both sides of (3.13) as

$$\Gamma(1+\beta) \times \ {}_1F_1[1+\beta, \tfrac{3}{2}; \tfrac{1}{4}]. \tag{3.14}$$

The special case $\beta = 1$ yields

$$\sum_{k=1}^\infty \frac{k \, k!}{(2k)!} = \frac{1}{8} \left(2 + 3 e^{\frac{1}{4}} \sqrt{\pi} \operatorname{erf}(1/2) \right). \tag{3.15}$$

4. The incomplete gamma function

The incomplete gamma function, defined by

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} \, dt, \tag{4.1}$$

admits the expansion

$$\Gamma(\alpha, x) = \Gamma(\alpha) + \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n!(\alpha+n)} x^{\alpha+n}. \tag{4.2}$$

Apply Corollary 1.2 to the function

$$G_\alpha(x) := \frac{\Gamma(\alpha, x) - \Gamma(\alpha)}{x^\alpha} = \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n!(\alpha+n)} x^n \tag{4.3}$$

to obtain

$$\int_0^\infty x^{\beta-1} e^{-x} G_\alpha(x) \, dx = \sum_{k=0}^\infty \frac{(-1)^{k+1} \, \Gamma(\beta+k)}{k!(\alpha+k)}. \tag{4.4}$$

Now simplify the integral and let $c = \beta - \alpha$ to produce

$$\int_0^\infty x^{c-1} e^{-x} \Gamma(\alpha, x)\, dx \quad = \quad \Gamma(\alpha)\Gamma(c) + \sum_{k=0}^\infty \frac{(-1)^{k+1}\, \Gamma(c+\alpha+k)}{k!\,(\alpha+k)}. \tag{4.5}$$

The special case $c = 1$ yields

$$\int_0^\infty e^{-x} \Gamma(\alpha, x)\, dx \quad = \quad \sum_{k=1}^\infty \frac{(-1)^{k+1}\, \Gamma(k+\alpha)}{k!}, \tag{4.6}$$

and the further specialization $\alpha = 0$ gives

$$\int_0^\infty e^{-x} \Gamma(0, x)\, dx \quad = \quad \ln 2. \tag{4.7}$$

Lemma 4.1.

$$\int_0^x \frac{e^{-t} - 1}{t} dt \quad = \quad -\gamma - \ln x - \Gamma(0, x) \tag{4.8}$$

Proof. We have

$$\int_0^x \frac{e^{-t}-1}{t} dt \quad = \quad - \left[\int_0^1 \frac{1-e^{-t}}{t} dt - \int_x^1 \frac{1-e^{-t}}{t} dt \right],$$

and using

$$\gamma \quad = \quad \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt,$$

we obtain

$$\begin{aligned} \int_0^x \frac{e^{-t}-1}{t} dt \quad &= \quad -\gamma - \int_1^\infty e^{-t} t dt + \int_x^1 \frac{dt}{t} - \int_x^1 \frac{e^{-t}}{t} dt \\ &= \quad -\gamma - \ln x - \int_x^\infty \frac{e^{-t}}{t} dt. \end{aligned}$$

□

Now apply Corollary 1.2 to the function

$$f(x) \quad = \quad \int_0^x \frac{e^{-t}-1}{t} dt = \sum_{k=1}^\infty \frac{(-1)^k x^k}{k k!} \tag{4.9}$$

to produce

$$\begin{aligned} \sum_{k=1}^\infty \frac{(-1)^k \Gamma(\beta+k)}{k\, k!} \quad &= \quad -\int_0^\infty x^{\beta-1} e^{-x} \left[\gamma + \ln x + \Gamma(0,x) \right] \, dx \\ &= \quad -\gamma\Gamma(\beta) - \Gamma'(\beta) - \int_0^\infty x^{\beta-1} e^{-x} \Gamma(0,x)\, dx. \end{aligned}$$

The case $\beta = 1$ recovers (4.7).

Reversing the order of integration in (4.7) yields

$$\int_0^\infty e^{-t} \frac{1-e^{-t}}{t} dt \quad = \quad \ln 2, \tag{4.10}$$

which can be written as

$$\int_0^1 \frac{1-v}{\ln v} dv = -\ln 2. \tag{4.11}$$

Lemma 4.2.

$$\int_0^\infty \frac{e^{-t}-1}{t} \Gamma(0,t)\, dt = \zeta(2) \tag{4.12}$$

Proof. Exchanging the order of integration gives

$$\begin{aligned} \int_0^\infty \frac{e^{-t}-1}{t} \int_t^\infty \frac{e^{-x}}{x} dx &= \int_0^\infty \frac{e^{-x}}{x} \int_0^x \frac{e^t-1}{t} dt\, dx \\ &= \int_0^\infty e^{-x} \sum_{k=1}^\infty \frac{x^{k-1}}{kk!}\, dx \\ &= \zeta(2). \end{aligned}$$

□

Example 4.1. The expansion of $\Gamma(\alpha, x)$ yields the identity

$$\sum_{k=1}^\infty \frac{(-1)^k \alpha^k x^k}{kk!} = -\gamma - \Gamma(0, \alpha x) - \ln \alpha - \ln x,$$

and applying Corollary 1.2 gives the Laplace transform of $\Gamma(0,t)$:

$$\int_0^\infty e^{-st} \Gamma(0,t) dt = \frac{\ln(1+s)}{s}. \tag{4.13}$$

Example 4.2. The function

$$f_1(x) = \int_0^x \frac{1-t-e^{-t}}{t^2} dt \tag{4.14}$$

admits the expansion

$$f_1(x) = \sum_{k=1}^\infty \frac{(-1)^{k-1} x^k}{k(k+1)!}.$$

Corollary 1.2 and the usual argument produces

$$\int_0^\infty e^{-t} \frac{(1-t-e^{-t})}{t^2}\, dt = 2\ln 2 - 1. \tag{4.15}$$

Lemma 4.3. Let $H_n = 1 + 1/2 + \cdots + 1/n$ be the harmonic number. Then

$$\sum_{k=1}^\infty \frac{H_k}{k!} \Gamma(k+\beta) = \frac{\Gamma(\beta)}{\beta} \tag{4.16}$$

for $-1 < \beta < 0$.

Proof. Corollary 1.2 applied to the function

$$f_2(x) = e^x \int_0^x \frac{1-e^{-t}}{t}\, dt = \sum_{k=1}^\infty \frac{H_k}{k!} x^k \tag{4.17}$$

yields, after reversing the order of integration,

$$\sum_{k=1}^{\infty} \frac{H_k}{k!}\Gamma(k+\beta) \quad = \quad -\frac{1}{\beta}\int_0^{\infty} t^{\beta}\frac{1-e^{-t}}{t}dt. \tag{4.18}$$

The last integral can be evaluated by integration by parts to obtain the result. □

5. Bessel functions

The Bessel function of order n is defined by

$$I_n(x) \quad = \quad \sum_{j=0}^{\infty} \frac{x^{2j+n}}{2^{2j+n}\, j!(j+n)!}, \tag{5.1}$$

Example 5.1.

$$\int_0^{\infty} x^{\beta-1}e^{-x}I_n(x)dx \quad = \quad \sum_{j=0}^{\infty} \frac{\Gamma(\beta+2j+n)}{2^{2j+n}j!(j+n)!} \tag{5.2}$$

Proof. The expansion of the Bessel function $I_n(x)$ shows that

$$A_k \quad = \quad \frac{1}{2^{2j+n}j!(j+n)!} \quad \text{if } k=2j+n \tag{5.3}$$

and $A_k = 0$ otherwise. The identity (5.2) then follows from Corollary 1.2.

□

Special case: $n=0$ yields

$$\int_0^{\infty} x^{\beta-1}e^{-x}I_0(x)dx \quad = \quad \sum_{j=0}^{\infty} \frac{\Gamma(\beta+2j)}{2^{2j}j!^2}. \tag{5.4}$$

Example 5.2. Let $a>0$. Then

$$\int_0^{\infty} x^{\beta-1}e^{-x}I_0(2\sqrt{ax})\,dx \quad = \quad \sum_{k=0}^{\infty} \frac{a^k}{k!^2}\,\Gamma(\beta+k). \tag{5.5}$$

Proof. The Taylor coefficients of the function

$$\varphi(x) \quad := \quad \sum_{k=0}^{\infty} \frac{(ax)^k}{k!^2} = I_0(2\sqrt{ax}) \tag{5.6}$$

are $A_k = a^k/k!^2$. The evaluation (5.5) now follows from Corollary 1.2. □

Note. This integral is similar to the evaluation

$$\int_0^{\infty} J_{\nu}(at)e^{-p^2t^2}t^{\mu-1}dt \quad = \quad \frac{\Gamma(\nu/2+\mu/2)(ap/2)^{\nu}}{2p^{\mu}\Gamma(\nu+1)} \times \exp\left(-\frac{a^2}{4p^2}\right)\,{}_1F_1\left(\nu/2-\mu/2+1;\nu+1;a^2/4p^2\right)$$

that can be found in [6], page 394, formula (3).

Special case: $\beta=1$ gives

$$\int_0^{\infty} e^{-x}I_0(2\sqrt{ax})\,dx \quad = \quad e^a. \tag{5.7}$$

This identity can also be written as

$$\int_0^\infty xe^{-x^2/4a} I_0(x)\, dx = 2ae^a. \tag{5.8}$$

The proof of the next theorem is outlined in the way we obtained it.

Theorem 5.1. Let $L_n(x)$ be the Laguerre polynomial of order n. Then

$$\int_0^\infty x^{2n+1} e^{-x^2/4a} I_0(x)\, dx = 2^{2n+1} a^{n+1} n! e^a L_n(-a). \tag{5.9}$$

Step 1. Differentiating (5.8) with respect to a produces

$$\int_0^\infty x^3 e^{-x^2/4a} I_0(x)\, dx = 8a^2(1+a)e^a. \tag{5.10}$$

Step 2. There exists a polynomial P_n such that

$$\int_0^\infty x^{2n-1} e^{-x^2/4a} I_0(x)\, dx = e^a P_n(a). \tag{5.11}$$

Moreover, P_n satisfies the recurrence

$$P_{n+1}(a) = 4a^2 \left(P_n'(a) + P_n(a)\right) \tag{5.12}$$

with initial value $P_1(a) = 2a$.

Proof. The recurrence (5.12) follows directly by an induction argument on the proposed form (5.11). □

Step 3. Motivated by the first few values of P_n we introduce a new sequence of polynomials.

Define $Q_n(a) = 2^{-(2n-1)} a^{-n} P_n(a)$. Then $Q_n(a)$ is a polynomial in a of degree $n-1$ that satisfies

$$\begin{aligned} Q_{n+1}(a) &= aQ_n'(a) + (a+n)Q_n(a), \\ Q_1(a) &= 1. \end{aligned} \tag{5.13}$$

Proof. The recurrence (5.13) follows directly from (5.12) and the definition of $Q_n(a)$. The fact that $Q_n(a)$ is a polynomial is a direct consequence of this recurrence.

□

Step 4. We now obtain a closed form for the coefficients of $Q_n(a)$. Write

$$Q_n(a) = \sum_{j=0}^{n-1} q_j(a) a^j. \tag{5.14}$$

The recurrence for Q_n yields

$$q_0(n+1) = nq_0(n), \tag{5.15}$$

$$q_j(n+1) = (n+j)q_j(n) + q_{j-1}(n), \qquad 1 \le j \le n-1, \tag{5.16}$$

$$q_n(n+1) = q_{n-1}(n). \tag{5.17}$$

We proceed to solve this system.

Observe that $Q_1(a) = 1$, so that $q_0(1) = 1$ and thus (5.17) yields $q_{n-1}(n) = 1$. Now $j = n-1$ in (5.16) produces

$$q_{n-1}(n+1) - q_{n-2}(n) = 2n-1, \tag{5.18}$$

and summing from 2 to n and using $q_0(2) = 1$ gives

$$q_{n-2}(n) = (n-1)^2. \tag{5.19}$$

Proceeding along the same lines we obtain

$$\begin{aligned} q_{n-3}(n) &= \frac{1}{2}(n-1)^2(n-2)^2, \\ q_{n-4}(n) &= \frac{1}{6}(n-1)^2(n-2)^2(n-3)^2. \end{aligned}$$

Step 5. The coefficients $q_j(n)$ are given by

$$q_j(n) = (n-j-1)! \binom{n-1}{j}^2. \tag{5.20}$$

Proof. The proposed formula satisfies the same recursion as $q_j(n)$ with the same initial conditions.

□

Step 6. We identify the polynomial $Q_n(a)$ in terms of the Laguerre polynomial $L_n(x)$. Indeed,

$$Q_n(a) = (n-1)! L_{n-1}(-a). \tag{5.21}$$

Proof. The expression for the coefficients $q_j(n)$ yields

$$Q_n(a) = \sum_{j=0}^{n-1} (n-1-j)! \binom{n-1}{j}^2 a^j, \tag{5.22}$$

and this is identified with the Laguerre polynomial in [5], 23 : 6 : 1. □

This completes the proof of Theorem 5.1.

Note. The series in (5.5) is

$$\sum_{k=0}^{\infty} \frac{a^k}{k!^2} \Gamma(\beta+k) = \Gamma(\beta) \, {}_1F_1[\beta, 1; a]. \tag{5.23}$$

The special case $\beta = n+1$ produces

$$\int_0^\infty x^{2n+1} e^{-x^2/4a} I_0(x) dx = 2^{2n+1} a^{n+1} n! \, {}_1F_1[n+1, 1; a]. \tag{5.24}$$

We thus have

$${}_1F_1[n+1, 1; a] = e^a L_n(-a). \tag{5.25}$$

Example 5.3. The product of two Bessel functions admits the expansion

$$J_\nu(x)J_{n-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+k+1)_k}{k!\Gamma(k+1+n-\nu)\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{n+2k}, \tag{5.26}$$

and Corollary 1.2 yields the identity

$$\int_0^\infty x^{\beta-1} e^{-x} J_\nu(x) J_{n-\nu}(x)dx =$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+2k+1)\Gamma(k+\beta)}{k!\Gamma(n+k+1)\Gamma(k+1+n-\nu)\Gamma(\nu+k+1)2^{2k+n}}.$$

Special case: $\nu = 0$ produces

$$\int_0^\infty x^{\beta-1} e^{-x} J_0(x) J_n(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+2k+1)\Gamma(k+\beta)}{k!^2 \Gamma^2(n+k+1)2^{2k+n}},$$

and further specialization to $n = 0$ yields

$$\int_0^\infty x^{\beta-1} e^{-x} J_0^2(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{k!^4 2^{2k}} \Gamma(k+\beta). \tag{5.27}$$

Special case: $\beta = 1$ yields

$$\int_0^\infty e^{-x} J_0(x) J_n(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n}(n+k)!} \binom{n+2k}{k}, \tag{5.28}$$

and $n = 0$ gives

$$\int_0^\infty e^{-x} J_0^2(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!} \binom{2k}{k}. \tag{5.29}$$

Note. The identity (5.29) is a special case of a formula of Gegenbauer,

$$\int_0^\infty e^{-2ax} J_0^2(bx)dx = \frac{1}{\pi\sqrt{a^2+b^2}} K\left(\frac{b}{\sqrt{a^2+b^2}}\right), \tag{5.30}$$

that can be found on page 391 of [6], formula (4).

Special case: $\beta = n+1$ gives

$$\int_0^\infty x^n e^{-x} J_0(x) J_n(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n} k!} \binom{n+2k}{k}. \tag{5.31}$$

6. Prescribing the coefficients A_n

In this section we prescribe the coefficients A_n in the basic identity (1.1). This is equivalent to prescribing the function φ.

Example 6.1. Take $\beta = 1,\ A_0 = 0$, and $A_k = 1/\Gamma(2k)$ for $k \geq 1$. Then

$$\varphi(x) \quad = \quad \sqrt{x}\sinh\sqrt{x}, \tag{6.1}$$

and Corollary 1.2 produces the identity

$$\int_0^\infty e^{-x}x^{\beta-1/2}\sinh(\sqrt{x})dx \quad = \quad \sum_{k=0}^\infty \frac{\Gamma(\beta+k)}{\Gamma(2k)} \tag{6.2}$$

given in Example 3.2.

Example 6.2. In this example we take $A_k = \Gamma(k)/\Gamma(2k)$, so that

$$\sum_{k=0}^\infty \frac{\Gamma(k)\,\Gamma(\beta+k)}{\Gamma(2k)} \quad = \quad 2\Gamma(\beta) + 2^{2\beta+2}\sqrt{\pi}\int_0^\infty t^{2\beta}e^{-3t^2}\operatorname{erf}(t)\,dt. \tag{6.3}$$

The integral can be evaluated by Mathematica as

$$\frac{\Gamma(1+\beta)}{\sqrt{\pi}\,3^{\beta+1}}\ {}_2F_1\left[\tfrac{1}{2}, 1+\beta, \tfrac{3}{2}; -\tfrac{1}{3}\right].$$

Special case: $\beta = 1$ yields

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^\infty \frac{\Gamma(k)}{\Gamma(2k)}x^k \\ &= 2 + \sqrt{\pi\,x}\ e^{x/4}\operatorname{erf}(\sqrt{x}/2). \end{aligned}$$

Corollary 1.2 then gives

$$\int_0^\infty e^{-x}\left(2+\sqrt{\pi\,x}e^{x/4}\operatorname{erf}(\sqrt{x}/2)\right)dx \quad = \quad \sum_{k=1}^\infty \frac{\Gamma(1+k)\,\Gamma(k)}{\Gamma(2k)}.$$

This series can be evaluated by Mathematica as

$$\frac{1}{2}\sum_{k=0}^\infty \frac{1}{\binom{2k}{k}} \quad = \quad \frac{2}{27}(18+\sqrt{3}\pi). \tag{6.4}$$

An elementary evaluation of this series is described in [2], exercise 16.d, page 384. We conclude that

$$\int_0^\infty e^{-3x/4}\sqrt{x}\operatorname{erf}(\sqrt{x}/2)dx \quad = \quad \frac{18+4\sqrt{3}\pi}{27\sqrt{\pi}}.$$

7. The arctangent family

In this section we consider the family

$$f_k(x) \quad = \quad \tan^{-1}\frac{2\theta^2x^2}{k^2}, \quad k > 0,\ \theta > 0, \tag{7.1}$$

with $f_0(x) = \pi/2$ and $X(x) = 4x/\pi$. The coefficients C_k are given by

$$\begin{aligned} C_k &= \frac{4}{\pi}\int_0^1 x\tan^{-1}\frac{\theta^2x^2}{k^2}dx \\ &= \frac{2}{\pi}\tan^{-1}\frac{2\theta^2}{k^2} - \frac{k^2}{2\pi\theta^2}\ln\left(1+4\theta^4/k^4\right). \end{aligned}$$

Example 7.1. Take $A_0 = 0$ and $A_k = 1$ for all $k \geq 1$ so that

$$\begin{aligned} \varphi(x) &= \sum_{k=1}^{\infty} \tan^{-1} \frac{2x^2}{k^2} \\ &= \frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi\theta x}{\tan \pi\theta x}. \end{aligned}$$

Theorem 1.1 then gives

$$\sum_{k=1}^{\infty} \frac{2}{\pi} \tan^{-1} \frac{2\theta^2}{k^2} - \frac{k^2}{2\pi\theta^2} \ln\left(1 + 4\theta^4/k^4\right) = \frac{4}{\pi} \int_0^1 x \left(\frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi\theta x}{\tan \pi\theta x} \right) dx. \tag{7.2}$$

Using the evaluation [4], p 578,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2\theta^2}{k^2} = \frac{\pi}{4} - \tan^{-1} \left(\frac{\tanh \pi\theta}{\tan \pi\theta} \right), \tag{7.3}$$

we obtain

$$\int_0^1 x \tan^{-1} \left(\frac{\tanh \pi\theta x}{\tan \pi\theta x} \right) dx = \frac{1}{2} \tan^{-1} \left(\frac{\tanh \pi\theta}{\tan \pi\theta} \right) + \frac{1}{8\theta^2} \sum_{k=1}^{\infty} k^2 \ln\left(1 + 4\theta^4/k^4\right). \tag{7.4}$$

We now provide an evaluation of the series in (7.4) that yields

$$\int_0^1 x \tan^{-1} \left(\frac{\tanh \pi\theta x}{\tan \pi\theta x} \right) dx = \frac{1}{2} \tan^{-1} \left(\frac{\tanh \pi\theta}{\tan \pi\theta} \right) + \frac{g(\theta)}{8\theta^2}, \tag{7.5}$$

where

$$\begin{aligned} g(\theta) &= \frac{4\pi\theta^3}{3} i - \frac{\zeta(3)}{\pi^2} + 2i\theta^2 \left(\log[1-u] - \log[1-v] \right) \\ &+ \frac{\theta}{\pi} \left\{ (1-i) \operatorname{PolyLog}[2,u] - (1+i) \operatorname{PolyLog}[2,v] \right\} \\ &+ \frac{1}{2\pi^2} \left(\operatorname{PolyLog}[3,u] + \operatorname{PolyLog}[3,v] \right) \end{aligned}$$

and

$$u = e^{(-2+2i)\pi\theta} \quad \text{and} \quad v = e^{(2+2i)\pi\theta}.$$

Step 1:

$$\sum_{k=1}^{\infty} k^2 \ln\left(1 + 4\theta^4/k^4\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} 2^{2j} \theta^{4j} \zeta(4j-2) \tag{7.6}$$

Proof. Use the expansion

$$\ln\left(1 + 4\theta^4/k^4\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2^{2j} \theta^{4j}}{j k^{4j}} \tag{7.7}$$

and reverse the order of summation. ∎

The identity (7.4) becomes

$$\int_0^1 x\tan^{-1}\left(\frac{\tanh\,\pi\theta x}{\tan\,\pi\theta x}\right)\,dx \quad = \quad \frac{1}{2}\tan^{-1}\left(\frac{\tanh\,\pi\theta}{\tan\pi\theta}\right) + \frac{1}{8\theta^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^{2k}\theta^{4k}}{k}\zeta(4k-2). \tag{7.8}$$

Step 2: Introduce the function

$$f(x) \quad = \quad \sum_{j=1}^{\infty}\zeta(4j-2)(-x)^{j-1}, \tag{7.9}$$

so that

$$\int_0^t f(x)\,dx \quad = \quad \sum_{j=1}^{\infty}(-1)^{j-1}\frac{\zeta(4j-2)}{j}t^j. \tag{7.10}$$

The result of Step 1 then becomes

$$\sum_{k=1}^{\infty}k^2\ln\left(1+4\theta^4/k^4\right) \quad = \quad \int_0^{4\theta^4} f(x)\,dx.$$

Step 3:

$$f(x) \quad = \quad \sum_{k=1}^{\infty}\frac{k^2}{k^4+x},$$

Proof. We have

$$f(x) \quad = \quad \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-x)^{j-1}}{k^{4j-2}}, \tag{7.11}$$

and the sum in j is a geometric series. □

Step 4:

$$\sum_{k=1}^{\infty}k^2\ln\left(1+4\theta^4/k^4\right) \quad = \quad \int_0^{\sqrt{2}\theta}\sum_{k=1}^{\infty}\frac{4k^2y^3}{k^4+y^4}\,dy$$

Step 5:

$$\sum_{k=1}^{\infty}k^2\ln\left(1+4\theta^4/k^4\right) \quad = \quad \int_0^{\sqrt{2}\theta}\pi y^3\coth(\pi w)\frac{dy}{w} + \int_0^{\sqrt{2}\theta}\pi y^3\coth(\pi z)\frac{dy}{z},$$

where $w = e^{\pi i/4}y$ and $z = e^{-\pi i/4}y$.

Proof. Observe that

$$\sum_{k=1}^{\infty}\frac{4k^2y^3}{k^4+y^4} \quad = \quad \sum_{k=-\infty}^{\infty}\frac{y^3}{k^2+w^2} + \sum_{k=-\infty}^{\infty}\frac{y^3}{k^2+z^2}, \tag{7.12}$$

and now use the partial fraction expansion of hyperbolic cotangent. □

Step 6: Direct symbolic evaluation of the integrals in Step 5 yields

$$\begin{aligned}\sum_{k=1}^{\infty} k^2 \ln\left(1+4\theta^4/k^4\right) &= \frac{4\pi\theta^3}{3} i - \frac{\zeta(3)}{\pi^2} + 2i\theta^2 \left(\log[1-u] - \log[1-v]\right) \\ &+ \frac{\theta}{\pi}\left\{(1-i)\,\text{PolyLog}[2,u] - (1+i)\,\text{PolyLog}[2,v]\right\} \\ &+ \frac{1}{2\pi^2}\left(\text{PolyLog}[3,u] + \text{PolyLog}[3,v]\right),\end{aligned}$$

where

$$u = e^{(-2+2i)\pi\theta} \quad \text{and} \quad v = e^{(2+2i)\pi\theta}.$$

Special case: $\theta = 1$ gives

$$\sum_{k=1}^{\infty} k^2 \ln(1+4/k^4) = \frac{4\pi}{3} - \ln 5 + \frac{2}{\pi}\text{PolyLog}[2, e^{-2\pi}] + \frac{1}{\pi^2}\left(\text{PolyLog}[3, e^{-2\pi}] - \zeta(3)\right),$$

so that

$$\begin{aligned}\int_0^1 x \tan^{-1}\left(\frac{\tanh \pi x}{\tan \pi x}\right) dx &= \frac{5\pi}{12} - \frac{\ln 5}{8} + \frac{1}{4\pi}\text{PolyLog}[2, e^{-2\pi}] \\ &+ \frac{1}{8\pi^2}\left(\text{PolyLog}[3, e^{-2\pi}] - \zeta(3)\right).\end{aligned} \tag{7.13}$$

Note: Differentiating (7.8) with respect to θ yields

$$\begin{aligned}\int_0^1 x^2 \frac{\sin(2\pi x\theta) - \sinh(2\pi x\theta)}{\cos(2\pi x\theta) - \cosh(2\pi x\theta)}\, dx &= \frac{1}{2}\frac{\sin(2\pi\theta) - \sinh(2\pi\theta)}{\cos(2\pi\theta) - \cosh(2\pi\theta)} - \frac{2\theta}{\pi}\sum_{k=1}^{\infty}\frac{k^2}{k^4+16\theta^4} + \\ &+ \frac{1}{4\pi\theta^3}\sum_{k=1}^{\infty} k^2 \ln\left(1+4\theta^4/k^4\right).\end{aligned} \tag{7.14}$$

The first sum can be evaluated directly by differentiating (7.3) with respect to θ:

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4+4\theta^2} = \frac{\pi}{4\theta} \times \frac{\sin(2\pi\theta) - \sinh(2\pi\theta)}{\cos(2\pi\theta) - \cosh(2\pi\theta)}. \tag{7.15}$$

The second sum is evaluated in Step 6.

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References

[1] B. Berndt and R. rankin. The books studied by ramanujan in india. *Amer. Math. Monthly*, 107:595–601, 2000.

[2] J. M. Borwein and P. B. Borwein. *Pi and the AGM- A study in analytic number theory and computational complexity*. Wiley, New York, 1st edition, 1987.

[3] G. S. Carr. *A synopsis of elementary results in pure mathematics: containing propositions, formulae, and methods of analysis, with abridged demonstrations. Supplemented by an index to the papers ... which are to be found in the principal journasls...* London: Francis Hodgson; Cambridge: MacMillan & Bowes., 1886.

[4] L. B. W. Jolley. *Summation of series*. Dover Publications, Inc., New York, 1961.

[5] J. Spanier and K. Oldham. *An atlas of functions*. Hemisphere Publishing Co., 1st edition, 1987.

[6] G.N. Watson. *A treatise on the Theory of Bessel Functions*. Cambridge University Press, 1966.

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