The Stability of Linear Discrete Time Delay Systems Over a Finite Time Interval: New Results

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Abstract - This paper gives sufficient conditions for the practical and finite time stability of a particular class of linear discrete time delay systems. Analyzing the finite time stability concept, these new delay-independent conditions are derived using an approach based on the Lyapunov-like functions. The practical and attractive practical stability for discrete time delay systems has been investigated. The above mentioned approach was supported by the classical Lyapunov technique to guarantee the attractivity properties of the system behavior.

Index Terms - Discrete systems, system stability, time delay system, finite time stability system, non-Lyapunov stability.

I. INTRODUCTION

The time delay systems have been investigated over many years. Time delay was often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

Numerous results have been reported on this matter, with a particular emphasis on the application of Lyapunov’s second method. The other solutions were based on the idea of the matrix measure as presented in [1-4].

From a practical point of view, the emphasis must be placed not only on the system stability (e.g. in the sense of Lyapunov), but also in the bounds of system trajectories. A system could be stable, but completely useless because it possesses undesirable transient performances.

It should be noticed that up to now, no results have been reported concerning the aforementioned problem of the non-Lyapunov stability for discrete time delay systems. Motivated by discussions on practical stability in [5-7] various notions of the stability over a finite time interval for continuous-time systems and constant set trajectory bounds have been introduced so far.

Some of the initial results completely based on the discrete fundamental matrix of the system have been reported in [8].

II. SYSTEM DESCRIPTION

A linear discrete system with state delay was considered. The system is described by:

\[ x(k+1) = A_0 x(k) + A_1 x(k-h), \]

(1.a)

with a known vector valued function of the initial conditions:

\[ x(\vartheta) = \psi(\vartheta), \quad \vartheta \in \{-h, -h+1, \ldots, 0\}, \]

(1.b)

where \( x(k) \in \mathbb{R}^n \) is a state vector and constant matrices \( A_0 \) and \( A_1 \) of the appropriate dimensions.

The time delay \( h \) is constant.

The solution is bounded for all bounded values of its arguments.

Let \( \mathbb{R}^n \) denote the state space of the systems given by (1) and \( \| \cdot \| \) the Euclidean norm.

The solutions of (1) are denoted by:

\[ x(k, k_0, x_0) \equiv x(k). \]

(2)

The discrete-time interval is denoted by \( K_N \), as a set of nonnegative integers:

\[ K_N = \{ k : k_0 \leq k \leq k_0 + k_N \}. \]

(3)

The quantity \( k_N \) can be a positive integer or the symbol \( +\infty \), so the finite time stability and practical stability can be treated simultaneously.

Let \( \mathbb{V} : K_N \times \mathbb{R}^n \to \mathbb{R} \), so that \( V(k, x(k)) \) is bounded and for which \( \| x(k) \| \) is also bounded.

The total difference \( \Delta V(k, x(k)) \) was defined along the trajectory of systems (1) as:

\[ \Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)). \]

(4)

For time-invariant sets it is assumed that \( S(k) \) is a bounded open set.

Let \( S_\beta \) be a given set of all allowable states of the system \( \forall k \in K_N \).

Set \( S_\alpha, S_\alpha \subset S_\beta \) denotes the set of all allowable initial states.

III. STABILITY DEFINITIONS

Practical stability

Definition 1. System (1) is attractively practically stable with respect to \( \{k_0, K_N, S_\alpha, S_\beta \} \), \( \alpha < \beta \), if and only
If \( \| \mathbf{x}(k) \|_{\ell^2(P,P_0)}^2 = \| \mathbf{x}_0 \|_{\ell^2(P,P_0)}^2 < \alpha \), implies \( \| \mathbf{x}(k) \|_{\ell^2(P,P_0)}^2 < \beta \), \( \forall k \in K_N \), with the property that: \( \lim_{k \to \infty} \| \mathbf{x}(k) \|_{\ell^2(P,P_0)}^2 \to 0 \).

**Definition 2.** System (1) is practically stable with respect to \( \{k_0, K_N, S_\alpha, S_\beta\} \), if and only if \( \| \mathbf{x} \|_{\ell^2(P,P_0)}^2 < \alpha \), implies \( \| \mathbf{x}(k) \|_{\ell^2(P,P_0)}^2 < \beta \), \( \forall k \in K_N \). 

**Definition 3.** System (1) is attractively practically unstable with respect to \( \{k_0, K_N, \alpha, \beta, \| (\cdot) \|_{\ell^2(P,P_0)}^2\} \), \( \alpha < \beta \), if and only if for \( \| \mathbf{x} \|_{\ell^2(P,P_0)}^2 < \alpha \), there exists a moment \( k = k^* \in K_N \), so that the condition \( \| \mathbf{x}(k^*) \|_{\ell^2(P,P_0)}^2 \geq \beta \) is fulfilled with the property \( \lim_{k \to \infty} \| \mathbf{x}(k) \|_{\ell^2(P,P_0)}^2 \to 0 \).

**Definition 4.** Linear discrete time delay system (1.a) is finite time stable with respect to \( \{\alpha, \beta, k_0, k_N, \| (\cdot) \|_{\ell^2(P,P_0)}^2\} \), \( \alpha \leq \beta \), if every trajectory \( \mathbf{x}(k) \) satisfies the initial function given by (1.b) such that \( \| \mathbf{x}(k) \| < \alpha \), \( k = 0, -1, -2, \ldots, -N \) which implies \( \| \mathbf{x}(k) \| < \beta \), \( \forall k \in K_N \).

**IV. SOME PREVIOUS RESULTS**

**Theorem 1.** For linear discrete time delay system (1) to be finite time stable with respect to \( \{\alpha, \beta, N, \| (\cdot) \|_{\ell^2(P,P_0)}^2\} \), \( \alpha < \beta \), \( \alpha, \beta \in \mathbb{R}_+ \), it is sufficient that:

\[
\| \Phi(k) \| < \frac{\beta}{\alpha} \cdot \frac{1}{1 + \sum_{j=1}^{M} |A_j|} , \quad \forall k = 0, 1, \ldots, M , \tag{5}
\]

[8].

**Theorem 2.** System (1), with \( \det A_1 \neq 0 \), is attractively practically stable with respect to \( \{k_0, K_N, \alpha, \beta, \| (\cdot) \|_{\ell^2(P,P_0)}^2\} \), \( \alpha < \beta \), if there exists \( P = P^T > 0 \), which is the solution of:

\[
2 A_1^T P A_0 - P = -Q , \tag{6}
\]

where \( Q = Q^T > 0 \) and if the following conditions are satisfied:

\[
\| A_1 \| < \sigma_{\min} \left( (Q - A_1^T P A_1)^{-1} \right)^{\frac{1}{2}} \sigma_{\max} \left( Q^{-1} A_0^T P \right) , \tag{7}
\]

where:

\[
\frac{1}{2} \lambda_{\max}^2 (\ ) < \beta/\alpha , \quad \forall k \in K_N , \tag{8}
\]

and if the following conditions are satisfied:

\[
\frac{1}{2} \lambda_{\max}^2 (\ ) = \max \left\{ \mathbf{x}^T(k) A_1^T P A_0 \mathbf{x}(k) : \mathbf{x}^T(k) A_1^T P A_0 \mathbf{x}(k) = 1 \right\} , \tag{9}
\]

as in [9].

**Remark 1.** The attractivity property is guaranteed by (6) and (7) and the system motion within pre-specified boundaries is guaranteed by condition (8).

**Theorem 3.** Suppose that the matrix \( A_1 \) fulfills \( (I - A_1^T A_1) > 0 \). The system given by (1) is finite time stable with respect to \( \{k_0, K_N, \alpha, \beta, \| (\cdot) \|_{\ell^2(P,P_0)}^2\} \), \( \alpha < \beta \), if there exists a positive real number \( p, p > 1 \), such that:

\[
\| \mathbf{x}(k-1) \|^2 < p^2 \| \mathbf{x}(k) \|^2 , \quad \forall k \in K_N , \quad \forall \mathbf{x}(k) \in S_\beta , \tag{10}
\]

and if the following condition is satisfied:

\[
\lambda_{\max} (\ ) < \beta/\alpha , \quad \forall k \in K_N , \tag{11}
\]

where:

\[
\lambda_{\max} (\ ) = \lambda_{\max} \left( A_0^T A_1 (I - A_1^T A_1) A_0^T A_0 + p^2 I \right) , \tag{12}
\]

as in [9].

**V. MAIN RESULTS**

Before presenting our crucial result, we need some preliminaries, discussions and explanations, as well some additional results.

The characteristic polynomial of system (1) is given by:

\[
f(\lambda) = \det M(\lambda) = \sum_{j=0}^{n(h+1)} a_j \lambda^j , \quad a_j \in \mathbb{R} . \tag{13}
\]

\[
M(\lambda) = I_n \lambda^{h+1} - A_0 \lambda^h - A_1
\]

Denote:

\[
\Omega \triangleq \left\{ \lambda \mid f(\lambda) = 0 \right\} = \lambda \left( A_{eq} \right) , \tag{14}
\]

the set of all characteristic roots of system (1).

The number of these roots amounts to \( n(h+1) \).

A root \( \lambda_m \) of \( \Omega \) with a maximum module:

\[
\lambda_m \in \Omega : |\lambda_m| = \max |\lambda (A_{eq})| . \tag{15}
\]

let us call it a maximum root (eigenvalue).

If the scalar variable \( \lambda \) in the characteristic polynomial is replaced by the matrix \( X \in \mathbb{R}^{n \times n} \), the two following monic matrix polynomials are obtained:
\[ M(X) = X^{h+1} - A_0 X^h - A_1, \quad (16) \]
\[ F(X) = X^{h+1} - X^h A_0 - A_1. \quad (17) \]

It is obvious that \( F(\lambda) = M(\lambda) \).

A matrix \( S \in \mathbb{R}^{n\times n} \) is a right solvent of \( M(X) \), as in [10], if:

\[ M(S) = 0. \quad (18) \]

If:

\[ F(R) = 0, \quad (19) \]

then \( R \in \mathbb{R}^{n\times n} \) is a left solvent of \( M(X) \), as in [16].

We will further use \( S \) to denote the right solvent and \( R \) to denote the left solvent of \( M(X) \).

In the present paper the majority of presented results start from the left solvents of \( M(X) \).

In contrast, in the existing literature the right solvents of \( M(X) \) were mainly studied.

The mentioned discrepancy can be overcome by the following Lemma.

Lemma 1. The conjugate transpose value of the left solvent of \( M(X) \) is also, at the same time, the right solvent of the following matrix polynomial:

\[ M^T(X) = X^{h+1} - A_0^T X^h - A_1^T. \quad (20) \]

Proof. Let \( R \) be the right solvent of \( M(X) \).

Then it holds:

\[ M^T(R^*) = (R^*)^{h+1} - A_0^T (R^*)^h - A_1^T = (R^{h+1} - R^h A_0 - A_1)^* = F^*(R) = 0, \quad (21) \]

so \( R^* \) is the right solvent of \( M^T(X) \).

Conclusion 1. Based on Lemma 1, all characteristics of the left solvents of \( M(X) \) can be obtained by the analysis of the conjugate transpose value of the right solvents of \( M^T(X) \).

The following proposed factorization of the matrix \( M(\lambda) \) will help us to understand better the relationship between the eigenvalues of left and right solvents and roots of the system.

Lemma 2. The matrix \( M(\lambda) \) can be factorized in the following way:

\[ M(\lambda) = (\lambda^h I_n + (S - A_0) \sum_{i=1}^h \lambda^{h-i} S^{i-1}) (\lambda I_n - S) \]
\[ = (\lambda I_n - R) \left( \lambda^h I_n + \sum_{i=0}^{h-1} \lambda^{h-i} R^{i-1} (R - A_0) \right). \quad (22) \]

Proof.

\[ M(\lambda) - M(X) = \lambda^h I_n - X^{h+1} - A_0 (\lambda^h I_n - X^h) = \left( \sum_{i=0}^{h} \lambda^{h-i} X^i - A_0 \sum_{i=0}^{h-1} \lambda^{h-i} X^i \right) (\lambda I_n - X). \quad (23) \]

If \( S \) is a right solvent of \( M(X) \) and then from (23) follows (22).

Similarly, if \( R \) is a left solvent of \( M(X) \), from:

\[ M(\lambda) - F(X) = \left( \lambda I_n - X \right) \left( \lambda^h I_n + \sum_{i=1}^h \lambda^{h-i} X^{i-1} (X - A_0) \right). \quad (24) \]

polynomial \( f(\lambda) \) is an annihilating polynomial for the right and left solvents of \( M(X) \).

The eigenvalues and eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of solvents of the matrix equation (18), as in [10] and [11].

Definition 5. Let \( M(\lambda) \) be a matrix polynomial in \( \lambda \).

If \( \lambda_i \in \mathbb{C} \) is such that \( \det(M(\lambda_i)) = 0 \), then we say that \( \lambda_i \)

is a latent root or an eigenvalue of \( M(\lambda) \).

If a nonzero \( v_i \in \mathbb{R}^n \) is such that:

\[ M(\lambda_i) v_i = 0, \quad (25) \]

then we say that \( v_i \) is a (right) latent vector or a (right)
eigenvector of \( M(\lambda) \), corresponding to the eigenvalue \( \lambda_{ij} \), as in [10] and [11].

The eigenvalues of the matrix \( M(\lambda) \) correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix \( A_{eq} \), as in [12].

Their number is \( n(h+1) \).

Since \( F^*(\lambda) = M^T(\lambda^*) \) holds, it is not difficult to show that the matrices \( M(\lambda) \) and \( M^T(\lambda) \) have the same spectrum.

In [13-14] some sufficient conditions for the existence, enumeration and characterization of right solvents of \( M(X) \) were derived.

\[ ^1 \text{See The Appendix A.} \]
They show that the number of solvents can be zero, finite or infinite.

For the needs of system stability (1) only the so-called maximum solvents are usable, the spectrums of which contain the maximum eigenvalue \( \lambda_m \).

A special case of the maximum solvent is the so-called dominant solvent, as in [13] and [14], which, unlike maximum solvents, can be computed in a simple way.

**Definition 6.** Every solvent \( S_m \) of \( M(X) \), whose spectrum \( \sigma(S_m) \) contains the maximum eigenvalue \( \lambda_m \) of \( \Omega \) is a maximum solvent.

**Definition 7.** The matrix \( A \) dominates the matrix \( B \) if all the eigenvalues of \( A \) are greater, in modulus, than those of \( B \).

In particular, if the solvent \( S_i \) of \( M(X) \) dominates the solvents \( S_j, \ldots, S_i \) we say it is a dominant solvent, as in [13] and [14].

(Note that a dominant solvent cannot be singular.)

**Conclusion 2.** The number of maximum solvents can be greater than one.

The dominant solvent is at the same time the maximum solvent too.

The dominant solvent \( S_i \) of \( M(X) \), under certain conditions, can be determined by the Traub iteration, as in [13] and Bernoulli iteration, as in [13] and [14].

The necessary and sufficient conditions for the asymptotic stability of linear discrete time delay systems (1) are given in the following result.

**Theorem 4.** Suppose that there exists at least one left solvent of \( M(X) \) and let \( R_m \) denote one of them. Then, linear discrete time delay system (1) is asymptotically stable if and only if for any matrix \( Q = Q^* > 0 \) there exists Hermitian matrix \( P = P^* > 0 \) such that:

\[
R_m^* P R_m - P = - Q.
\]

[15] and [16].

**Proof.** (Sufficient condition) Define the following vector discrete functions:

\[
x_k = x(k + \vartheta), \quad \vartheta \in \{-h, -h + 1, \ldots, 0\},
\]

\[
z(x_k) = x(k) + \sum_{j=1}^{h} \Xi(j)x(k-j),
\]

where, \( \Xi(k) \in \mathbb{C}^{m \times m} \) is, in general, some time varying discrete matrix function.

The conclusion of the theorem follows immediately by defining the Lyapunov functional for the system (1), as:

\[
V(x_k) = z^*(x_k)Pz(x_k), \quad P = P^* > 0.
\]

It is obvious that \( z(x_k) = 0 \) if and only if \( x_k = 0 \), so it follows that \( V(x_k) > 0 \) for \( \forall x_k \neq 0 \).

The forward difference of (29), along the solutions of system (1) is:

\[
\Delta V(x_k) = \Delta z^*(x_k)Pz(x_k)
\]

\[
+ z^*(x_k)P \Delta z(x_k) + \Delta z^*(x_k)P \Delta z(x_k).
\]

The difference of \( \Delta z(x_k) \) can be determined in the following manner:

\[
\Delta z(x_k) = \Delta x(k) + \sum_{j=1}^{h} \Xi(j) \Delta x(k-j),
\]

with:

\[
\Delta x(k) = (A_0 - I_a)x(k) + A_1x(k-h),
\]

and:

\[
\sum_{j=1}^{h} \Xi(j) \Delta x(k-j) = \Xi(1) [x(k) - x(k-1)] + \cdots + \Xi(h) [x(k-h+1) - x(k-h)]
\]

Then simple manipulations lead to:

\[
\sum_{j=1}^{h} \Xi(j) \Delta x(k-j) = \Xi(1) x(k) - \Xi(h) x(k-h)
\]

\[
+ (\Xi(2) - \Xi(1)) x(k-1) + \cdots + (\Xi(h) - \Xi(h-1)) x(k-h+1)
\]

Define a new matrix \( \Pi \) by:

\[
\Pi = A_0 + \Xi(1).
\]

If:

\[
\Delta \Xi(h) = A_1 - \Xi(h),
\]

then \( \Delta z(x_k) \) has a form:

\[
\Delta z(x_k) = (\Pi - I_a)z(x_k) + \sum_{j=1}^{h} \Delta \Xi(j) \cdot x(k-j).
\]

If one adopts:

\[
\Delta \Xi(j) = (\Pi - I_a) \Xi(j), \quad j = 1, 2, \ldots, h,
\]

then \( \Delta z(x_k) \) becomes:

\[
\Delta z(x_k) = (\Pi - I_a)z(x_k).
\]

Therefore, (33) becomes:

\[
\Delta V(x_k) = z^*(x_k) [\Pi^*P \Pi - P] z(x_k).
\]

It is obvious that if the following equation is satisfied:

\[
\Pi^*P \Pi - P = - Q, \quad Q = Q^* > 0,
\]

then \( \Delta V(x_k) < 0, \ x_k \neq 0 \).
In the Lyapunov matrix equation (41), of all possible solvents $R$ of $M(X)$, only one of the maximum solvents is of importance, for it is the only one that contains the maximum eigenvalue $\lambda_m \in \Omega$ (Conclusion 2), which has dominant influence on the stability of the system.

So, (26) represents the stability sufficient condition for the system given by (1).

The matrix $\Xi(1)$ can be determined in the following way.

From (38) follows:
\[
\Xi(h+1) = R^h \Xi(1),
\]
and using (35) and (36) one can get (19), and for the sake of brevity, instead of the matrix $\Xi(1)$, one introduces a simple notation $\Xi$.

If a solvent which is not maximal is integrated into the Lyapunov equation, it may happen that there is a positive definite solution of Lyapunov matrix equation (26), although the system is not stable.

Conversely, if system (1) is asymptotically stable then all roots $\lambda \in \Omega$ are located within the unit circle.

Since $\sigma(R_m) \subset \Omega$, $\rho(R_m) < 1$ follows, so the positive definite solution of Lyapunov matrix equation (26) exists (necessary condition).

**Theorem 5.** Suppose that there exists at least one left solvent of $M(X)$ and let $R_m$ denote one of them.

Then, linear discrete time delay system (1), with
\[
\det A_i \neq 0,
\]
is attractively practically stable with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \|()\|\}$, $\alpha < \beta$, if there exists a positive real number $\varphi$, $\varphi > 1$, such that:
\[
\|x(k-1)\|^2 \leq \varphi^2 \|x(k)\|^2,
\]
and for all $k \in \mathcal{K}_N$, $\forall x(k) \in \mathcal{S}_\beta$.

Proof. Let:
\[
V(x(k)) = x^T(k)x(k) + x^T(k-1)x(k-1).
\]

Following the classical procedure, as in [15], one can get:
\[
\ln x^T(k+1)x(k+1) - \ln x^T(k)x(k) > \ln \lambda_{\min}( ) ,
\]
where:
\[
\lambda_{\min}( ) = \lambda_{\min}(A_0^TA_1(I-A_0A_1^{-1})A_0^T + \rho^2I).
\]

If the summing $\sum_{j=0}^{k-1}$ is applied on both sides of (51) for $\forall k \in \mathcal{K}_N$, one can obtain:
\[
\ln x^T(k_0+k)x(k_0+k) \geq \ln \lambda_{\max}( ) + \ln x^T(k_0)x(k_0).
\]

It is clear that for any $x_0$, $\delta < \|x_0\|^2 < \alpha$ follows and for some $k^* \in \mathcal{K}_N$, taking into account the basic condition of Theorem 4, eq. (42), it can be concluded:
\[
\ln x^T (k_o + k^*) x (k_o + k^*) > \\
> \ln \lambda^{(k^*)}_{\min} (A_o, A_1, \phi(t)) + \ln x^T (k_o) x (k_o) > \\
> \ln \delta \cdot \lambda^{(k^*)}_{\min} (\delta > \ln \beta, \text{ for some } k^* \in K_N)
\] (54)

VI. CONCLUSION

New definitions and theorems have been established and proved for a particular class of the discrete time delay systems.

The conditions guarantee the practical attractivity and only practical stability within the pre-specified time-invariant sets in the state space.

Moreover, based on a classical definition, new theorems have been derived for the so-called finite time stability as well as the corresponding results for discrete time delay systems.

It is necessary to underline the difference between Theorem 5 and all the others.

The former belongs to the class of so-called time delay dependent conditions and all the others to the criteria which do not include the value of time delay in the final result.

The later are easier to apply for technical purposes.

NOTATION

\[ \begin{align*}
\mathbb{R} & \quad \text{Real vector space} \\
\mathbb{R}_+ & \quad \text{All the nonnegative integers} \\
\mathbb{C} & \quad \text{Complex vector space} \\
\lambda^* & \quad \text{Conjugate of } \lambda \in \mathbb{C} \\
F^* & \quad \text{Conjugate transpose of matrix } F \in \mathbb{R}^{n \times n} \\
F & \quad > 0 \quad \text{Positive definite matrix} \\
\det(F) & \quad \text{Determinant of matrix } F \\
\lambda_i(F) & \quad \text{Eigenvalue of matrix } F \\
\lambda(F) & \quad \{ \lambda | \det(F - \lambda I) = 0 \} \\
\sigma(F) & \quad \text{Spectrum of matrix } F \\
\end{align*} \]

APPENDIX A

System (1) can be expressed with the following representation without delay, as in [2]:

\[
x_{eq} (k) = \left[ x^T (k - h) \quad x^T (k - h + 1) \quad \cdots x^T (k) \right] \in \mathbb{R}^N
\]

\[
x_{eq} (k + 1) = A_{eq} x_{eq} (k), \quad N \approx n(h + 1)
\]

\[
A_{eq} = \begin{pmatrix}
0 & I_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_n \\
A_1 & 0 & \cdots & A_0
\end{pmatrix} \in \mathbb{R}^{N \times N}
\]

(B.1)

The system defined by (B.1) is called the equivalent system, while the matrix \( A_{eq} \), is the matrix of equivalent system.

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