CALCULATION OF ALL STABILIZING FRACTIONAL-ORDER PD CONTROLLERS FOR INTEGRATING TIME DELAY SYSTEMS

Serdar Ethem Hamamci¹,* and Muhammet Koksal²

¹Inonu University, Electrical-Electronics Eng. Department, 44280 Malatya, Turkey.
²Fatih University, Electrical-Electronics Eng. Department, 34500 Istanbul, Turkey.

Abstract: In this paper, a simple and effective stabilization method for integrating time delay systems using fractional order PD controllers $C(s) = k_p + k_d s^\mu$ is proposed. The presented method is based on finding the stability regions according to the fractional orders of the derivative element in the range of (0, 2). These regions are computed by using three stability boundaries: Real Root Boundary (RRB), Complex Root Boundary (CRB) and Infinite Root Boundary (IRB). The method gives the explicit formulae corresponding to these boundaries in terms of fractional order PD controller (PD$^\mu$ controller) parameters. Thus, the complete set of stabilizing controllers for an arbitrary integrating time delay system can be obtained. In order to demonstrate the effectiveness in solution accuracy and the simplicity of this method, two simulation studies are given. The simulation results indicate that the PD$^\mu$ controllers can provide the larger stability regions than the integer order PD controllers.

Keywords: Fractional-order control, PD controllers, integrating systems, time delay, stabilization.

* Corresponding author (Tel: +90 422 3774796, fax: +90 422 3410046, e-mail: shamamci@inonu.edu.tr)
1. INTRODUCTION

A recent trend shows that the control of integrating systems with time delay has been one of the active areas of control researches in the literature [1-4]. This type of systems is frequently encountered in the process industries. The integrating systems have the pole or poles at the origin which characterize the open-loop instability. In the time domain, this means that a bounded input will lead to unbounded outputs and, hence, saturation. The combined effects of poles at the origin and the time delay make the controller design task very difficult [5].

Despite great development in control strategies, most industrial systems are still controlled by proportional integral derivative (PID) controllers. This popularity in the industrial practice stems from their structural simplicity and robust performance in a wide range of operating conditions [6,7]. For PID controllers, there are some cases so that the three-term action is not necessarily desirable and a two-term (specifically PD) action is chosen. The control of integrating systems is a typical example for this. If it is not possible to well tune the integral term of the PID controller and if the robustness issue is not important, the integral element in the controller can be eliminated. For these systems, only a little study is available for tuning the controllers in the PD action (see [8], and references therein). In these papers, some easy-to-use formulae for the optimal PD controller parameters are derived.

One of the possibilities to enhance the performance of a PD controller for the integrating systems with time delay is to extend its integer order of derivative element to fractional order. In recent years, an increasing number of studies can be found related to the application of fractional controllers in many areas of science and engineering [9-11]. Podlubny [12] has proposed a generalization of the PID controller, namely the $\text{PI}^\lambda \text{D}^\mu$ controller, involving an integrator of order $\lambda$ and differentiator of order $\mu$ (the orders $\lambda$
and $\mu$ may assume real noninteger and nonnegative values) [13]. More recently, some results on the control of integrating systems with time delay using the $\text{PI}^d$ controller by Chen et al. [11] and the $\text{PI}^dD^{\mu}$ controller by Monje et al. [14] have appeared. However, to the best knowledge of the authors, the control of integrating systems with time delay using the $\text{PD}^\mu$ controllers has not yet been studied.

Since the minimal requirement for a controller is to make the control system stable, it is desirable to know the complete set of stabilizing controller parameters for a given plant before controller design and tuning [15]. Many important methods have been recently reported on computation of all stabilizing PI, PD and PID controllers for the linear, time-invariant systems with time delay. Hermite-Biehler Theorem [16], parameter-space approach [17] and stability boundary locus approach [18] are the most important methods for this purpose. However, the stabilization problems considered in these methods completely deal with system’s dynamics whose behavior are described by integer-order differential equations. Furthermore, no systematic study currently exists for obtaining the stability regions of the integrating systems with time delay using the fractional-order PD controllers. The formulation, numerical scheme and numerical results for the computation of stabilizing $\text{PD}^\mu$ controllers for the integrating time delay systems presented in this paper are attempts to fill this gap.

In this paper, using the results of stability boundary locus method [18], a solution to the problem of finding all $\text{PD}^\mu$ controllers that stabilize an arbitrary integrating system with time delay is given. The method for the $\text{PD}^\mu$ controller stabilization presented here is based on first obtaining the stability region in the $(k_d, k_p)$-plane for a fixed value of $\mu$ by using the stability domain boundaries. To achieve this, analytical and straightforward expressions for describing the stability boundaries are derived. Then, for the range of $(0, 2)$ of $\mu$, a set of stability regions are computed. Finally, the
biggest stability region which has the most various behaviors of the control system in this set is obtained. Furthermore, once the set of stability regions is obtained, a new region which is intersection of these stability regions can be determined. This region is independent from the $\mu$ parameter and is called reliable stability region. Moreover, the approach presented in this paper provides several considerable advantages, for example, it can be applied to the fractional-order integrating systems without and with time delay. However, these cases are not considered in this paper.

2. FRACTIONAL-ORDER PD$^\mu$ CONTROL SYSTEM

A fractional-order PD$^\mu$ control system is shown in Fig. 1. In this figure, $y$ is the output, $r$ is the reference input, $e$ is the error and $u$ is the control signal. $G(s)$ is the transfer function of the system and $C(s)$ is the transfer function of the controller in the type of PD$^\mu$ controller.

- Figure 1 -

Definition 2.1. A fractional-order PD$^\mu$ controller can be considered as the generalization of the conventional integer-order PD controllers because of involving a differentiator of order $\mu$. The transfer function of the PD$^\mu$ controller has the form

$$C(s) = k_p + k_ds^\mu$$

where $\mu$ is the fractional order whose value can change in the range of $(0, 2)$. Taking $\mu = 1$ in Eq. (1), a classical PD controller is obtained. If $\mu \geq 2$, the controller is transformed to higher order structure which is different in form compared to the PD structure [19].

3. STABILIZATION USING PD$^\mu$ CONTROLLERS

Consider the unity feedback fractional-order control system shown in Fig. 1, where $G(s)$ is the integrating time delay system given as
\[ G(s) = \frac{K}{a_0 s^n + a_{n-1}s^{n-1} + \ldots + a_1 s} e^{-\theta s} = \left( \frac{K}{\sum_{i=1}^n a_i s^i} \right) e^{-\theta s} \]  

(2)

and \( C(s) \) is a PD\(^\theta \) controller which has form of Eq. (1). The problem is to compute all PD\(^\theta \) controllers which stabilize the integrating system in Eq. (2).

The transfer function of the overall fractional order control system is given by

\[ \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} . \]  

(3)

**Definition 3.1.** The denominator of Eq. (3) is described as *fractional-order characteristic equation* of the closed loop system [20]. In general, the fractional-order characteristic equation can be represented as

\[ P(s) = p_1 s^{q_1} + \ldots + p_k s^{q_k} + p_0 \]  

(4)

where \( p_i \) are the coefficients and \( q_i \) are the fractional orders \((i=1\sim k)\).

**Definition 3.2.** In the parameter space \( P \), the boundaries between the stability and instability domains are defined by the following three parts [17,21-23]:

(i) **Real Root Boundary (RRB):** A real root crosses over the imaginary axis at \( s=0 \). Thus, the real root boundary is obtained by substituting \( s=0 \) in \( P(s) \) of Eq. (4). As a result, the RRB is determined as \( p_0=0 \).

(ii) **Complex Root Boundary (CRB):** A pair of complex roots crosses over the imaginary axis at \( s=j\omega \). Therefore, the system becomes unstable which means that the real and the imaginary parts of \( P(s) \) in Eq. (4) become zero simultaneously. This boundary is also called as *stability boundary locus*.

(iii) **Infinite Root Boundary (IRB):** A real root crosses over the imaginary axis at \( s=j\infty \). Thus, the IRB can be characterized by taking \( p_k=0 \) from Eq. (4).

From Eqs. (1)-(3), the characteristic equation of the control system is obtained as

\[ P(s) = Ke^{-\theta s} (k_p + k_d s^\mu) + \sum_{i=1}^n a_i s^i . \]  

(5)
For a given PD\(^\mu\) controller parameters \(k_p\), \(k_d\) and \(\mu\) the closed-loop system is said to be stable if the quasipolynomial \(P(s)\) has no roots in the closed right-half of the s-plane (RHP). The stability domain \(S\) in the parameter space \(P\) with \(k_p\) and \(k_d\) being coordinates is the region that for \((k_p, k_d, \mu)\in S\) the roots of quasipolynomial \(P(s)\) all lie in open left-half of the s-plane (LHP) [21]. Therefore, the determination of stability domain \(S\) is an important task for the design of the PD\(^\mu\) controllers.

In applying the above descriptions to the characteristic equation in Eq. (5), it follows from Part (i) that the RRB is determined as

\[
k_p = 0.
\] (6)

To construct the stability boundary locus (or CRB), we substitute \(s=j\omega\) into Eq. (5) to obtain

\[
P(j\omega) = Ke^{-j\omega\theta}[k_p + k_d (j\omega)^\mu] + \sum_{i=1}^{n} a_i (j\omega)^i
\] (7)

The noninteger power of a complex number \((j\omega)^\gamma\) can be easily calculated by MATLAB or another scientific package. Hence, \(P(j\omega)\) can be written as

\[
P(j\omega) = K(\cos \omega\theta - j \sin \omega\theta)[k_p + k_d (e + jf)] + \sum_{i=1}^{n} a_i (c_i + jd_i)
\]

\[
= \Re\{P(j\omega)\} + j\Im\{P(j\omega)\} = 0
\] (8)

where \(\Re\{P(j\omega)\}\) and \(\Im\{P(j\omega)\}\) denote the real and imaginary parts of the characteristic equation, respectively, and

\[
e = \Re\{j\omega)^\mu\}, \quad f = \Im\{j\omega)^\mu\}\]
\[
c_i = \Re\{j\omega)^i\}, \quad d_i = \Im\{j\omega)^i\}, \quad i = 1, \ldots, n
\] (9)
(10)

Then, equating the real and imaginary parts of Eq. (8) to zero, one obtains

\[
k_p E(\omega) + k_d C(\omega) = -\sum_{i=1}^{n} a_i c_i
\]
\[
k_p F(\omega) + k_d D(\omega) = -\sum_{i=1}^{n} a_i d_i
\] (11)
where
\[
E(\omega) = K \cos \omega \theta 
\]
\[
F(\omega) = -K \sin \omega \theta 
\]
\[
C(\omega) = Ke \cos \omega \theta + Kf \sin \omega \theta 
\]
\[
D(\omega) = Kf \cos \omega \theta - Ke \sin \omega \theta 
\]

Finally, by solving the 2-dimensional system of Eq. (11) the parameters of PD\(^\mu\) controller are obtained as
\[
k_p = \frac{-1}{K^2 f} \left[ \sum_{i=1}^{n} a_i c_i D(\omega) - \sum_{i=1}^{n} a_i d_i C(\omega) \right] 
\]
\[
k_d = \frac{-1}{K^2 f} \left[ \sum_{i=1}^{n} a_i d_i E(\omega) - \sum_{i=1}^{n} a_i c_i F(\omega) \right] 
\]

Changing \(\omega\) from 0 to \(\infty\), a stability boundary locus is constructed in the \((k_d, k_p)\)-plane using Eqs. (16)-(17).

There are more theoretical difficulties for the calculating of the IRB due to time delay. The characteristic equation possesses an infinite number of roots, which can not be calculated analytically in the general case. However, the asymptotic location of roots far from the origin is well known, which may lead to IRB \([17]\). It can be shown from the results of \([17]\) that the IRB does not exist for this case.

The stability boundary locus and the RRB line divide the entire parameter plane \((k_d, k_p)\) into stable and unstable regions. The stable region can be determined by choosing a test point within each region. The characteristic polynomial belonging to arbitrary point in the stable region have no RHP roots until the characteristic polynomial of any point in the unstable region have a certain number of RHP roots. This procedure is repeated for the values of \(\mu\) in the range of \((0, 2)\).

For some or all \(\mu\) values, a reliable stability region which is the intersection of
their stability regions can be obtained. Thus, a new subregion giving all stable \( k_p \) and \( k_d \) values for the chosen \( \mu \) values is determined. If this subregion is constructed for all \( \mu \) values, the pairs of \((k_p, k_d)\) in the region result the stable closed loop systems independent from \( \mu \).

4. SIMULATION EXAMPLES

In this section, the results of simulations of two examples to illustrate the value of the presented method are given. The first example considers a typical first order integrating plant with time delay for which \((\theta/a_1)\) is small. The second example is chosen for the stabilization of a second order integrating system with time delay. These types of integrating systems are often encountered in the industry.

4.1 Example 1

Consider an integrating system of first order with time delay [24] described by the transfer function

\[
G(s) = \frac{K}{a_1 s} e^{-\theta s}
\]  

(18)

where \( K=1, a_1=5 \) and \( \theta=1 \). Here, it is aimed to find the stability regions for the various values of \( \mu \) in the PD\(^\mu\) controller.

The fractional-order characteristic equation of the control system is derived as

\[
P(s) = 5s + e^{-s}(k_d s^\mu + k_p)
\]

(19)

In applying the stabilization procedure given in Section 3, the stability boundaries can be summarized as follows

**RRB line:** \( k_p = 0 \),  

(20)

**CRB curve:**

\[
\begin{align*}
  k_p &= \frac{(5\omega \cos \omega + 5\omega \sin \omega)}{f} \\
  k_d &= \frac{(-5\omega \cos \omega)}{f}
\end{align*}
\]

(21)
where \( e \) and \( f \) are calculated by Eq. (9). For the simplest case \( \mu = 1 \), the stability region in the \((k_d, k_p)\)-plane is obtained for the integer order classical PD controller. Taking \( \mu = 1 \) in Eq. (21), the \( k_p \) and \( k_d \) values are determined by

\[
\begin{align*}
  k_p &= 5 \omega \sin \omega, \quad (22) \\
  k_d &= -5 \cos \omega. \quad (23)
\end{align*}
\]

Fig. 2 shows the stability boundary locus for a range of frequency \( \omega \in (0, 11.5) \) and the RRB line. It can be observed from this figure that the parameter plane is divided into five regions, namely \( R_1 \), \( R_2 \), \( R_3 \), \( R_4 \) and \( R_5 \). By choosing one arbitrary test point in each regions and using the stability test method in [25], the stability region which is the shaded region \( (R_3) \) shown in Fig. 2 is determined. Fig. 3 shows more clearly the stability region which gives the stable characteristic polynomials. In this figure, the stability boundary locus is computed for the range of \( \omega \in [0, \omega_i] \). Equating (20) to (22), the intersection frequency \( \omega_i \) is calculated as 3.14.

- **Figure 2** -

- **Figure 3** -

To demonstrate the effect of the fractional derivatives, we choose the value of \( \mu \) as 0.5. By repeating the above procedure, the stability region for the PD\(^{0.5}\) controller is shown in Fig. 4. The accuracy of the stability region can be easily tested using the unit step responses of the closed loop system. For calculating the unit step response of a fractional-order system, there are different methods in the literature which use generally digital approximation approaches [26-29]. Since these methods are based on some approximations, the simulation results can be different. In this paper, fractional-order operators were approximated by continued fraction expansion of the bilinear transform [28]. Unit step responses of the PD\(^{0.5}\) control system when \( k_d \) is chosen as 0.5 and \( k_p \) is changed from 0.5, 2.5, 5.5 to 7.668 can be shown in Fig. 5. From this figure, it is seen
that the control system has more oscillatory response when the value of $k_p$ is increased from 0 to boundary value, $k_p=7.668$. If the $k_p$ is bigger than the boundary value or smaller than zero, the control system is unstable. Similarly, the stability region for the PD$^{1.5}$ controller is also shown in Fig. 6.

- Figure 4 -

- Figure 5 -

- Figure 6 -

Fig. 7 shows the stability regions obtained by using some different $\mu$ values for the PD$^{\mu}$ controller. As can be seen from Fig. 7, the stability region for $\mu=0.1$ gives the largest stability region in the parameter plane. Furthermore, the intersection of the stability regions in Fig. 7 gives the reliable stability region for the plant of Eq. (18). Thus, any $(k_d, k_p)$ value in this region provides the stable unit step response for $\mu=0.1, 0.5, 1, 1.5$ and $1.8$.

- Figure 7 -

4.2 Example 2

Consider a second order integrating system with the transfer function

$$G(s) = \frac{K}{a_2 s^2 + a_1 s} e^{-\theta s}$$

(24)

where $K=1$, $a_2=1$, $a_1=1$ and $\theta=1$. It follows from Eqs. (16) and (17) that the equations for $k_p$ and $k_d$ have been obtained as

$$k_p = \left[ (e + af) \omega \cos \omega + (f - \omega e) \omega \sin \omega \right] / f,$$

(25)

$$k_d = \left( \omega^2 \sin \omega - \omega \cos \omega \right) / f.$$

(26)

where $e$ and $f$ are calculated by Eq. (9). Fig. 8 shows the stability regions for $\mu=0.1$, $\mu=0.5$, $\mu=1$ and $\mu=1.5$. It is noted from this figure that the value of $\mu=1.5$ leads to bigger stability region than integer value of $\mu$. The reliable stability region which is the
intersection of these stability regions is shown in Fig. 9. Unit step responses of the PD$^{1.5}$ control system when $k_d$ is chosen as 0.5 and $k_p$ is changed from 0.1, 0.2, 1 to 1.789 can be shown in Fig. 10.

- Figure 8 -
- Figure 9 -
- Figure 10 -

In order to show the effect of time delay, we choose the different values of $\theta$. The stability regions for $\theta=0.5$, $\theta=1$ and $\theta=1.5$ have been shown in Fig.11 for $\mu=0.5$ and in Fig. 12 for $\mu=1.5$ where it can be seen that the time delay has an important effect on the stability region. The greatness of the stability region is inversely proportional with the time delay of the integrating plant.

- Figure 11 -
- Figure 12 -

5. CONCLUSIONS

In the control of an integrating time delay system, it is often necessary to stabilize the system with a simple P or PD controller in the inner loop [21]. In this paper, without using an inner loop we have investigated the stabilization problem of integrating time-delay systems with the PD$^\mu$ controllers $C(s) = k_p + k_ds^\mu$ for a unity feedback control system structure. Based on the stability boundary locus characterization of stability boundaries, the effect of fractional derivative order on the stabilization of integrating time delay systems has been carried out. Numerical results have shown that a fractional-order PD controller can be indeed better than an integer-order one for the stabilization of integrating time delay systems.
REFERENCES


Figure 1
Figure 2

The diagram illustrates the stability boundary locus for a control system. The axes are labeled $k_p$ and $k_d$, with $-60$ to $60$ on the vertical axis and $-5$ to $5$ on the horizontal axis. The stability boundary locus is denoted by $R_1$ to $R_5$. The values $\omega = 0$ and $\omega = 11.5$ are marked on the diagram, indicating specific critical points along the boundary. The shaded area represents the stability region of the system.
Figure 3
Figure 4
Figure 5
Figure 6
Figure 7
Figure 8
Figure 9
Figure 10
Figure 11
Figure 12
FIGURE CAPTIONS

**Figure 1.** Fractional-order control system structure.

**Figure 2.** Stability boundary locus and RRB line ($\mu=1$).

**Figure 3.** Stability region for the PD controller.

**Figure 4.** Stability region for the $PD^{0.5}$ controller.

**Figure 5.** Unit step responses for different values of $(k_d, k_p)$ for $PD^{0.5}$ controller.

**Figure 6.** Stability region for the $PD^{1.5}$ controller.

**Figure 7.** Stability regions for the $PD^\theta$ controllers.

**Figure 8.** Stability regions for the various $PD^\theta$ controllers.

**Figure 9.** Reliable stability region for $PD^\theta$ controllers ($\mu=0.1, 0.5, 1, 1.5$).

**Figure 10.** Unit step responses for different values of $(k_d, k_p)$ for $PD^{1.5}$ controller.

**Figure 11.** Stability regions for different values of $\theta$ ($\mu=0.5$).

**Figure 12.** Stability regions for different values of $\theta$ ($\mu=1.5$).