Variational Calculation of the Limit Cycle and its Frequency in a Two-Neuron Model with Delay

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Introduction:

Divergent Perturbation Series

Basic Principles of Variational Perturbation Theory (VPT)

Applications of VPT:

Anharmonic Oscillator

Soft Matter Physics, Critical Phenomena

Two-Neuron Model:

Perturbative Solution

Shohat Resummation and VPT

Review: My Talk in 2005

Large-D Expansion for the Ground-State Energy of a D-dimensional Anharmonic Oscillator:

$$E^{(N)}(\Omega_0^{(N)}) = D^{4/3} g^{1/3} \left(B_0^{(N)} + \frac{B_1^{(N)}}{D} + \dots \right) + \dots$$

$$B_1 = \frac{3^{1/2} - 2^{1/2}}{2^{1/6}} \approx 0.28316079432217911884466460479488203651 23$$

$$B_1^{(\mathrm{extrap})} = 0.2831607943221791188446646047948820369(24)$$

$$B_2 = -\frac{239}{18 \cdot 2^{2/3}\left(25 + 12 \cdot 6^{1/2}\right)} \approx -0.1537760559399284913195761085499705701590$$

$$B_2^{(\mathrm{extrap})} = -0.153776055939928491319576108549961(60)$$

What does this have to do with neurons???

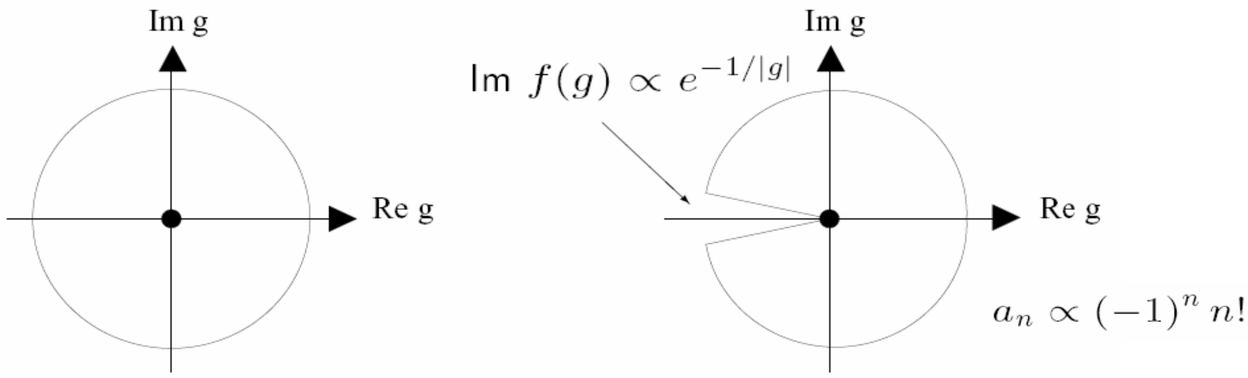
Divergent Series

Perturbation Theory:

weak-coupling series: $f_N(g) = \sum_{n=0}^{N} a_n \, g^n$

Example: anomalous magnetic moment of electron (QED)

Analytic Properties:



convergent series — divergent series

Variational Perturbation Theory:

$\Longrightarrow$ resummation of weak-coupling series

$\Longrightarrow$ optimization with respect to variational parameters

Basic Principles of VPT

weak-coupling series: $$f_N(g) = \sum_{n=0}^{N} a_n \; g^n$$

strong-coupling series: $$f_M(g) = g^{\,p/q} \sum_{m=0}^{M} b_m \; g^{-2m/q}$$

Example:

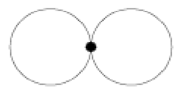
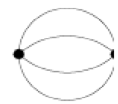
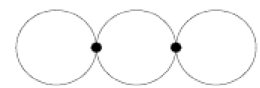
	p	q	p/q	$2/q$
anharmonic oscillator	1	3	1/3	2/3

Diagrammatic Approach to Perturbation Series

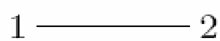
Anharmonic Oscillator:

$$V(x) = \frac{1}{2}\,\omega^2 x^2 + g\,x^4$$

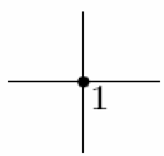
Ground-State Energy:

$E_0 \;=\; \frac{\omega}{2} - \lim_{\beta\to\infty} \frac{1}{\beta}\Big[\frac{1}{8}$  $+\;\frac{1}{48}$  $+\;\frac{1}{16}$  $+\;\ldots\Big]$

Feynman Rules:



$$\longrightarrow \quad \frac{1}{2\omega}\, e^{-\omega|\tau_1-\tau_2|}$$



$$\longrightarrow \quad -24g \int_0^{\beta} d\tau_1$$

Recursive Approach

Schrödinger Equation: $$-\frac{1}{2}\,\psi''(x) + \left(\frac{1}{2}\,\omega^2 x^2 + g x^4\right)\psi(x) = E_0\,\psi(x)$$

Bender-Wu Expansion:

$$E_0 = \frac{\omega}{2} + \sum_{k=1}^{\infty} \epsilon_k\, g^k\,, \qquad \psi(x) = N \exp\left[-\frac{\omega}{2}x^2 + \phi(x)\right]$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k(x) g^k\,, \qquad \phi_k(x) = \sum_{m=1}^{\infty} c_m^{(k)}\, x^m$$

Recursion Relations:

$$c_m^{(k)} = \frac{(m+1)(2m+1)}{2m\omega}\, c_{m+1}^{(k)} + \sum_{l=1}^{k-1}\sum_{n=1}^{m} \frac{n(m-n+1)}{m\omega}\, c_n^{(l)} c_{m-n+1}^{(k-l)}$$

$$\epsilon_k = -c_1^{(k)} \qquad\qquad c_m^{(k)} \equiv 0 \quad \text{for } m > k+1$$

Starting Values: $$c_1^{(1)} = -\frac{3}{4\omega^2}\,, \qquad c_2^{(1)} = -\frac{1}{4\omega}$$

Variational Perturbation Theory

Weak-Coupling Series of Ground-State Energy:

$$E_0 = \frac{\omega}{2} + g\,\frac{3}{4\omega^2} - g^2\,\frac{21}{8\omega^5} + g^3\,\frac{333}{16\omega^8} + \ldots$$

Identity:

$$V(x) = \underbrace{\frac{1}{2}\,\Omega^2 x^2}_{\text{effective harmonic oscillator}} + \underbrace{g\,x^4 + \frac{1}{2}\left(\omega^2 - \Omega^2\right)x^2}_{\text{perturbation}}$$

effective harmonic oscillator perturbation

Ω: variational parameter

Substitution:

$$V(x) = \frac{1}{2}\left(\Omega\sqrt{1 + g\,\frac{\omega^2 - \Omega^2}{g\Omega^2}}\right)^2 x^2 + g\,x^4$$

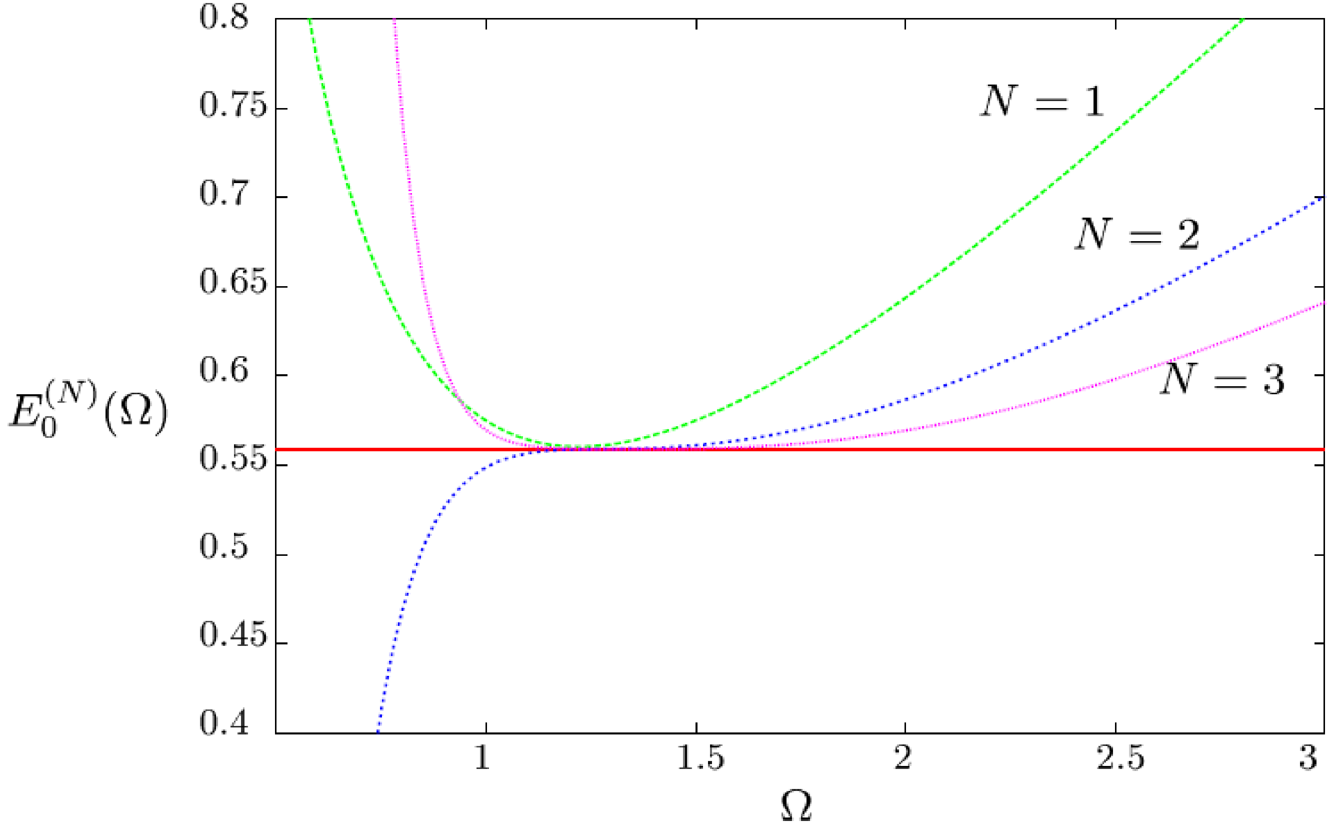
$$\omega \longrightarrow \Omega\sqrt{1 + gr}\,, \qquad r = \frac{\omega^2 - \Omega^2}{g\Omega^2}$$

Example: First Order

$$E_0^{(1)} = \frac{\omega}{2} + g\,\frac{3}{4\omega^2}$$

$$E_0^{(1)}(\Omega) = \frac{\Omega}{4} + \frac{\omega^2}{4\Omega} + g\,\frac{3}{4\Omega^2}$$

Principle of Minimal Sensitivity:



Conditions:

$$\frac{\partial E_0^{(1)}(\Omega)}{\partial \Omega} = 0 \quad \Rightarrow \quad \Omega^{(1)} \quad \Rightarrow \quad E_0^{(1)} = E_0^{(1)}(\Omega^{(1)})$$

$$\frac{\partial^2 E_0^{(2)}(\Omega)}{\partial \Omega^2} = 0 \quad \Rightarrow \quad \Omega^{(2)} \quad \Rightarrow \quad E_0^{(2)} = E_0^{(2)}(\Omega^{(2)})$$

$$\frac{\partial E_0^{(3)}(\Omega)}{\partial \Omega} = 0 \quad \Rightarrow \quad \Omega^{(3)} \quad \Rightarrow \quad E_0^{(3)} = E_0^{(3)}(\Omega^{(3)})$$

Results

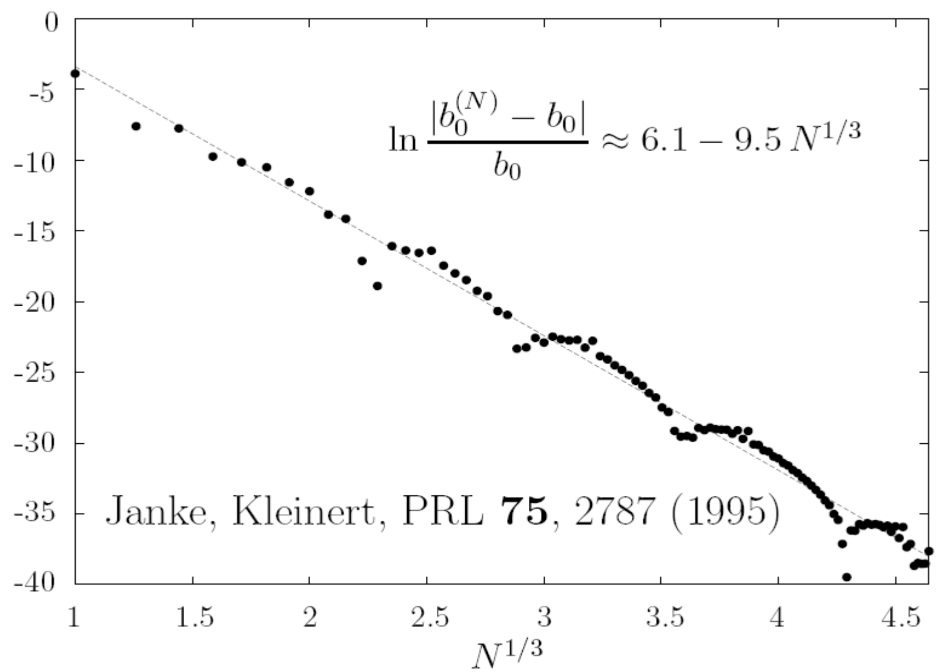
Strong-Coupling Limit $g \to \infty$:

$$\frac{\partial E_0^{(1)}(\Omega)}{\partial \Omega} = \frac{1}{4} - \frac{\omega^2}{4\Omega^2} - g\frac{3}{2\Omega^3} = 0$$

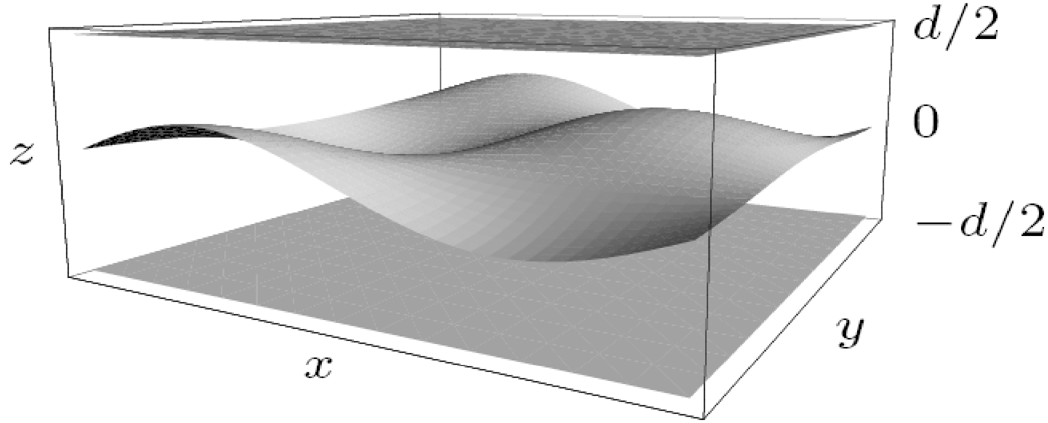
$$\Rightarrow \Omega^{(1)} = g^{1/3}\left(\Omega_0^{(1)} + \Omega_1^{(1)} g^{-2/3} + \Omega_2^{(1)} g^{-4/3} + \dots\right)$$

$$\Rightarrow E_0^{(1)} = g^{1/3}\left(b_0^{(1)} + b_1^{(1)} g^{-2/3} + b_2^{(1)} g^{-4/3} + \dots\right)$$

Exponential Convergence:



Soft Matter Physics



Pressure Law: $$p = \alpha \frac{(k_B T)^2}{\kappa (d/2)^3}$$

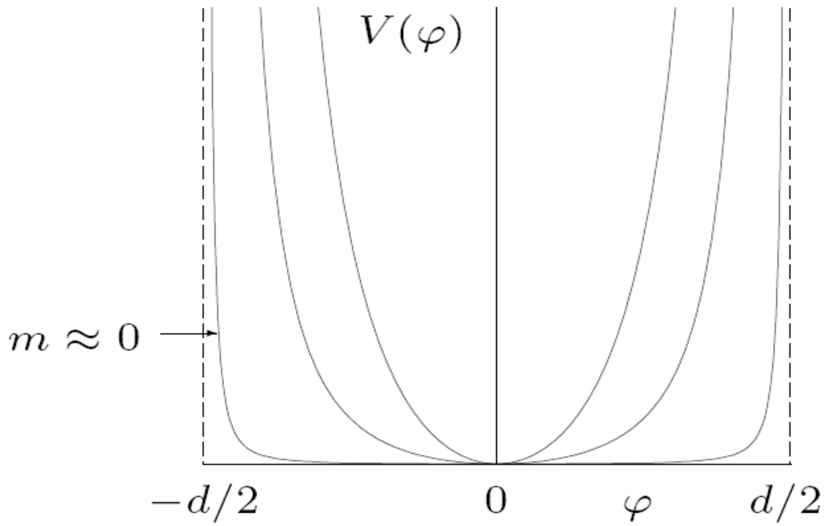
W. Helfrich: Z. Naturforschung **A 33**, 305 (1978)

Harmonic Bending Energy: $$E = \frac{\kappa}{2} \int d^2x \, \left[\boldsymbol{\nabla}^2 \varphi(\mathbf{x}) \right]^2$$

Partition Function:

$$Z = \prod_{\mathbf{x}} \left[\int_{-d/2}^{+d/2} \frac{d\varphi(\mathbf{x})}{\sqrt{2\pi k_B T/\kappa}} \right] \times \exp\left\{ -\frac{\kappa}{2k_B T} \int d^2x \, [\boldsymbol{\nabla}^2 \varphi(\mathbf{x})]^2 \right\}$$

Potential Restricting Fluctuations: $V(\varphi) = m^4 \frac{d^2}{\pi^2} \tan^2\left(\frac{\pi\varphi}{d}\right)$



Weak-Coupling Series of Free Energy: $F^{(L)} = m^2 \sum_{n=0}^{L-1} a_n \left(\frac{\pi^2}{m^2 d^2}\right)^n$

Strong-Coupling Limit ($m \to 0$): $\alpha_{\rm VPT}^{(4)} \approx 0.0797$

M. Bachmann, H. Kleinert, A. Pelster: Phys. Lett. **A 261**, 127 (1999)

Monte-Carlo Result: $\alpha_{\rm MC} = 0.0798 \pm 0.0003$

W. Janke, H. Kleinert: Phys. Lett. **A 117**, 353 (1986)
G. Gompper, D.M. Kroll: Europhys. Lett. **9**, 59 (1989)

Critical Phenomena

Specific Heat of Liquid Helium: $C \propto (T - T_\lambda)^{-\alpha}$; $T \gtrapprox T_\lambda$

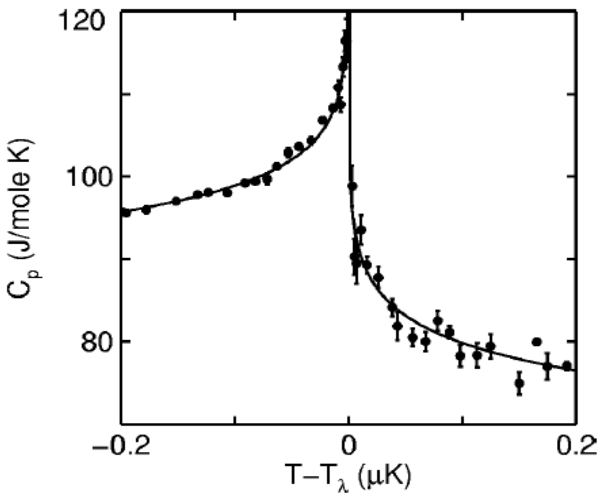
Results for α:

H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4-Theories* (World Scientific, Singapore, 2001).

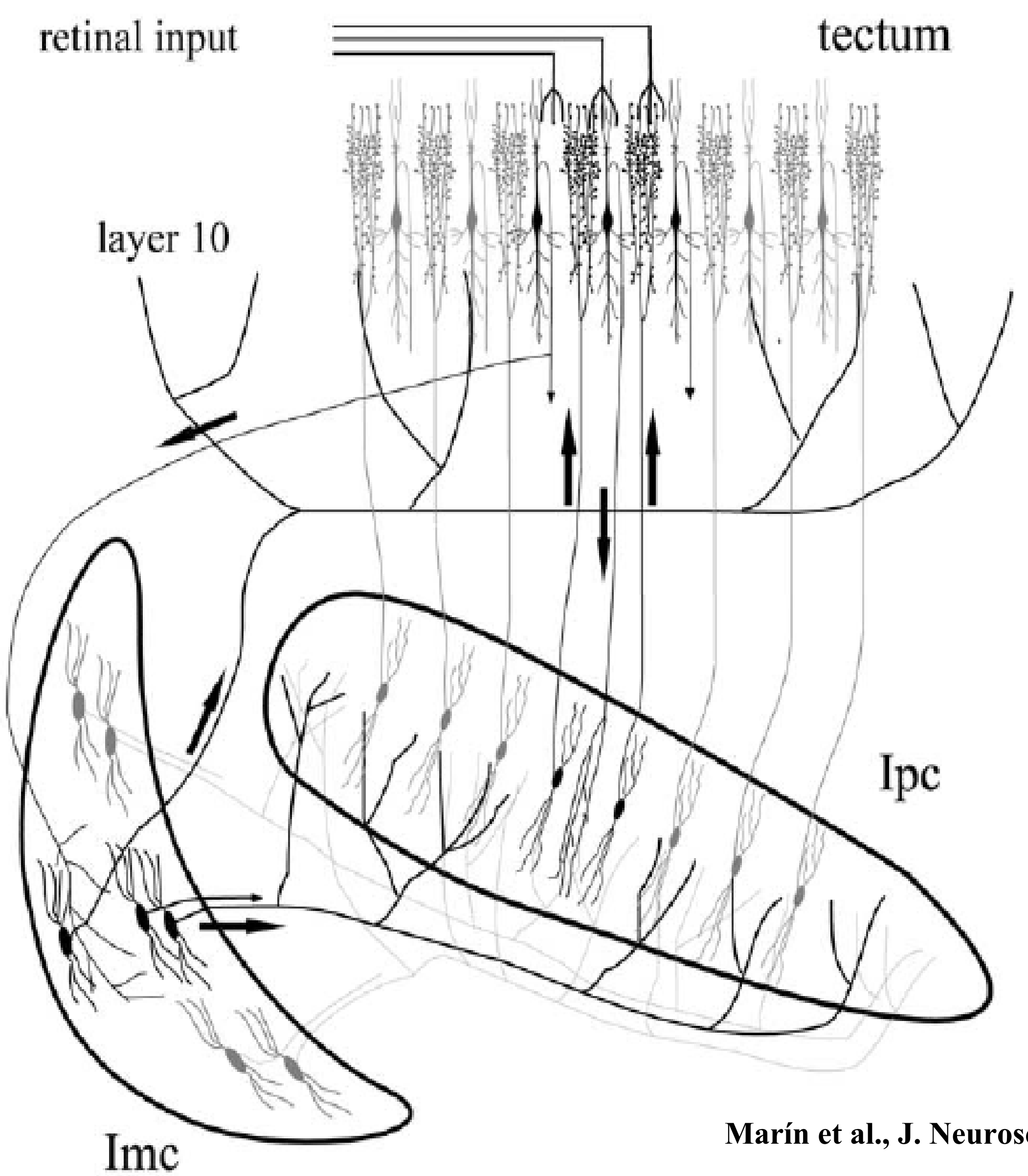
H. Kleinert, PRD **60**, 085001 (1999): $\alpha = -0.0129 \pm 0.0006$

K. H. Mueller, G. Ahlers, F. Pobell, PRB **14**, 2096 (1976): $\alpha = -0.022 \pm 0.006$

J. Lipa et al., PRB **68**, 174518 (2003): $\alpha = -0.0127 \pm 0.0003$



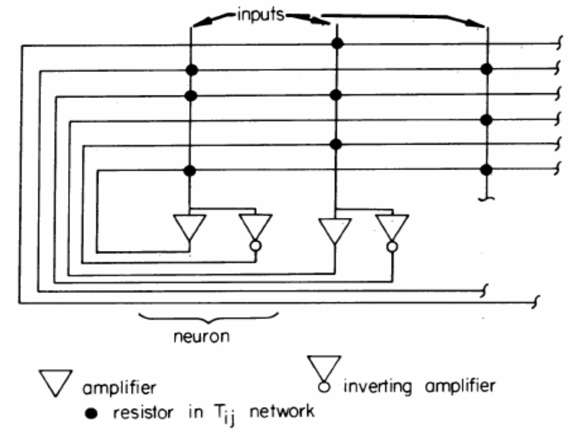
Neurons, Finally



Marín et al., J. Neurosci. (2005)

Time-Continuous Model of Neuron Dynamics by Hopfield

- Leaky neuron with external input and input from other neurons:



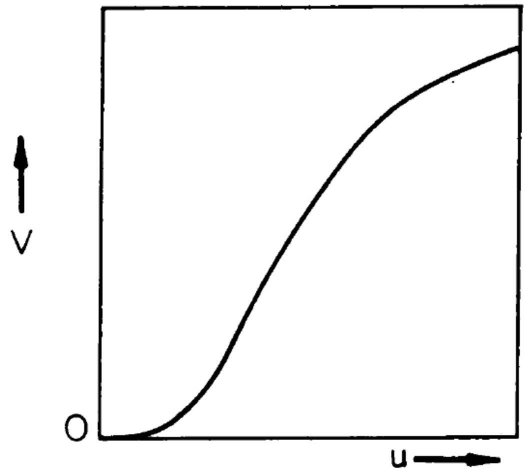
$$C_i \frac{du_i}{dt} = \sum_j T_{ij} V_j - \frac{u_i}{R_i} + I_i$$

Hopfield (1984)

u_i : input voltage, V_j : output voltage, T_{ij}: synaptic interconnection matrix

- Nonlinear Transfer Function:

$$V_i = g(u_i)$$



Two-Neuron Model With Delay

- Model Equation:

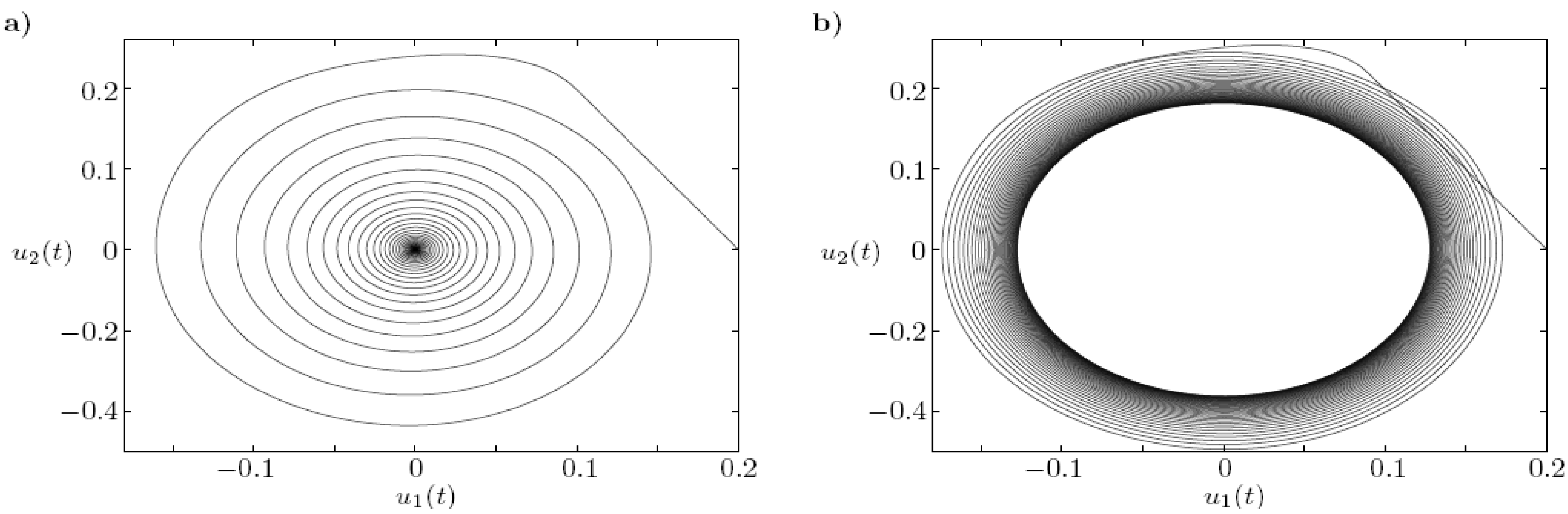
$$\frac{du_1(t)}{dt} = -u_1(t) + a_1 \tanh[u_2(t - \tau^{(2)})]$$
$$\frac{du_2(t)}{dt} = -u_2(t) + a_2 \tanh[u_1(t - \tau^{(1)})]$$

- Characteristic Equation:

$$(\lambda + 1)^2 - a_1 a_2 e^{-\lambda(\tau^{(1)} + \tau^{(2)})} = 0$$

- Supercritical Hopf-Bifurcation:

$$\tau^{(1)} + \tau^{(2)} > \tau_0 = \frac{1}{\omega_0} \sin^{-1}\left(\frac{-2\omega_0}{a_1 a_2}\right)$$
$$\omega_0 = \sqrt{|a_1 a_2| - 1}$$



Poincaré-Lindstedt Method

- Expansions:

$$
\begin{aligned}
\epsilon &= \sqrt{\tau - \tau_0} \\
\mathbf{u}(t) &= \epsilon \mathbf{U}(t) = \epsilon \left[\mathbf{U}^{(0)}(t) + \epsilon \mathbf{U}^{(1)}(t) + \ldots \right] \\
\omega(\epsilon) &= \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots .
\end{aligned}
$$

- Rescaling:

$$\xi = \omega(\epsilon)t \,, \qquad \mathbf{U}(t) = \mathbf{V}(\xi)$$

- Zeroth Order:

$$
\begin{aligned}
\frac{dV_1^{(0)}(\xi)}{d\xi} &= -\frac{V_1^{(0)}(\xi)}{\omega_0} + \frac{a_1}{\omega_0} V_2^{(0)}(\xi - \omega_0\tau_0) \\
\frac{dV_2^{(0)}(\xi)}{d\xi} &= -\frac{V_2^{(0)}(\xi)}{\omega_0} + \frac{a_2}{\omega_0} V_1^{(0)}(\xi - \omega_0\tau_0)
\end{aligned}
$$

- Solutions:

$$
\begin{aligned}
V_1^{(0)}(\xi) &= A_0 \cos\xi + B_0 a_1 \sin(\omega_0\tau_0) \sin\xi \\
V_2^{(0)}(\xi) &= B_0 \cos\xi - \frac{A_0}{a_1 \sin(\omega_0\tau_0)} \sin\xi
\end{aligned}
$$

Vanishing of Secular Terms

- Inhomogeneity:

$$\frac{dV_1^{(n)}(\xi)}{d\xi} = -\frac{V_1^{(n)}(\xi)}{\omega_0} + \frac{a_1}{\omega_0}V_2^{(n)}(\xi - \omega_0\tau_0) + f_1^{(n)}(\xi)$$

$$\frac{dV_2^{(n)}(\xi)}{d\xi} = -\frac{V_2^{(n)}(\xi)}{\omega_0} + \frac{a_2}{\omega_0}V_1^{(n)}(\xi - \omega_0\tau_0) + f_2^{(n)}(\xi)$$

- Fourier Decompositions:

$$\begin{pmatrix} V_1^{(n)}(\xi) \\ V_2^{(n)}(\xi) \end{pmatrix} = \sum_{k=1}^{\infty} \left[\begin{pmatrix} a_{1,k}^{(n)} \\ a_{2,k}^{(n)} \end{pmatrix} \cos k\xi + \begin{pmatrix} b_{1,k}^{(n)} \\ b_{2,k}^{(n)} \end{pmatrix} \sin k\xi \right]$$

$$\begin{pmatrix} f_1^{(n)}(\xi) \\ f_2^{(n)}(\xi) \end{pmatrix} = \sum_{k=1}^{\infty} \left[\begin{pmatrix} \alpha_{1,k}^{(n)} \\ \alpha_{2,k}^{(n)} \end{pmatrix} \cos k\xi + \begin{pmatrix} \beta_{1,k}^{(n)} \\ \beta_{2,k}^{(n)} \end{pmatrix} \sin k\xi \right]$$

- Condition:

$$a_2 \sin(\omega_0\tau_0)\alpha_{1,1}^{(n)} + \beta_{2,1}^{n} = 0$$

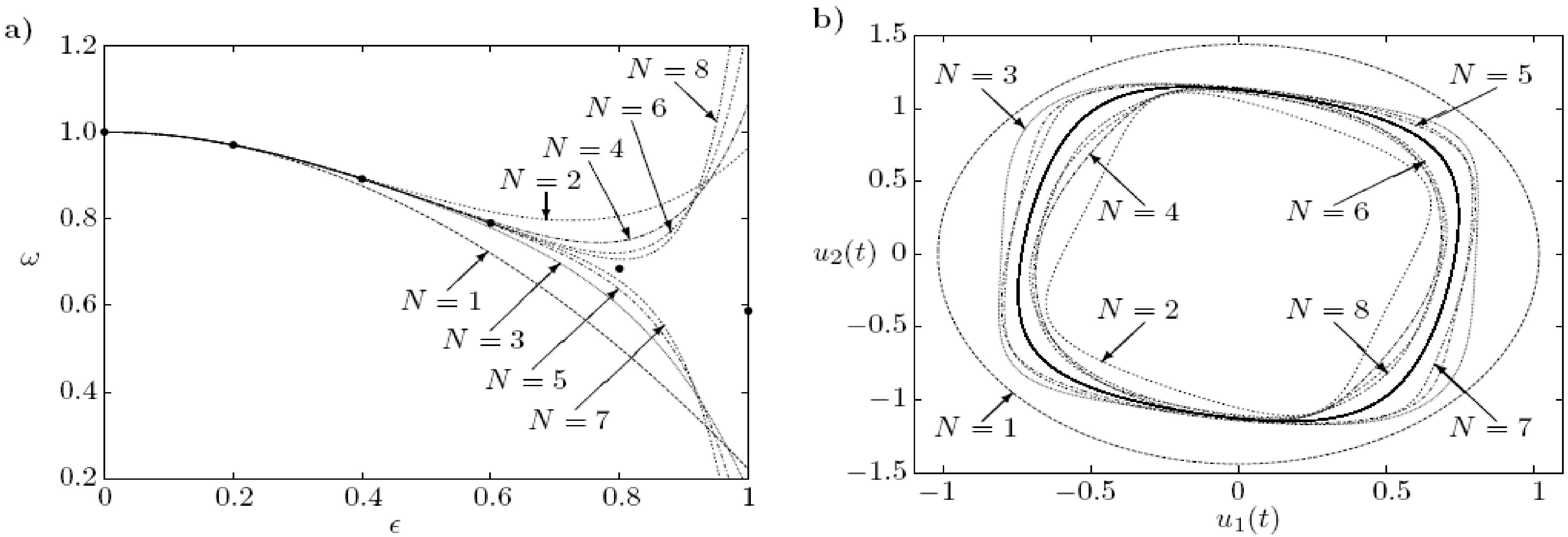
$$\alpha_{2,1}^{(n)} - a_2 \sin(\omega_0\tau_0)\beta_{1,1}^{(n)} = 0$$

- Second-Order Results:

$$\omega_2 = -\frac{\omega_0 + \omega_0^3}{1 + \tau_0 + \tau_0\omega_0^2}$$

$$A_0 = \pm\sqrt{\frac{8a_1^2\omega_0^2}{(1 + a_1^2 + \omega_0^2)(1 + \tau_0 + \omega_0^2\tau_0)}}$$

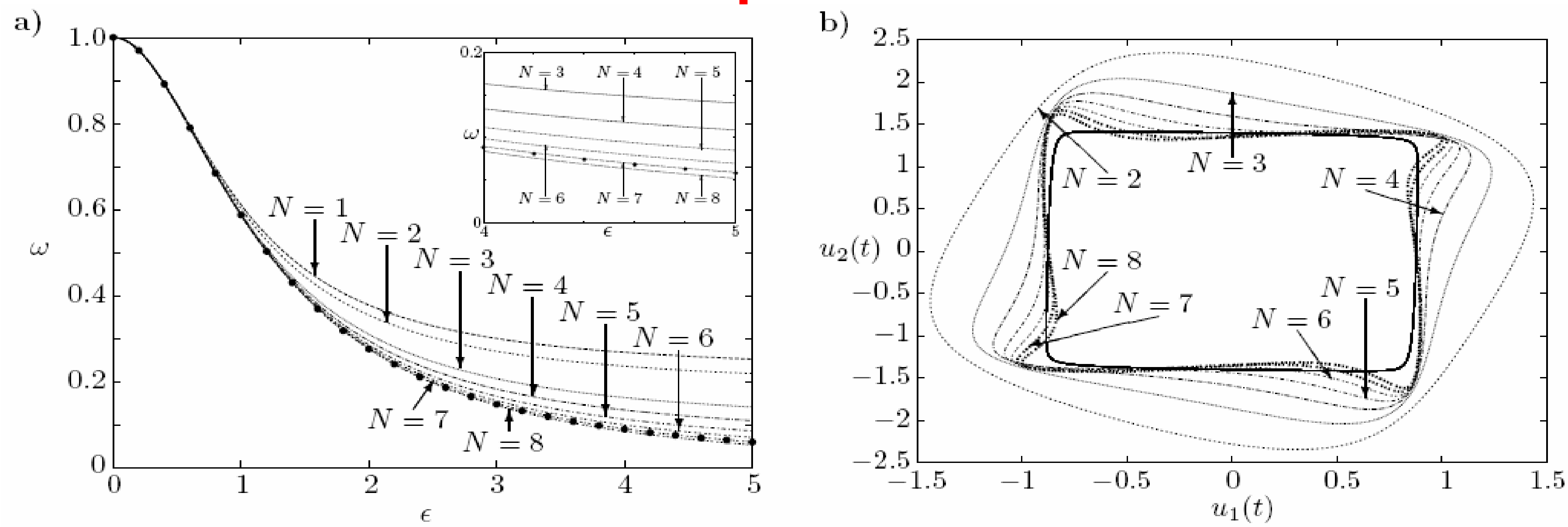
Perturbative Results



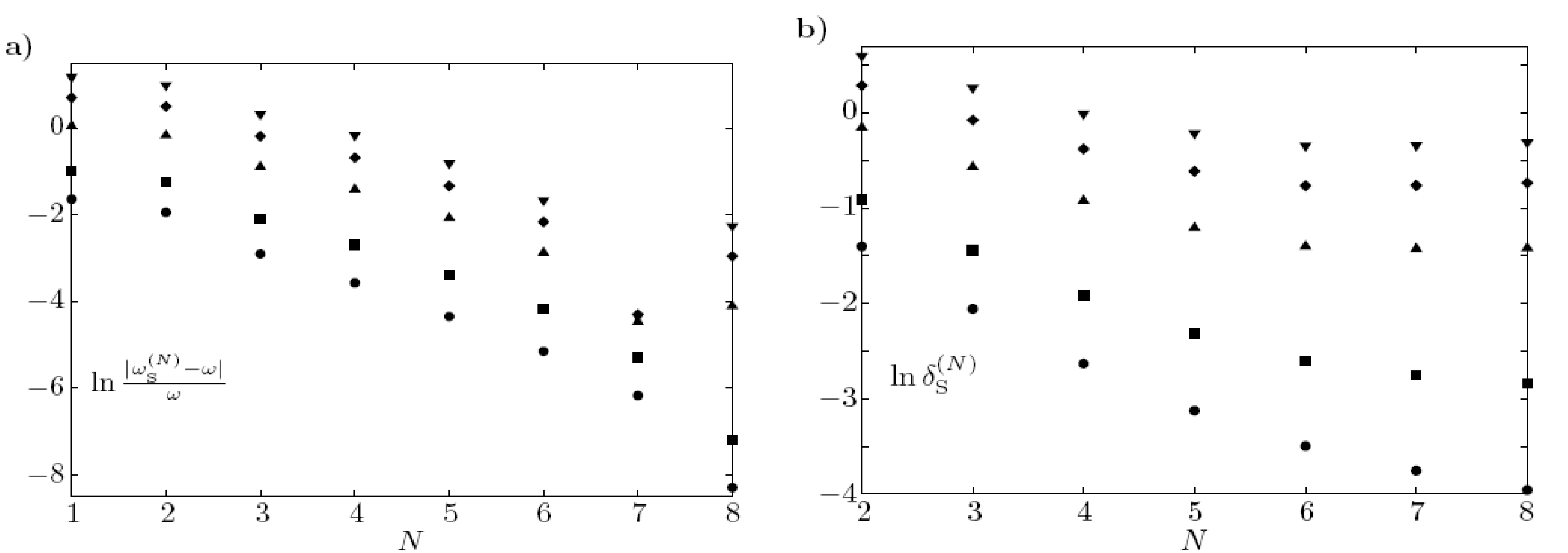
Shohat Resummation

- New Expansion Parameter: $$\mu = \frac{\epsilon^2}{1+\epsilon^2}$$
- Angular Frequency: $$\omega^{(N)} = \sum_{n=0}^{N} \omega_{2n}\epsilon^{2n} \;\longrightarrow\; \omega_S^{(N)} = \sum_{n=0}^{N} \mu^n \sum_{k=0}^{n} \binom{n-1}{k} \omega_{2(n-k)}$$
- Limit Cycle: $$\mathbf{u}^{(N)}(t) = \epsilon \sum_{n=0}^{N-1} \mathbf{V}^{(2n)}(\xi/\omega)\epsilon^{2n} \;\longrightarrow\; \mathbf{u}_S^{(N)}(t) = \epsilon \sum_{n=0}^{N-1} \mu^n \sum_{k=0}^{n} \binom{n-1}{k} \mathbf{V}^{(2(n-k))}(t)$$

Shohat Expansion Results



Convergence



Variational Perturbation Theory (VPT)

- **Angular Frequency**
- Perturbation Expansion:

$$\omega^{(N)} = \sum_{n=0}^{N} \omega_{2n}\epsilon^{2n} \qquad\qquad g = \epsilon^2$$

- Introduction of Variational Parameter to the Perturbation Expansion:

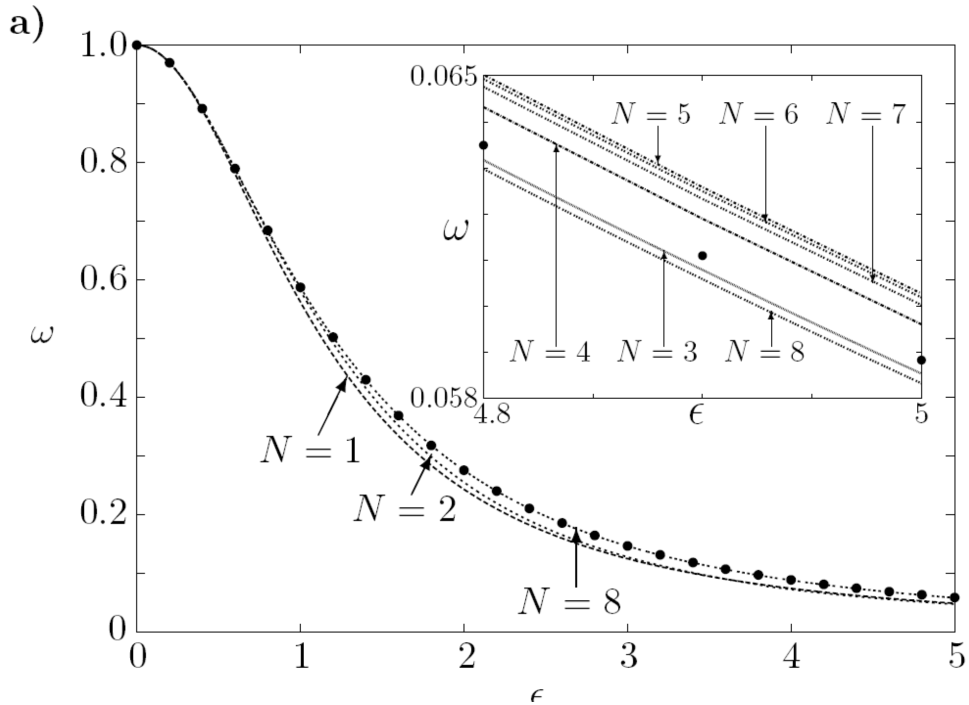
$$\omega_{\mathrm{VPT}}^{(N)}(g,K) = \sum_{n=0}^{N} \omega_{2n} g^n K^{p-nq} \sum_{k=0}^{N-n} \binom{(p-nq)/2}{k} \left(\frac{1}{K^2} - 1\right)^k$$

- First Order: $\omega = 1 - \frac{4}{2+\pi} g + \mathcal{O}(g^2)$

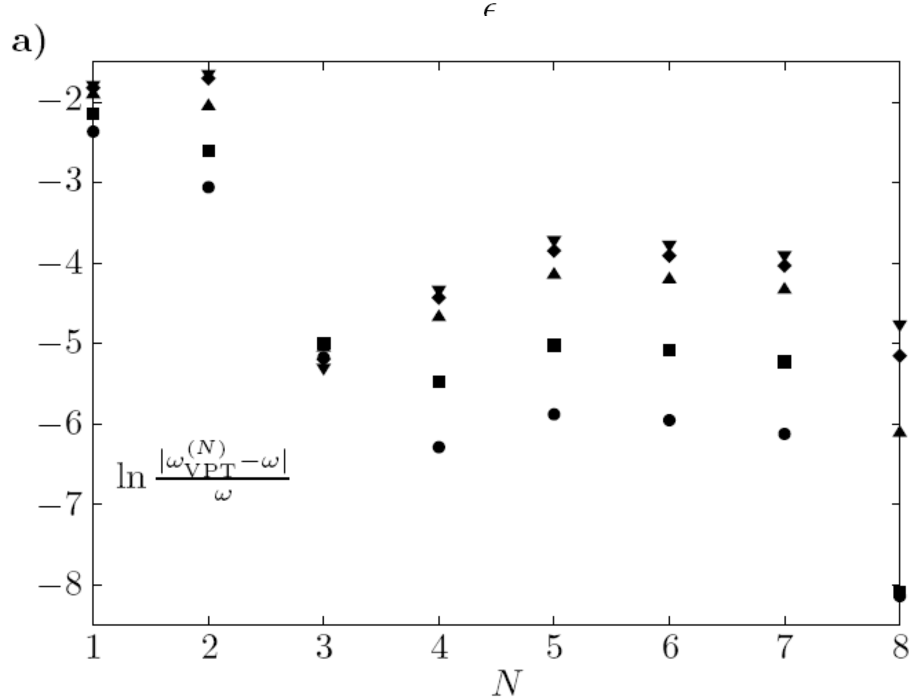
$$\omega_{\mathrm{VPT}}^{(1)}(g,K) = \frac{(2+\pi)(2K^2-1) - 4g}{K^4(2+\pi)} \qquad K^{(1)} = \sqrt{1 + \frac{4g}{2+\pi}}$$

- Result: $\omega_{\mathrm{VPT}}^{(1)}(g,K^{(1)}) = \frac{2+\pi}{4g+2+\pi}$

- **VPT Results:**



- **Convergence:**



- **Limit Cycle**

- Fourier Series:

$$\begin{pmatrix} V_1^{(n)}(\xi) \\ V_2^{(n)}(\xi) \end{pmatrix} = \sum_{k=1}^{\infty} \left[\begin{pmatrix} a_{1,k}^{(n)} \\ a_{2,k}^{(n)} \end{pmatrix} \cos k\xi + \begin{pmatrix} b_{1,k}^{(n)} \\ b_{2,k}^{(n)} \end{pmatrix} \sin k\xi \right]$$

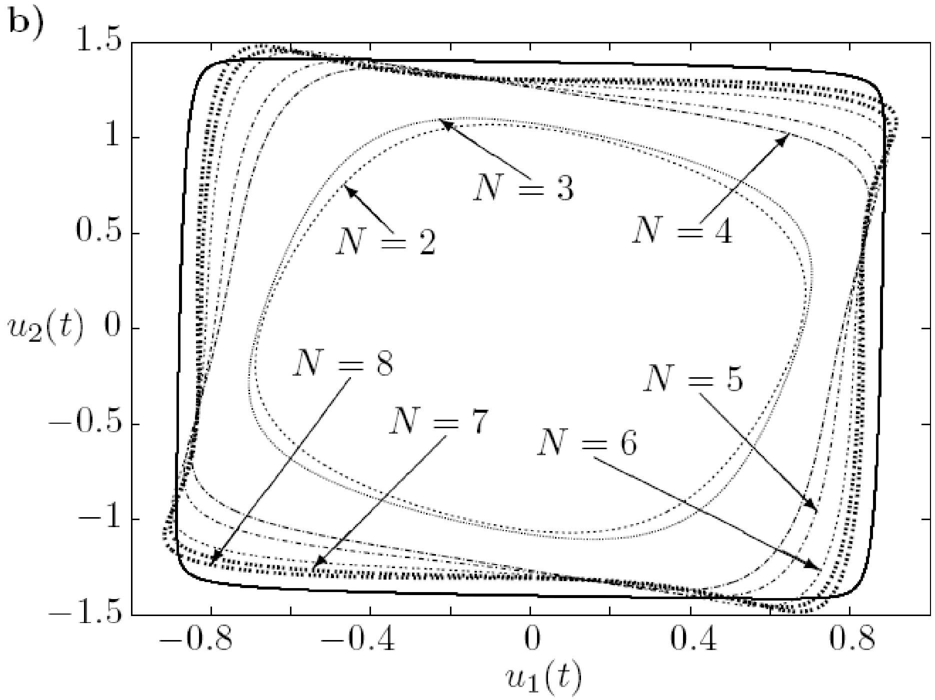
- Introduction of Variational Parameter to the Fourier Series:

$$(A/B)_{1/2,k}^{(N)} = \sum_{n=0}^{N-1} (a/b)_{1/2}^{(2n)} g^n \quad \longrightarrow \quad (A/B)_{1/2,k,\mathrm{VPT}}^{(N)}(g, K)$$

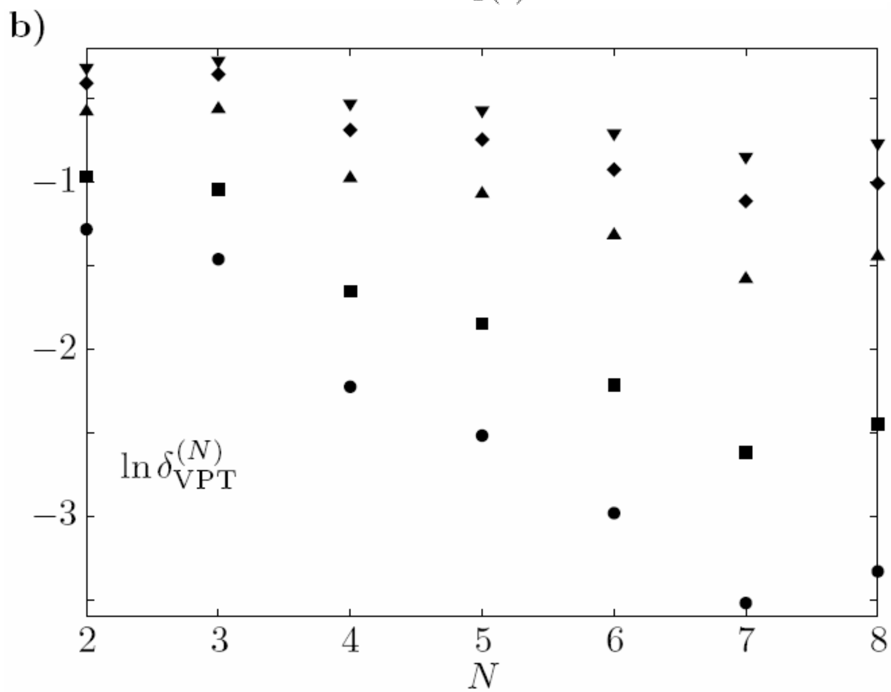
- Evaluation for Optimal Value from Angular Frequency:

$$\begin{pmatrix} V_{1,\mathrm{VPT}}^{(N)}(\xi) \\ V_{2,\mathrm{VPT}}^{(N)}(\xi) \end{pmatrix} = \sum_{k=1}^{\infty} \left[\begin{pmatrix} A_{1,k,\mathrm{VPT}}^{(N)}(g, K^{(N-1)}) \\ A_{2,k,\mathrm{VPT}}^{(N)}(g, K^{(N-1)}) \end{pmatrix} \cos k\xi + \begin{pmatrix} B_{1,k,\mathrm{VPT}}^{(N)}(g, K^{(N-1)}) \\ B_{2,k,\mathrm{VPT}}^{(N)}(g, K^{(N-1)}) \end{pmatrix} \sin k\xi \right]$$

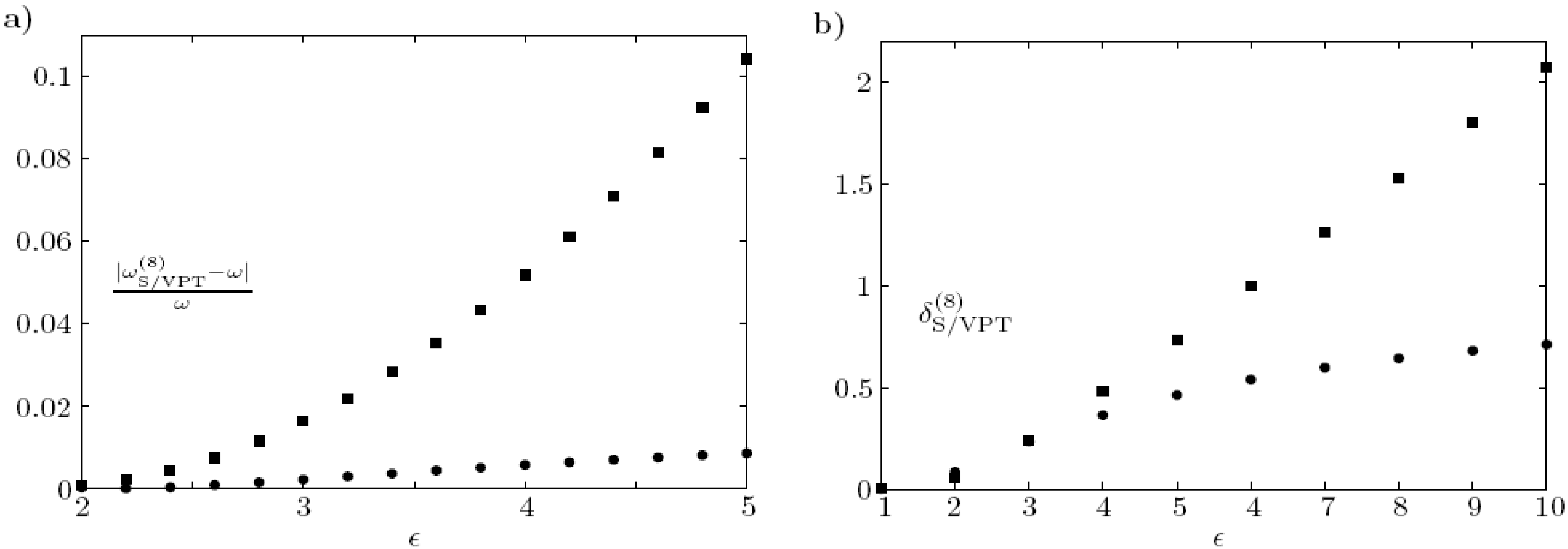
- **VPT Results:**



- **Convergence:**



Comparison VPT/Shohat



Brandt, Pelster, and Wessel (2006)

Outlook

- **Conduction-Velocity Asymmetry in Biological Systems:** $\tau^{(1)} \neq \tau^{(2)}$