Discrete Software Reliability Assessment with Discretized NHPP Models

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Abstract—A software reliability growth model is one of the fundamental techniques to assess software reliability quantitatively. The software reliability growth model is required to have a good performance in terms of goodness-of-fit, predictability, and so forth. In this paper, we propose discretized software reliability growth models. As to the software reliability growth modeling, discretized nonhomogeneous Poisson process models are investigated particularly for accurate software reliability assessment. We show that the discrete nonhomogeneous Poisson process models have better performance than discretized deterministic software reliability growth models which have been proposed so far. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

As one of the technologies to assess software reliability quantitatively, software reliability growth models (abbreviated as SRGMs) [1–5] have been researched in several research institutions and enterprises from the 1970s. By applying the SRGMs in the testing phase of software development process, software development managers can grasp the progress of software testing, decide a shipping time, and estimate the maintenance cost for undiscovered faults. In particular, nonhomogeneous Poisson process models (abbreviated as NHPP models) are known as one of the useful SRGMs for assessing and forecasting software reliability of the developed software systems. At present, the NHPP models contribute to software reliability assessment in many computer manufactures and software houses.

In recent years, because the size, complexity, and diversification of computer systems have grown dramatically, software development managers require SRGMs which enable us to assess software reliability more accurately than conventional SRGMs proposed so far. As one of the efficient techniques to improve the performance of the SRGMs, discretized software reliability growth

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modeling technique has been proposed. The discretized SRGMs can be derived by discretizing the original continuous SRGMs via using Hirota’s bilinearization methods [6]. As to deterministic SRGMs, Satoh [7] has proposed a discrete Gompertz curve model. Satoh and Yamada [8] have discussed a software reliability assessment method for discrete logistic curve models. And they have performed goodness-of-fit comparisons with these discrete statistical data analysis models (abbreviated as discrete SDA models) by using a new goodness-of-fit evaluation criterion [9].

In this paper we propose discrete NHPP models which have better performance than the discretized deterministic SRGMs in terms of goodness-of-fit and predictability, and derive several software reliability assessment measures such as a software reliability function and so on. Our discrete NHPP models have more advantages in terms of numerical calculations. The remainder of this paper is organized as follows. In Section 2, we discuss the discrete NHPP models and the estimation method of their parameters. And then, we perform goodness-of-fit comparisons of our discrete NHPP models in Section 3. In Section 4, we present numerical application for software reliability assessment measures. We conclude with a summary in Section 5.

2. DISCRETE NHPP MODELING

First, we assume that a discrete counting process \{N_n, n \geq 0\} \ (n = 0, 1, 2, \ldots) representing the cumulative number of faults detected up to the \(n^{\text{th}}\) testing-period has the following properties based on a continuous NHPP [10]:

\[
\Pr\{N_n = x \mid N_0 = 0\} = \frac{(\Lambda_n)^x}{x!} \exp[-\Lambda_n], \quad n, x = 0, 1, 2, \ldots,
\]

where \(\Pr\{A\}\) means the probability of event \(A\). \(\Lambda_n\) in equation (1) is a mean value function of the discrete counting process, which represents the expected cumulative number of faults detected up to the \(n^{\text{th}}\) testing-period. The discrete counting process \{N_n, n \geq 0\} \ (n = 0, 1, 2, \ldots) obeying the above properties is called a discrete NHPP in this paper.

In this section, discrete NHPP models derived by discretizing the original NHPP models are discussed. Our discrete NHPP models discussed in this paper are derived by using special difference methods applying Hirota’s bilinearization methods. Generally, the ordinary forward or central difference equations do not have exact solutions, i.e., the difference equations do not conserve the properties of the continuous NHPP models. However, the discretization method by using the Hirota’s bilinearization method can overcome the problems above. And, our discrete NHPP models can easily obtain the parameter estimates by using the method of least-squares.

2.1. Discrete Exponential SRGM

We propose a discrete analog of the original exponential SRGM of which the mean value function is the simplest form among the SRGMs. The difference equation for this model has an exact solution. Let \(H_n\) denote the expected cumulative number of faults detected up to \(n^{\text{th}}\) testing-period. A discrete analog of the exponential SRGM is derived as

\[
H_{n+1} - H_n = \delta b (a - H_n),
\]

from the assumptions of the continuous NHPP model. Solving the above equation, we can obtain an exact solution \(H_n\) in equation (2) as

\[
H_n = a [1 - (1 - \delta b)^n], \quad a > 0, \quad 0 < b < 1,
\]

where \(\delta\) represents the constant time-interval, \(a\) the expected initial fault content, and \(b\) the fault detection rate per fault. As \(\delta \to 0\), equation (3) converges to an exact solution of the original exponential SRGM which is described by the differential equation.
The parameter estimates $\hat{a}$ and $\hat{b}$ which are the estimated values of $a$ and $b$ can be obtained by the following procedure using the method of least-squares. First, we derive the following regression equation from equation (2):

$$Y_n = A + BH_n,$$

where

$$Y_n = H_{n+1} - H_n,$$

$$A = \delta ab,$$

$$B = -\delta b.$$\hspace{1cm}(5)

Based on the regression analysis, we can estimate $\hat{A}$ and $\hat{B}$ which are the estimates of $A$ and $B$ in equation (5). Thus, the parameter estimates $\hat{a}$ and $\hat{b}$ can be obtained as

$$\hat{a} = \frac{\hat{A}}{\hat{B}},$$

$$\hat{b} = -\frac{\hat{B}}{\delta}.$$\hspace{1cm}(6)

$Y_n$ in equation (4) is independent of $\delta$ because $\delta$ is not used in calculating $Y_n$ in equation (5). Hence, we can obtain the same parameter estimates $\hat{a}$ and $\hat{b}$, respectively, when we choose any value of $\delta$.

2.2. Discrete Inflection S-Shaped SRGM

We also propose a discrete analog of the original inflection S-shaped SRGM which is the continuous one. Let $I_n$ denote the expected cumulative number of faults detected up to $n^{th}$ testing-period. A discrete analog of the inflection S-shaped SRGM is derived as

$$I_{n+1} - I_n = \delta abl + \frac{\delta b(1-2l)}{2} [I_n - I_{n+1}] - \frac{\delta b(1-l)}{a} I_n I_{n+1},$$\hspace{1cm}(7)

from the assumptions of the continuous NHPP model. Solving the above equation, we can obtain an exact solution $I_n$ in equation (7) as

$$I_n = \frac{a[1 - ((1 - (1/2)\delta b)/(1 + (1/2)\delta b))^n]}{1 + c((1 - (1/2)\delta b)/(1 + (1/2)\delta b))^n}, \quad a > 0, \quad 0 < b < 1, \quad c > 0, \quad 0 \leq l \leq 1,$$

where $\delta$ represents the constant time-interval, $a$ the expected initial fault content, $b$ the fault detection rate per fault, and $c$ the inflection parameter. The inflection parameter is defined as $c = (1 - l)/l$, where $l$ is the inflection rate which indicates the ratio of the number of detectable faults to the total number of faults in the software system. As $\delta \to 0$, equation (8) converges to an exact solution of the original inflection S-shaped SRGM which is described by the differential equation.

The inflection point occurs when

$$\bar{n} = \begin{cases}    [n'], & \text{if } \Delta I_{[n']} \geq \Delta I_{[n'] + 1}, \\    [n'] + 1, & \text{otherwise}, \end{cases}$$\hspace{1cm}(9)

where the difference operator $\Delta I_n$ in equation (9) is defined as

$$\Delta I_n \equiv \frac{I_{n+1} - I_n}{\delta}. $$\hspace{1cm}(10)
\([x]\) represents the Gaussian symbol for any real number \(x\), and
\[
n' = -\log c \frac{\log c}{\log ((1 - (1/2)\delta b)/(1 + (1/2)\delta b))} - 1.
\] (11)

Moreover, we define \(t^{**}\) as
\[
t^{**} = n'\delta.
\] (12)

When \(n'\) is an integer, we can see that \(t^{**}\) converges the inflection point of the inflection S-shaped SRGM which is described by the differential equation as \(\delta \to 0\), that is,
\[
t^{**} = -\delta \log c \frac{\log c}{\log ((1 - (1/2)\delta b)/(1 + (1/2)\delta b))} - \delta \to \frac{\log c}{b}, \quad \text{as } \delta \to 0.
\] (13)

Incidentally, the inflection S-shaped SRGM is regarded as a Riccati equation. Hirota [6] proposed a discrete Riccati equation which has an exact solution. A Bass model [11] which forecasts the innovation diffusion of products is also a Riccati equation. Satoh [12] proposed a discrete Bass model which can overcome the shortcomings of the ordinary least-square procedures in the continuous Bass model.

We can derive a regression equation to estimate the model parameters from equation (7). The regression equation is obtained as
\[
Y_n = A + BK_n + CL_n,
\] (14)
where
\[
Y_n = I_{n+1} - I_n,
K_n = I_n + I_{n+1},
L_n = I_n I_{n+1},
A = \delta abl,
B = \delta b \left(\frac{1 - 2l}{2}\right),
C = \left(-\delta b \frac{1 - l}{a}\right).
\] (15)

Based on the regression analysis, we can estimate \(\hat{A}\), \(\hat{B}\), and \(\hat{C}\) by using the observed data, which are the estimates of \(A\), \(B\), and \(C\), respectively. Therefore, we can obtain the parameter estimates \(\hat{a}\), \(\hat{b}\), and \(\hat{l}\) from equation (15) as follows:
\[
\hat{a} = \frac{\hat{A}}{\sqrt{\hat{B}^2 - \hat{A}\hat{C} - \hat{B}}},
\hat{b} = \frac{2\sqrt{\hat{B}^2 - \hat{A}\hat{C}}}{\delta},
\hat{l} = \frac{1 - \hat{B}/\sqrt{\hat{B}^2 - \hat{A}\hat{C}}}{2}.
\] (16)

\(Y_n\), \(K_n\), and \(L_n\) in equation (14) are independent of \(\delta\) because \(\delta\) is not used in calculating \(Y_n\), \(K_n\), and \(L_n\) in equation (15). Hence, we can obtain the same parameter estimates \(\hat{a}\), \(\hat{b}\), and \(\hat{l}\), respectively, when we choose any value of \(\delta\).

### 3. MODEL COMPARISONS

We perform goodness-of-fit comparisons of our discrete NHPP models in the preceding section with the discrete SDA models such as logistic and Gompertz curve models [7–9] in terms of a predicted relative error [3], a mean square error (MSE) [4], and Akaike’s information criterion (AIC) [13]. First, we arrange four data sets (DS1–DS4) used in the model comparisons. The data sets DS1 [3] and DS2 [14] indicate exponential growth curves in terms of the cumulative number of detected faults versus testing time, and DS3 [7] and DS4 [14] indicate S-shaped ones, respectively.
3.1. Comparison Criteria

We introduce the predicted relative error and the MSE as comparison criteria in this section. The predicted relative error at arbitrary testing time $t_e$, $R_e[t_e]$, can be expressed as follows:

$$R_e[t_e] = \frac{\hat{y}(t_e; t_q) - q}{q},$$  \hspace{1cm} (17)\

where $\hat{y}(t_e; t_q)$ is the estimated value of the mean value function at the termination time $t_q$ by using the observed data by the arbitrary testing time $t_e$ ($0 \leq t_e \leq t_q$), and $q$ is the observed cumulative number of faults detected by the termination time $t_q$.

And the MSE is obtained by dividing the number of data pairs into the sum of squared errors between the observed and estimated cumulative numbers of detected faults. Thus, the MSE value is obtained by

$$\text{MSE} = \frac{1}{K} \sum_{k=1}^{K} [y_k - \hat{y}(t_k)]^2,$$  \hspace{1cm} (18)\

where $\hat{y}(t_k)$ and $y_k$ denote the estimated and observed values of the expected cumulative number of faults detected during $(0, t_k]$ ($k = 1, 2, \ldots, K$), respectively.

3.2. Numerical Results

Figures 1–4 show the results of the model comparisons based on the predicted relative error for DS1–DS4, respectively. And Table 1 shows the result of model comparison based on the MSE for each model. In the model comparison based on the MSE, the results depend on the number of model parameters of each model. Accordingly, we provide Table 2 which shows the result of model comparison based on the AIC for the discrete exponential and inflection S-shaped SRGMs which fit better to the actual data sets. From Table 2, the model comparison based on the MSE can be validated.

From these three results of goodness-of-fit comparisons, we can conclude that the discrete exponential SRGM is a more useful model for software reliability assessment for the observed data which indicates an exponential growth curve, and the discrete inflection S-shaped SRGM.
Figure 2. The predicted relative errors for DS2.

Figure 3. The predicted relative errors for DS3.

Figure 4. The predicted relative errors for DS4.
Table 1. The result of model comparison based on the MSE. (SRGM: software reliability growth model.)

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Discrete Exponential SRGM</th>
<th>Discrete Inflection S-shaped SRGM</th>
<th>Discrete Logistic Curve Model</th>
<th>Discrete Gompertz Curve Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS1</td>
<td>39.643</td>
<td>12.141</td>
<td>101.92</td>
<td>72.854</td>
</tr>
<tr>
<td>DS2</td>
<td>1762.5</td>
<td>2484.0</td>
<td>27961</td>
<td>13899</td>
</tr>
<tr>
<td>DS3</td>
<td>25631</td>
<td>9598.1</td>
<td>149441</td>
<td>19579</td>
</tr>
<tr>
<td>DS4</td>
<td>11722</td>
<td>438.59</td>
<td>49741</td>
<td>27312</td>
</tr>
</tbody>
</table>

Table 2. The result of model comparison between the discrete exponential SRGM and the discrete inflection S-shaped SRGM based on the AIC.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Discrete Exponential SRGM</th>
<th>Discrete Inflection S-shaped SRGM</th>
<th>Absolute Value of Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS1</td>
<td>110.031</td>
<td>109.195</td>
<td>0.836</td>
</tr>
<tr>
<td>DS2</td>
<td>115.735</td>
<td>118.752</td>
<td>3.017</td>
</tr>
<tr>
<td>DS3</td>
<td>617.434</td>
<td>606.132</td>
<td>11.30</td>
</tr>
<tr>
<td>DS4</td>
<td>315.069</td>
<td>274.818</td>
<td>40.25</td>
</tr>
</tbody>
</table>

is a more useful one for assessment after 60% of the testing progress ratio for the observed data which indicates an S-shaped growth curve.

4. SOFTWARE RELIABILITY ASSESSMENT

We can derive useful metrics for quantitative software reliability assessment from the proposed SRGMs by using the properties of the discrete NHPP. In this section we adopt DS1 in which 25 pairs \((t_k, y_k)\) \((k = 1, 2, \ldots, 25; t_{25} = 25, y_{25} = 136)\) have been observed, for the discrete exponential SRGM, and DS3 in which 59 pairs \((t_k, y_k)\) \((k = 1, 2, \ldots, 59; t_{59} = 59, y_{59} = 5186)\) have been observed, for the discrete inflection S-shaped SRGM, where \(y_k\) is the cumulative number of faults detected by the execution of testing time \(t_k\). The observation time unit of DS1 is CPU hours, and that of DS3 the number of weeks. Figures 5 and 6 depict the estimated mean value functions of \(H_n\) in equation (3) and \(I_n\) in equation (8), respectively, where the related quantities for \(H_n\) in equation (3) are \(\tilde{a} = 139.9\) and \(\tilde{b} = 0.113\) and for \(I_n\) in equation (8) are \(\tilde{a} = 5217, \tilde{b} = 0.091, \tilde{c} = 2.350, n^* = 8.383, (n^*) = 8,\) and \(\tilde{n} = 9\).

Figure 5. The estimated discrete mean value function, \(\hat{H}_n\), for DS1.
4.1. Reliability Function

We can derive the software reliability function by using the properties of the discrete NHPP. Given that the testing has been going on up to \( n \)th testing-period by which \( x \) software faults have been detected, the probability that a software failure does not occur in the time interval \((n, n + h]\) \((h = 0, 1, 2, \ldots)\) is given as

\[
R(n, h) \equiv \Pr\{N_{n+h} - N_n = 0 \mid N_n = x\}
= \exp[-\{A_{n+h} - A_n\}], \quad n, h = 0, 1, 2, \ldots,
\]

(19)

\(R(n, h)\) is a so-called software reliability based on the discrete NHPP.

4.2. Reliability Growth Rate

In addition to the software reliability function in equation (19), we consider a software reliability growth rate which can reflect the influence of debugging effort during each testing-period to analyze the testing effect of each period on the software reliability. In order to assess the influence
of debugging effort during each testing-period on the software reliability quantitatively, we newly propose a quantitative measure by using the discrete NHPP models, which is named as a software reliability growth rate. The software reliability growth rate is formulated from equation (19) as follows:

\[ r(n, h) = R(n, h) - R(n - 1, h). \] 

For \( h = 1 \), the software reliability growth rate for \( H_n \) (DS1) and for \( I_n \) (DS3) are shown in Figures 7 and 8, respectively.

In Figure 7, we can see that the software reliability growth rate for given \( h = 1 \) per period increases by around \( n = 25 \). We can estimate the software reliability growth rate at the testing termination time of DS1 to be about \( r(25, 1) \approx 4.353 \times 10^{-2} \), i.e., it is gained as \( r(25, 1) = 4.353 \times 10^{-2} \) of the value of software reliability during a period between the 24\textsuperscript{th} and 25\textsuperscript{th} testing.

In Figure 8, we can also see that the testing by about the 40\textsuperscript{th} period does not contribute to software reliability growth. Therefore, we conclude that it is necessary to test more after around the 40\textsuperscript{th} testing-period. We can also estimate it at the testing termination time of DS3 to be about \( r(59, 1) \approx 2.155 \times 10^{-2} \).

5. CONCLUDING REMARKS

We have proposed two discrete NHPP models derived from the exponential SRGM and inflection S-shaped SRGM which are the continuous NHPP models. And we have verified that the discrete NHPP models have better performance for software reliability assessment in terms of the predicted relative error and the MSE in the goodness-of-fit comparisons in Section 3. And then, we have derived new software reliability measures for the discrete NHPP models, and have shown numerical examples for these measures. Comparing with the continuous NHPP models, we consider that the proposed procedures in this paper contribute to decreasing the labor of software development managers because the proposed procedures yield accurate estimates of software reliability measures simply. In particular, the inflection parameter \( l \) in equation (7) can be obtained along with the other parameters \( a \) and \( b \) by applying the discrete inflection S-shaped SRGM and its parameter estimation procedure to the observed data, simultaneously.

Further studies are needed to examine the performance of the proposed two discrete NHPP models more by using many other observed data sets, and to deal with problems of software development management such as estimating the optimal software release time [15] and the optimal allocation of testing-effort [16].
REFERENCES


