Computation in an Asymptotic Expansion Method

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Outlines

Motivations

- Difficulties in recent markets
  - Increasing liquidity of hybrid securities such as
    - options on Currencies
    - PRDC and other complex securities
  - Longterm securities
  - Extremely volatile market
  - Practical demands for
    - Fast valuation for trading
    - Fast calibration
    - Fast computation of Greeks
- Needs for
  - general models
  - an (exact or)approximate analytical formula
  - higher accuracy

An Asymptotic Expansion Approach

- **An asymptotic expansion approach**
  - **is initiated by Kunitomo and Takahashi[1992], Takahashi[1995,1999] and Yoshida[1992]**
  - **provides us the unified methodology for evaluation in a general model**
  - **is mathematically justified by Watanabe theory in Malliavin calculus**
    - **Watanabe[1987], Yoshida[1992], Kunitomo and Takahashi[2003]**

An Asymptotic Expansion Approach(cont'd)

- There are many applications of the asymptotic expansion approach in finance.
    - Average options: Yoshida[1992], Kunitomo and Takahashi[1992], Takahashi[1999]
    - Optimal portfolio: Takahashi and Yoshida[2004,2005]
    - Cross currency models: Takahashi[1995], Osajima[2006,2007], Jackel and Kawai[2007], Takahashi and Takehara[2007,2008a,2008b]
    - Other applications: Kunitomo and Takahashi[2001], Kawai[2003], Takahashi and Saito[2003], Muroi[2005], Kunitomo and Kim[2005], Takahashi and Uchida[2006], Matsuoka, Takahashi and Uchida[2006], etc.

Objectives

- **In this talk we will present**
  - **basics of an asymptotic expansion**
  - **two alternative approach for the key step of computation**
  - **some practically important examples**

An Underlying Model: a Black-Schloes Economy

Hereafter we will focus on the following particular case of a Black-Scholes(B-S)-type economy. For general cases, see Takahashi, Takehara and Toda[2009].

- Settings
  - (W, P): a one-dimensional Wiener Space
  - P: a risk-neutral measure
  - r: a constant risk-free rate, assumed to be zero
- $S^{(\epsilon)}$: a single risky asset price process dependent on $\epsilon \in (0, 1]$

$$S_t^{(\epsilon)} = S_0 + \epsilon \int_0^t \sigma(S_s^{(\epsilon)}, s) dW_s \quad (1)$$

  - some regularity conditions on σ: $\mathrm{R}_+^2 \mapsto \mathrm{R}$
- Consider the following pricing problem;

$$V(0, T) = \mathrm{E}\left[\Phi(S_T^{(\epsilon)})\right] \quad (2)$$

An Asymptotic Expansion in a B-S Economy

- Then, $S_t^{(\epsilon)}$ has its asymptotic expansion.

An Asymptotic Expansion of $S_t^{(\epsilon)}$

$$S_t^{(\epsilon)} = S_0 + \sum_{n=1}^{N} \frac{\epsilon^n}{n!} A_{nt} + o(\epsilon^N) \tag{3}$$

for every $N = 1, 2, \cdots$ as $\epsilon \downarrow 0$.

- $S_t^{(0)} = \lim_{\epsilon \downarrow 0} S_t^{(\epsilon)} = S_0$ for all t
- A_{1t} is given by

$$A_{1t} = \int_0^t \sigma(S_s^{(0)}, s) dW_s =: \int_0^t \sigma_s^{(0)} dW_s \tag{4}$$

which follows $N(0, \Sigma_t)$;

$$\Sigma_t = \int_0^t (\sigma_s^{(0)})^2 ds. \tag{5}$$

- Assume that for all t, $\Sigma_t > 0$.

An Asymptotic Expansion in a B-S Economy(cont'd)

- Moreover, up to the third order we have

$$A_{2t} = 2\int_0^t \partial\sigma_s^{(0)} A_{1s} dW_s \tag{6}$$

$$A_{3t} = 3\int_0^t \left(\partial^2\sigma_s^{(0)} (A_{1s})^2 + \partial\sigma_s^{(0)} A_{2s}\right) dW_s \tag{7}$$

where $\partial^n\sigma_s^{(0)} = \frac{\partial^n}{\partial x^n}\sigma(x,t)|_{x=S_t^{(0)}}$.

- Normalize $S_T^{(\epsilon)}$ as $G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon}$. Then,

$$G^{(\epsilon)} = A_{1T} + \sum_{n=1}^{N} \frac{\epsilon^n}{(n+1)!} A_{(n+1)t} + o(\epsilon^N). \tag{8}$$

An Asymptotic Expansion in a B-S Economy(cont'd)

- Then, $\mathbf{E}\left[\Phi(G^{(\epsilon)})\right]$ has its asymptotic expansion in the sense of Watanabe[1987] or Yoshida[1992]. Especially, taking $N=2$, we have

$$
\begin{aligned}
& \mathbf{E}\left[\Phi(G^{(\epsilon)})\right] \\
= \; & \mathbf{E}\left[\Phi(A_{1T})\right] + \epsilon \mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{2T}\right] \\
& + \epsilon^2 \left\{ \mathbf{E}\left[\Phi^{(1)}(A_{1T})A_{3T}\right] + \frac{1}{2}\mathbf{E}\left[\Phi^{(2)}(A_{1T})(A_{2T})^2\right] \right\} + o(\epsilon^2)
\end{aligned}
$$

An Asymptotic Expansion in a B-S Economy(cont'd)

- Then, $\mathrm{E}\left[\Phi(G^{(\epsilon)})\right]$ has its asymptotic expansion in the sense of Watanabe[1987] or Yoshida[1992]. Especially, taking $N = 2$, we have

$$
\begin{aligned}
&\mathrm{E}\left[\Phi(G^{(\epsilon)})\right] \\
=\ & \mathrm{E}\left[\Phi(A_{1T})\right] + \epsilon \mathrm{E}\left[\Phi^{(1)}(A_{1T})A_{2T}\right] \\
& + \epsilon^2 \left\{ \mathrm{E}\left[\Phi^{(1)}(A_{1T})A_{3T}\right] + \frac{1}{2}\mathrm{E}\left[\Phi^{(2)}(A_{1T})(A_{2T})^2\right] \right\} + o(\epsilon^2) \\
=\ & \mathrm{E}\left[\Phi(A_{1T})\right] + \epsilon \mathrm{E}\left[\Phi^{(1)}(A_{1T})\mathrm{E}\left[A_{2T}|A_{1T}\right]\right] \\
& + \epsilon^2 \left\{ \mathrm{E}\left[\Phi^{(1)}(A_{1T})\mathrm{E}\left[A_{3T}|A_{1T}\right]\right] \right. \\
& \qquad \left. + \frac{1}{2}\mathrm{E}\left[\Phi^{(2)}(A_{1T})\mathrm{E}\left[(A_{2T})^2|A_{1T}\right]\right] \right\} + o(\epsilon^2). \qquad (9)
\end{aligned}
$$

An Asymptotic Expansion in a B-S Economy(cont'd)

- Since A_{1T}'s distribution has already been specified as a normal dist.,

$$\begin{aligned} \mathbf{E}\left[\Phi(G^{(\epsilon)})\right] &= \int_{\mathbf{R}} \Phi(x)f(x)dx \\ &+ \epsilon \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{2T}|A_{1T}=x]f(x)\}dx \\ &+ \epsilon^2 \left\{ \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{3T}|A_{1T}=x]f(x)\}dx \right. \\ &+ \left. \frac{1}{2}\int_{\mathbf{R}} \Phi(x)(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}[(A_{2T})^2|A_{1T}=x]f(x)\}dx \right\} + o(\epsilon^2) \end{aligned} \tag{10}$$

where $f(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(-\frac{x^2}{2\Sigma_T}\right)$.

- Then, once you evaluate these conditional expectations, the expansion of $\mathbf{E}\left[\Phi(G^{(\epsilon)})\right]$ is obtained.

An Asymptotic Expansion in a B-S Economy(cont'd)

- In previous works applying the asymptotic expansion approach
  - they used some formulas given in Takahashi[1995,1999] or Takahashi and Takehara[2007] for computation of these conditional expectations.
    - formulas for the expansion up to the third order
    - a general computation scheme has not been given explicitly so far
- We will give two alternative but equivalent computation schemes for these quantities.

Computation of Conditional Expectations

- Here a computation algorithm for the key step of our expansion, that is evaluation of conditional expectations, is proposed.
  - based on further expansions into iterated Itô integrals
- Recall that

$$
\begin{aligned}
A_{2T} &= 2\int_0^T \partial\sigma_s^{(0)} A_{1s} dW_s \\
&= 2\int_0^T \int_0^{t_1} \partial\sigma_{t_1}^{(0)} \sigma_{t_2}^{(0)} dW_{t_2} dW_{t_1} \\
A_{3T} &= 3\int_0^T \left(\partial^2\sigma_s^{(0)} (A_{1s})^2 + \partial\sigma_s^{(0)} A_{2s}\right) dW_s.
\end{aligned}
$$

Expansions into iterated Itô integrals

- **Then, with (iterated)application of Itô's formula, we obtain**

$$\begin{aligned} A_{3T} &= 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma_{t_1}^{(0)}\partial\sigma_{t_2}^{(0)}\sigma_{t_3}^{(0)}dW_{t_3}dW_{t_2}dW_{t_1} \\ &+ 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial^2\sigma_{t_1}^{(0)}\sigma_{t_2}^{(0)}\sigma_{t_3}^{(0)}dW_{t_3}dW_{t_2}dW_{t_1} \\ &+ 3\int_0^T\int_0^{t_1}\partial^2\sigma_{t_1}^{(0)}(\sigma_{t_2}^{(0)})^2dt_2dW_{t_1}. \end{aligned} \tag{11}$$

Expansions into iterated Itô integrals(cont'd)

- **Similarly,**

$$
\begin{aligned}
& (A_{2T})^2 \\
= \; & 16 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma_{t_1}^{(0)} \partial\sigma_{t_2}^{(0)} \sigma_{t_3}^{(0)} \sigma_{t_4}^{(0)} dW_{t_4} dW_{t_3} dW_{t_2} dW_{t_1} \\
+ \; & 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma_{t_1}^{(0)} \sigma_{t_2}^{(0)} \partial\sigma_{t_3}^{(0)} \sigma_{t_4}^{(0)} dW_{t_4} dW_{t_3} dW_{t_2} dW_{t_1} \\
+ \; & 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \partial\sigma_{t_1}^{(0)} \partial\sigma_{t_2}^{(0)} (\sigma_{t_3}^{(0)})^2 dt_3 dW_{t_2} dW_{t_1} \\
+ \; & 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \partial\sigma_{t_1}^{(0)} \partial\sigma_{t_2}^{(0)} \sigma_{t_2}^{(0)} \sigma_{t_3}^{(0)} dW_{t_3} dt_2 dW_{t_1} \\
+ \; & 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \left(\partial\sigma_{t_1}^{(0)}\right)^2 \sigma_{t_2}^{(0)} \sigma_{t_3}^{(0)} dW_{t_3} dW_{t_2} dt_1 \\
+ \; & 4 \int_0^T \int_0^{t_1} \left(\partial\sigma_{t_1}^{(0)}\right)^2 (\sigma_{t_2}^{(0)})^2 dt_2 dt_1. \qquad (12)
\end{aligned}
$$

- A_{2T}, A_{3T} **and** $(A_{2T})^2$ **are now further expanded into iterated Itô integrals.**

Conditional Expectations as a sum of Hermite polynomials(cont'd)

- **Then, the following lemma is essential.**

Lemma 1: a version of Proposition 3 of Nualart, Üstünel and Zakai[1988]

$$\mathbf{E}\left[\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f_n(t_1,\cdots,t_n)dW_{t_n}\cdots dW_{t_1} \,\Big|\, \int_0^T q(t)dW_t = x\right]$$
$$= \left(\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f_n(t_1,\cdots,t_n)q(t_1),\cdots q(t_n)d_{t_n}\cdots d_{t_1}\right) \times \frac{H_n(x;\Sigma_q)}{\Sigma_q^n} \tag{13}$$

where $\Sigma_q = \int_0^T q^2(t)dt$ and $H_n(x;\Sigma)$ is the n-th order Hermite polynomial defined by

$$H_n(x;\Sigma) = (-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma}.$$

Conditional Expectations as a sum of Hermite polynomials(cont'd)

- **By Lemma 1, you obtain**

$$\begin{aligned} & \mathbf{E}[A_{2T}|A_{1T}=x] \\ = & \left(2\int_0^T\int_0^{t_1}\partial\sigma_{t_1}^{(0)}\sigma_{t_1}^{(0)}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2} \end{aligned} \quad (14)$$

$$(15)$$

$$\begin{aligned} & \mathbf{E}[A_{3T}|A_{1T}=x] \\ = & \left(6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma_{t_1}^{(0)}\sigma_{t_1}^{(0)}\partial\sigma_{t_2}^{(0)}\sigma_{t_2}^{(0)}(\sigma_{t_3}^{(0)})^2 dt_3 dt_2 dt_1\right. \\ + & \left. 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial^2\sigma_{t_1}^{(0)}\sigma_{t_1}^{(0)}(\sigma_{t_2}^{(0)})^2(\sigma_{t_3}^{(0)})^2 dt_3 dt_2 dt_1\right)\frac{H_3(x;\Sigma_T)}{\Sigma_T^3} \\ + & \left(3\int_0^T\int_0^{t_1}\partial^2\sigma_{t_1}^{(0)}\sigma_{t_1}^{(0)}(\sigma_{t_2}^{(0)})^2 dt_2 dW_{t_1}\right)\frac{H_1(x;\Sigma_T)}{\Sigma_T}. \end{aligned} \quad (16)$$

Conditional Expectations as a sum of Hermite polynomials(cont'd)

$$
\begin{aligned}
& \mathbf{E}[(A_{2T})^2 | A_{1T} = x] \\
= & \Big(16 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma_{t_1}^{(0)} \sigma_{t_1}^{(0)} \partial\sigma_{t_2}^{(0)} \sigma_{t_2}^{(0)} (\sigma_{t_3}^{(0)})^2 (\sigma_{t_4}^{(0)})^2 dt_4 dt_3 dt_2 dt_1 \\
+ & \; 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \partial\sigma_{t_1}^{(0)} \sigma_{t_1}^{(0)} (\sigma_{t_2}^{(0)})^2 \partial\sigma_{t_3}^{(0)} \sigma_{t_3}^{(0)} (\sigma_{t_4}^{(0)})^2 dt_4 dt_3 dt_2 dt_1 \Big) \\
& \times \frac{H_4(x; \Sigma_T)}{\Sigma_T^4} \\
+ & \Big(16 \int_0^T \int_0^{t_1} \int_0^{t_2} \partial\sigma_{t_1}^{(0)} \sigma_{t_1}^{(0)} \partial\sigma_{t_2}^{(0)} \sigma_{t_2}^{(0)} (\sigma_{t_3}^{(0)})^2 dt_3 dt_2 dt_1 \\
+ & \; 8 \int_0^T \int_0^{t_1} \int_0^{t_2} \left(\partial\sigma_{t_1}^{(0)} \right)^2 (\sigma_{t_2}^{(0)})^2 (\sigma_{t_3}^{(0)})^2 dt_3 dt_2 dt_1 \Big) \frac{H_2(x; \Sigma_T)}{\Sigma_T^2} \\
+ & \; 4 \int_0^T \int_0^{t_1} \left(\partial\sigma_{t_1}^{(0)} \right)^2 (\sigma_{t_2}^{(0)})^2 dt_2 dt_1 . \qquad (17)
\end{aligned}
$$

Conditional Expectations as a sum of Hermite polynomials(cont'd)

- **Substituting these results into**

$$\begin{aligned} \mathbf{E}\left[\Phi(G^{(\epsilon)})\right] &= \int_{\mathbf{R}} \Phi(x) f(x) dx \\ &+ \epsilon \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{2T}|A_{1T}=x]f(x)\}dx \\ &+ \epsilon^2 \left\{ \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{3T}|A_{1T}=x]f(x)\}dx \right. \\ &+ \left. \frac{1}{2}\int_{\mathbf{R}} \Phi(x)(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}[(A_{2T})^2|A_{1T}=x]f(x)\}dx \right\} + o(\epsilon^2), \end{aligned} \tag{18}$$

the analytically tractable asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\epsilon)})\right]$ **can be obtained.**

  - **since these conditional expectations are sums of Hermite polynomials, the computation can be easily implemented.**

An Alternative Approach

- In this section we introduce an alternative but equivalent approach to the method explained so far.
  - transforming computation of conditional expectations to solving some system of O.D.E.s
  - linking the conditional expectations to unconditional ones.

Conditional and Unconditional Expectations

Lemma 2: the Expansion of Conditional Expectations

Let $X \in L^2(\Omega, P)$ and Z be a Gaussian r.v. with mean 0 and variance Σ. Then,

$$\mathbf{E}\left[X|Z\right] = \sum_{n=0}^{\infty} a_n H_n(Z;\Sigma) \tag{19}$$

where a_n is given by

$$a_n = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0}\left\{e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}\left[e^{i\xi Z}X\right]\right\}. \tag{20}$$

- **This lemma shows the equivalence between computation of conditional expectations and of unconditional ones.**

Conditional and Unconditional Expectations(cont'd)

- Thanks to Lemma 2, we can compute the unconditional expectations instead of the conditional ones;

$$
\begin{aligned}
\mathrm{E}[A_{2T}|A_{1T}=x] &= \sum_{n=0}^{2} a_n^{2,1} H_n(x;\Sigma_T), \\
\mathrm{E}[A_{3T}|A_{1T}=x] &= \sum_{n=0}^{3} a_n^{3,1} H_n(x;\Sigma_T), \\
\mathrm{E}[(A_{2T})^2|A_{1T}=x] &= \sum_{n=0}^{4} a_n^{2,2} H_n(x;\Sigma_T).
\end{aligned}
$$

Conditional and Unconditional Expectations(cont'd)

- Coefficients in these expressions are given by

$$a_n^{2,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\Big|_{\xi=0}\left\{\mathbf{E}\left[Z_T^{\langle\xi\rangle}A_{2T}\right]\right\}$$
$$a_n^{3,1} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\Big|_{\xi=0}\left\{\mathbf{E}\left[Z_T^{\langle\xi\rangle}A_{3T}\right]\right\}$$
$$a_n^{2,2} = \frac{1}{n!}\frac{1}{(i\Sigma)^n}\frac{\partial^n}{\partial\xi^n}\Big|_{\xi=0}\left\{\mathbf{E}\left[Z_T^{\langle\xi\rangle}(A_{2T})^2\right]\right\}$$

where $Z_t^{\langle\xi\rangle} = \exp\left(i\xi A_{1t} + \frac{\xi^2}{2}\Sigma_t\right)$.

- Note that $Z^{\langle\xi\rangle}$ is a martingale with $Z_0^{\langle\xi\rangle} = 1$.

A system O.D.E.s

- First, applying Itô's formula to $\left(Z_t^{\langle\xi\rangle} A_{2t}\right)$ we have

$$
\begin{aligned}
\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{2t}\right] &= \mathbf{E}\Bigg[\underbrace{\int_0^t Z_s^{\langle\xi\rangle} dA_{2s}}_{=0} + \underbrace{\int_0^t A_{2s} dZ_s^{\langle\xi\rangle}}_{=0} \\
&\qquad + \left\langle A_2, Z^{\langle\xi\rangle} \right\rangle_t \Bigg] \\
&= 2(i\xi) \int_0^t \partial\sigma_s^{(0)} \sigma_s^{(0)} \mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds
\end{aligned}
\tag{21}
$$

A system O.D.E.s(cont'd)

- Then, applying Itô's formula to $\left(Z_t^{\langle\xi\rangle} A_{1t}\right)$ again, we also have

$$
\begin{aligned}
\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{1t}\right] &= \mathbf{E}\left[\underbrace{\int_0^t Z_s^{\langle\xi\rangle} dA_{1s}}_{=0} + \underbrace{\int_0^t A_{1s} dZ_s^{\langle\xi\rangle}}_{=0} \right. \\
&\qquad \left. + \left\langle A_1, Z^{\langle\xi\rangle}\right\rangle_t\right] \\
&= (i\xi)\int_0^t (\sigma_s^{(0)})^2 \mathbf{E}\left[Z_s^{\langle\xi\rangle}\right] ds \\
&= (i\xi)\int_0^t (\sigma_s^{(0)})^2 ds \qquad (22)
\end{aligned}
$$

since $\mathbf{E}\left[Z_t^{\langle\xi\rangle}\right] = 1$ for all t.

A System O.D.E.s(cont'd)

- Similarly, we obtain the following system of ordinary differential equations(or integral equations);

$$
\begin{aligned}
\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{1t}\right] &= (i\xi)\int_0^t (\sigma_s^{(0)})^2 ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{2t}\right] &= 2(i\xi)\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle} (A_{1t})^2\right] &= \int_0^t (\sigma_s^{(0)})^2 ds \\
&\quad +2(i\xi)\int_0^t (\sigma_s^{(0)})^2\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{3t}\right] &= 3(i\xi)\left(\int_0^t \partial^2\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} (A_{1s})^2\right] ds \right. \\
&\quad \left. +\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{2s}\right] ds\right)
\end{aligned}
$$

A System O.D.E.s(cont'd)

- $$\begin{aligned} \mathrm{E}\left[Z_t^{\langle\xi\rangle} A_{2t} A_{1t}\right] &= 2\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathrm{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds \\ &\quad +(i\xi)\int_0^t (\sigma_s^{(0)})^2\mathrm{E}\left[Z_s^{\langle\xi\rangle} A_{2s}\right] ds \\ &\quad +2(i\xi)\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathrm{E}\left[Z_s^{\langle\xi\rangle} (A_{1s})^2\right] ds \\ \mathrm{E}\left[Z_t^{\langle\xi\rangle} (A_{2t})^2\right] &= 4\int_0^t (\partial\sigma_s^{(0)})^2\mathrm{E}\left[Z_t^{\langle\xi\rangle} (A_{1s})^2\right] ds \\ &\quad +4(i\xi)\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathrm{E}\left[Z_t^{\langle\xi\rangle} A_{2s}A_{1s}\right] ds. \end{aligned}$$

- Note that this system has a grading structure: high-order equations depend only lower ones.
- Their solutions can be derived analytically by substituting lower-order solutions into higher-order equations recursively.

A System O.D.E.s(cont'd)

- Actually, these equations are solved by recursive substitution of lower-order solutions.

$$
\begin{aligned}
\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{1t}\right] &= (i\xi)\int_0^t(\sigma_s^{(0)})^2ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{2t}\right] &= 2(i\xi)\int_0^t\partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{1s}\right]ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle}(A_{1t})^2\right] &= \int_0^t(\sigma_s^{(0)})^2ds \\
&\quad +2(i\xi)\int_0^t(\sigma_s^{(0)})^2\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{1s}\right]ds \\
\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{3t}\right] &= 3(i\xi)\left(\int_0^t\partial^2\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}(A_{1s})^2\right]ds\right. \\
&\quad \left.+\int_0^t\partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{2s}\right]ds\right)
\end{aligned}
$$

A System O.D.E.s(cont'd)

- **Actually, these equations are solved by recursive substitution of lower-order solutions.**

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{1t}\right] = (i\xi)\int_0^t(\sigma_s^{(0)})^2ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{2t}\right] = 2(i\xi)\int_0^t\partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{1s}\right]ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle}(A_{1t})^2\right] = \int_0^t(\sigma_s^{(0)})^2ds + 2(i\xi)\int_0^t(\sigma_s^{(0)})^2\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{1s}\right]ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle}A_{3t}\right] = 3(i\xi)\left(\int_0^t\partial^2\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}(A_{1s})^2\right]ds + \int_0^t\partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle}A_{2s}\right]ds\right)$$

A System O.D.E.s(cont'd)

- **Actually, these equations are solved by recursive substitution of lower-order solutions.**

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{1t}\right] = (i\xi)\int_0^t (\sigma_s^{(0)})^2 ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{2t}\right] = 2(i\xi)\int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle} (A_{1t})^2\right] = \int_0^t (\sigma_s^{(0)})^2 ds + 2(i\xi)\int_0^t (\sigma_s^{(0)})^2\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{1s}\right] ds$$

$$\mathbf{E}\left[Z_t^{\langle\xi\rangle} A_{3t}\right] = 3(i\xi)\left(\int_0^t \partial^2\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} (A_{1s})^2\right] ds + \int_0^t \partial\sigma_s^{(0)}\sigma_s^{(0)}\mathbf{E}\left[Z_s^{\langle\xi\rangle} A_{2s}\right] ds\right)$$

The Expansion of the Functional on $S^{(\epsilon)}$ Revisited

- You can compute the conditional expectations explicitly by
  - expanding them into iterated Itô integrals

or

  - converting them to the solution of the graded system of O.D.E.s
- Then you can also derive the explicit expression of

$$
\begin{aligned}
\mathbf{E}\left[\Phi(G^{(\epsilon)})\right] &= \int_{\mathbf{R}} \Phi(x) f(x) dx \qquad (23)\\
&+ \epsilon \int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{2T}|A_{1T}=x]f(x)\}dx\\
&+ \epsilon^2 \left\{\int_{\mathbf{R}} \Phi(x)(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{3T}|A_{1T}=x]f(x)\}dx\right.\\
&+ \left.\frac{1}{2}\int_{\mathbf{R}} \Phi(x)(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}[(A_{2T})^2|A_{1T}=x]f(x)\}dx\right\} + o(\epsilon^2).
\end{aligned}
$$

- It cannot be decided which is more useful than the other for practical computation in general, especially for computation with high-order expansions.

Numerical Examples(1)

- Here some practically important examples of application of our method are shown.
  - λ-SABR model
  - Cross-currency Libor market model with a stochastic volatility of the spot exchange rate

Numerical Examples(1)(cont'd)

- First, for λ-SABR model the following processes are considered;

$$\begin{aligned} dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta}dW_t^1, \\ d\sigma^{(\epsilon)}(t) &= \lambda(\theta-\sigma^{(\epsilon)}(t))dt+\epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1+\epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2 \end{aligned}$$

  where $\nu_1=\rho\nu$, $\nu_2=(\sqrt{1-\rho^2})\nu$. (The correlation between S and σ is $\rho\in[-1,1]$.)

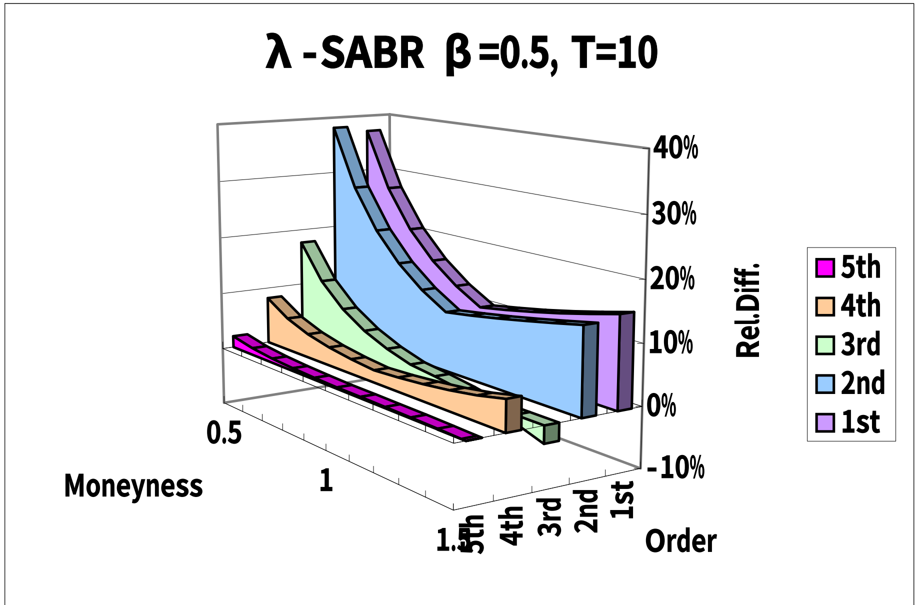
- When $\beta=1$, we assume the log-transformed process instead of $S^{(\epsilon)}$ and apply the asymptotic expansion to $X_T^{(\epsilon)}$;

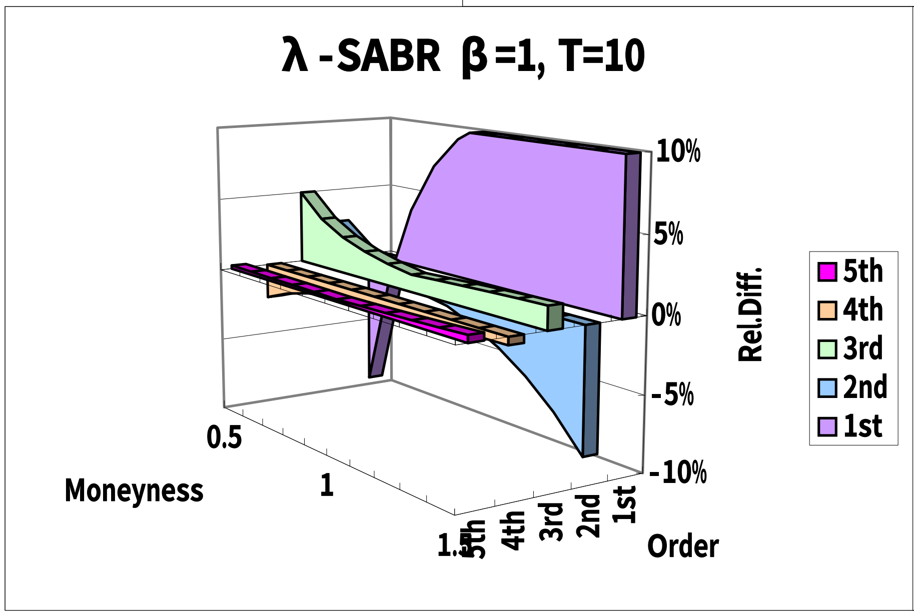
$$dX^{(\epsilon)}(t) = -\frac{1}{2}(\sigma^{(\epsilon)}(t))^2dt+\sigma^{(\epsilon)}(t)dW_t^1.$$

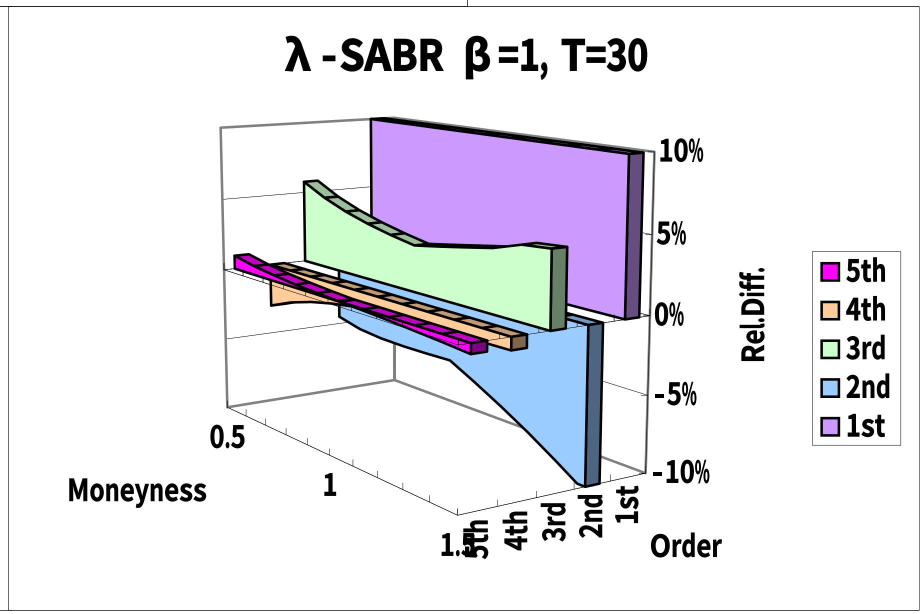
Numerical Examples(1)(cont'd)

- **For parameters, the followings are assumed;**

Parameter	$S(0)$	λ	$\sigma(0)$	β	ρ	θ	ν	T
i	100	0.1	3.0	0.5	-0.7	3.0	0.3	10
ii	100	0.1	0.3	1.0	-0.7	0.3	0.3	10
iii	100	0.1	0.3	1.0	-0.7	0.3	0.3	30


λ -SABR β =0.5, T=10
40%
30%
20%
10%
0%
-10%
Rel.Diff.
5th
4th
3rd
2nd
1st
0.5
1
1.5
Moneyness
5th
4th
3rd
2nd
1st
Order


λ -SABR β =1, T=10
10%
5%
0%
-5%
-10%
Rel.Diff.
5th
4th
3rd
2nd
1st
0.5
1
1.5
Moneyness
5th
4th
3rd
2nd
1st
Order


λ -SABR β =1, T=30
10%
5%
0%
-5%
-10%
Rel.Diff.
5th
4th
3rd
2nd
1st
0.5
1
1.5
Moneyness
5th
4th
3rd
2nd
1st
Order

Numerical Examples(2)

- Next, a cross-currency Libor market model is considered.
- In particular evaluate

$$\begin{aligned} V^C(0;T,K) &= P_d(0,T) \times \mathrm{E}^P\left[(S(T)-K)^+\right] \\ &= P_d(0,T) \times \mathrm{E}^P\left[(F_T(T)-K)^+\right] \end{aligned}$$

for call options and

$$\begin{aligned} V^P(0;T,K) &= P_d(0,T) \times \mathrm{E}^P\left[(K-S(T))^+\right] \\ &= P_d(0,T) \times \mathrm{E}^P\left[(K-F_T(T))^+\right]. \end{aligned}$$

for put options.

- $S(T)$: the spot exchange rate
- $F_T(t)$: the forex forward rate with maturity T.

Numerical Examples(2)(cont'd)

- Then, in a standard cross-currency Libor market model, the underlying variables are specified as
  - for domestic forward Libor rates

$$f_{dj}^{(\epsilon)}(t) = f_{dj}(0) + \epsilon^2 \sum_{i=j+1}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{dj}(u) f_{dj}^{(\epsilon)}(u) du + \epsilon \int_0^t f_{dj}^{(\epsilon)}(u) \gamma_{dj}'(u) dW_u,$$

for $j = n(t) - 1, n(t), n(t) + 1, \cdots, N$, where

$$g_{dj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{dj}^{(\epsilon)}(t)}{1 + \tau_j f_{dj}^{(\epsilon)}(t)} \gamma_{dj}(t)$$

and W is a r-dimensional standard Wiener process.

Numerical Examples(2)(cont'd)

- **Then, in a standard cross-currency Libor market model, the underlying variables are specified as**
  - **for foreign forward Libor rates**

$$\begin{aligned} f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) - \epsilon^2 \sum_{i=0}^{j} \int_0^t g_{fi}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du \\ &+ \epsilon^2 \sum_{i=0}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du \\ &- \epsilon^2 \int_0^t \sigma^{(\epsilon)}(u) \bar{\sigma}' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du \\ &+ \epsilon \int_0^t f_{fj}^{(\epsilon)}(u) \gamma_{fj}'(u) dW_u, \end{aligned}$$

for $j = n(t)-1, n(t), n(t)+1, \cdots, N$**, where**

$$g_{fj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{fj}^{(\epsilon)}(t)}{1 + \tau_j f_{fj}^{(\epsilon)}(t)} \gamma_{fj}(t).$$

Numerical Examples(2)(cont'd)

- Then, in a standard cross-currency Libor market model, the underlying variables are specified as
  - for the volatility of the spot exchange rate

$$\begin{aligned}\sigma(t) &= \sigma^{(\epsilon)}(0) + \int_0^t \mu(u, \sigma^{(\epsilon)}(u))du \\ &+ \epsilon^2 \sum_{j=1}^{N} \int_0^t g_{dj}^{0,(\epsilon)}(u)' \omega(u, \sigma^{(\epsilon)}(u))du + \epsilon \int_0^t \omega'(u, \sigma^{(\epsilon)}(u))dW_u\end{aligned}$$

  - for the forex forward

$$F_{N+1}^{(\epsilon)}(t) = F_{N+1}(0) + \epsilon \int_0^t \sigma_F^{(\epsilon)}(u)' F_{N+1}^{(\epsilon)}(u)dW_u$$

where

$$\sigma_F^{(\epsilon)}(t) := \sum_{j=0}^{N} \left(g_{fj}^{0,(\epsilon)}(t) - g_{dj}^{0,(\epsilon)}(t) \right) + \sigma^{(\epsilon)}(t)\bar{\sigma}.$$

Numerical Examples(2)(cont'd)

- In particular, we specify parameters as
  - Dimension of W: $r = 4$
  - Maturity: $T = 10$
  - Intervals between Libor's resetting: $\tau_j \equiv 0.5$
  - Flat structures of interest rates
    - Initial term structure for domestic Libors: $f_{dj}(0) = \bar{f}_d \ \forall j$
    - Vol. term structure for domestic Libors: $\gamma_{dj}(t) = \bar{\gamma}_d \ \forall j, t \leq T_j$
    - Initial term structure for foreign Libors: $f_{fj}(0) = \bar{f}_f \ \forall j$
    - Vol. term structure for foreign Libors: $\gamma_{fj}(t) = \bar{\gamma}_f \ \forall j, t \leq T_j$
    - $\bar{f}_d = 2\%$, $\bar{\gamma}_d = 30\%$, $\bar{f}_f = 5\%$, $\bar{\gamma}_f = 12\%$

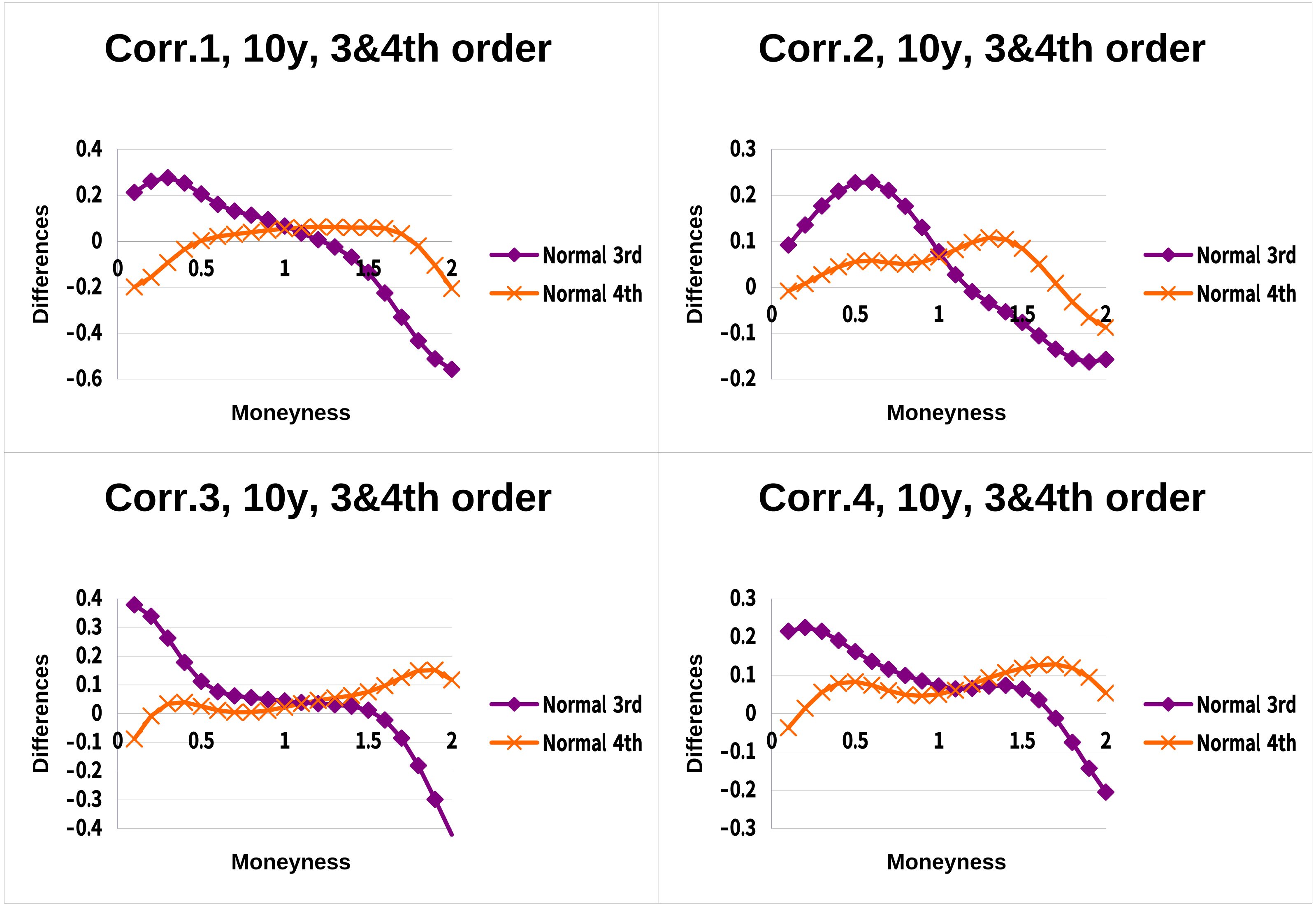
Numerical Examples(2)(cont'd)

- a stochastic volatility process

$$\sigma^{(\epsilon)} = \sigma(0) + \kappa \int_0^t (\theta - \sigma^{(\epsilon)}(s))ds + \epsilon\omega \int_0^t \sigma^{(\epsilon)}(s)dW_s$$

  - $\sigma(0) = \theta = 0.1,\ \kappa = 0.1,\ \omega = 0.3$

- Instantaneous correlation structures: any specification is available
  - Corr.1: all factors are independent
  - Corr.2: the correlation between $S(t)$ and $\sigma(t)$ is -0.5
  - Corr.3: correlations between interest rates and the spot forex
    - $f_{dj}(t)$ and $S(t)$: 0.5
    - $f_{fj}(t)$ and $S(t)$: -0.5
  - Corr.4: many correlated relations
    - $S(t)$ and $\sigma(t)$: -0.5
    - $f_{dj}(t)$ and $S(t)$: 0.5
    - $f_{fj}(t)$ and $S(t)$: -0.5



Corr.1, 10y, 3&4th order
0.4
0.2
0
-0.2
-0.4
-0.6
Differences
0
0.5
1
1.5
2
Moneyness
Normal 3rd
Normal 4th
Corr.2, 10y, 3&4th order
0.3
0.2
0.1
0
-0.1
-0.2
Differences
0
0.5
1
1.5
2
Moneyness
Normal 3rd
Normal 4th
Corr.3, 10y, 3&4th order
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
Differences
0
0.5
1
1.5
2
Moneyness
Normal 3rd
Normal 4th
Corr.4, 10y, 3&4th order
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
Differences
0
0.5
1
1.5
2
Moneyness
Normal 3rd
Normal 4th

Conclusions

- In this talk the followings were presented;
  - basics of an asymptotic expansion approach
  - two alternative but equivalent schemes for the key computation
    - conditional expectations' expansions into iterated Itô integrals
    - unconditional expectations as solutions of O.D.E.s
- Moreover, the accuracy and importance of our method was confirmed through
  - λ-SABR model
  - a general cross-currency Libor market model with a stochastic volatility of the spot forex

  even with maturities longer than ten years.

Conclusions

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  even with maturities longer than ten years.

Thank you for your kind attention!!

An Asymptotic Expansion in a B-S Economy(cont'd)

- In particular, letting Φ be δ_x, the delta function with a mass at x, we have the expansion of the density function of $G^{(\epsilon)}$;

$$
\begin{aligned}
& f_{G^{(\epsilon)}}(x) = \mathbf{E}\left[\delta_x(G^{(\epsilon)})\right] \\
= \quad & f(x) + \epsilon(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{2T}|A_{1T}=x]f(x)\} \\
+ \quad & \epsilon^2\left\{(-1)\frac{\partial}{\partial x}\{\mathbf{E}[A_{3T}|A_{1T}=x]f(x)\}\right. \\
+ \quad & \left.\frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\mathbf{E}[(A_{2T})^2|A_{1T}=x]f(x)\}\right\} + o(\epsilon^2).
\end{aligned} \tag{24}
$$

An Asymptotic Expansion in a B-S Economy(cont'd)

- Alternatively, Takahashi[1999] proposed heuristic but intuitive formal expansion of the characteristic function(ch.f.) of $G^{(\epsilon)}$ as

$$\Psi_{G^{(\epsilon)}}(\xi) = \mathbf{E}\left[e^{i\xi G^{(\epsilon)}}\right]$$
$$= \Psi_{0,\Sigma_T}(\xi)\mathbf{E}\left[Z_T^{\langle\xi\rangle}\left\{1+\epsilon(i\xi)\mathbf{E}\left[A_{2T}|_{A_{1T}}\right]\right.\right. \qquad (25)$$
$$+ \left.\left.\epsilon^2\left((i\xi)\mathbf{E}\left[A_{3T}|_{A_{1T}}\right]+\frac{1}{2}(i\xi)^2\mathbf{E}\left[(A_{2T})^2|_{A_{1T}}\right]\right)\right\}\right]+o(\epsilon^2)$$

where

$$Z_T^{\langle\xi\rangle} = e^{i\xi A_{1T}+\frac{\xi^2}{2}\Sigma_T}, \quad \Psi_{0,\Sigma}(\xi) = e^{-\frac{\xi^2}{2}\Sigma}.$$

- Then, he derived its (approximate) density function by Fourier inversion;

$$f_{G^{(\epsilon)}}(x) = \mathcal{F}^{-1}(\Psi_{G^{(\epsilon)}}) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ix\xi}\Psi_{G^{(\epsilon)}}(\xi)d\xi. \qquad (26)$$

- As seen later, this approach is equivalent to the previous one (??).

Expansions into iterated Itô integrals(cont'd)

- Or equivalently, you can use this relationship on a Wiener-Chaos expansion;

Lemma 1: a Wiener-Chaos Expansion and Malliavin Derivatives

Let $\mathcal{F}_t^W = \sigma\{W_s; s \leq t\}$ and X in $L^2(\Omega, P)$ be $\mathcal{F}_T^W$-measurable. Then,

$$X = \sum_{n=0}^{\infty} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) dW_{t_n} \cdots dW_{t_1}. \quad (27)$$

where f_n is symmetric and in $L^2(0,T]^n$. Moreover, each integrand can be obtained via the Malliavin derivative of X;

$$f_n(t_1, \cdots, t_n) = \mathrm{E}\left[D^n_{t_1,\cdots,t_n} X\right]. \quad (28)$$

- As we have seen, the Wiener-Chaos expansions of A_{2T}, A_{3T} and $(A_{2T})^2$ consist of a finite number of terms.

The Expansion of the Functional on $S^{(\epsilon)}$ Revisited(cont'd)

- As mentioned earlier, this stream of computation is completely equivalent to the ch.f.-expansion approach in Takahashi[1999].
- In fact, the density function $f_{G^{(\epsilon)}} = \mathrm{E}[\delta_x(G^{(\epsilon)})]$ is

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= f(x) + \epsilon(-1)\frac{\partial}{\partial x}\{\mathrm{E}[A_{2T}|A_{1T}=x]f(x)\} \\ &+\epsilon^2\left\{(-1)\frac{\partial}{\partial x}\{\mathrm{E}[A_{3T}|A_{1T}=x]f(x)\}\right. \\ &\left.+(-1)^2\frac{\partial^2}{\partial x^2}\{\mathrm{E}[(A_{2T})^2|A_{1T}=x]f(x)\}\right\} + o(\epsilon^2) \\ &= f(x) + \epsilon(-1)\frac{\partial}{\partial x}\{\sum_{n=0}^{2} a_n^{2,1} H_n(x;\Sigma_T) f(x)\} \\ &+\epsilon^2\left\{(-1)\frac{\partial}{\partial x}\{\sum_{n=0}^{3} a_n^{3,1} H_n(x;\Sigma_T) f(x)\}\right. \\ &\left.+\frac{1}{2}(-1)^2\frac{\partial^2}{\partial x^2}\{\sum_{n=0}^{4} a_n^{2,2} H_n(x;\Sigma_T) f(x)\}\right\} + o(\epsilon^2) \end{aligned}$$

The Expansion of the Functional on $S^{(\epsilon)}$ Revisited(cont'd)

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \mathcal{F}^{-1}(\Psi_{0,\Sigma_T}(\xi)) + \epsilon \sum_{n=0}^{2} a_n^{2,1} \mathcal{F}^{-1}\left((i\xi)(i\xi\Sigma_T)^n \Psi_{0,\Sigma_T}(\xi)\right) \\
&+\epsilon^2 \left\{ \sum_{n=0}^{3} a_n^{3,1} \mathcal{F}^{-1}\left((i\xi)(i\xi\Sigma_T)^n \Psi_{0,\Sigma_T}(\xi)\right) \right. \\
&\left. + \frac{1}{2} \sum_{n=0}^{4} a_n^{2,2} \mathcal{F}^{-1}\left((i\xi)^2(i\xi\Sigma_T)^n \Psi_{0,\Sigma_T}(\xi)\right) \right\} + o(\epsilon^2) \\
&= \mathcal{F}^{-1}\left((\Psi_{0,\Sigma_T}(\xi)) \times \left\{ 1 + \epsilon(i\xi) \sum_{n=0}^{2} a_n^{2,1}(i\Sigma_T)^n \xi^n \right. \right. \\
&+ \left. \left. \epsilon^2 \left((i\xi) \sum_{n=0}^{3} a_n^{3,1}(i\Sigma_T)^n \xi^n + \frac{1}{2}(i\xi)^2 \sum_{n=0}^{4} a_n^{2,2}(i\Sigma_T)^n \xi^n \right) \right\} \right) \\
&+o(\epsilon^2)
\end{aligned}
$$

The Expansion of the Functional on $S^{(\epsilon)}$ Revisited(cont'd)

- By definition of $a_n^{2,1}$, $a_n^{3,1}$ and $a_n^{2,2}$, this can be written as

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \mathcal{F}^{-1}\Bigg((\Psi_{0,\Sigma_T}(\xi)) \times \Bigg\{ 1 + \epsilon(i\xi) \sum_{n=0}^{2} a_n^{2,1}(i\Sigma_T)^n \xi^n \\
&+ \epsilon^2 \Bigg((i\xi) \sum_{n=0}^{3} a_n^{3,1}(i\Sigma_T)^n \xi^n + \frac{1}{2}(i\xi)^2 \sum_{n=0}^{4} a_n^{2,2}(i\Sigma_T)^n \xi^n \Bigg) \Bigg\} \\
&+ o(\epsilon^2) \\
&= \mathcal{F}^{-1}\Bigg((\Psi_{0,\Sigma_T}(\xi)) \times \Bigg\{ 1 + \epsilon(i\xi)\mathbf{E}\left[Z_T^{\langle\xi\rangle} A_{2T} \right] \\
&+ \epsilon^2 \Bigg((i\xi)\mathbf{E}\left[Z_T^{\langle\xi\rangle} A_{3T} \right] + \frac{1}{2}(i\xi)^2 \mathbf{E}\left[Z_T^{\langle\xi\rangle} (A_{2T})^2 \right] \Bigg) \Bigg\} \Bigg) \\
&+ o(\epsilon^2).
\end{aligned}
$$

- Thus, the approximate density function obtained by our method coincides with the inversion of formally expanded ch.f. of $G^{(\epsilon)}$, $\Psi_{G^{(\epsilon)}}(\xi) = \mathbf{E}\left[e^{i\xi G^{(\epsilon)}} \right]$.