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Philosophy and Foundations of Mathematics: Epistemological and Ontological Aspects

A conference dedicated to Per Martin-Löf on the occasion of his retirement

May 5 – May 8

Swedish Collegium for Advanced Study
Linneanum, Thunbergsvägen 2, Uppsala
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Philosophy and Foundations of Mathematics: Epistemological and Ontological Aspects -

A Conference Dedicated to Per Martin-Löf on the Occasion of his Retirement,

May 5-8, 2009

Swedish Collegium for Advanced Study, Uppsala, Sweden

PROGRAMME

Venue: Thunbergssalen, SCAS, Linneanum,

Botaniska trädgården, Thunbergsvägen 2, 752 38 Uppsala.

Tuesday, May 5

9.00 Registration (Linnésalen)

Coffee/tea

10.00 Opening by Björn Wittrock, Principal of SCAS

10.20-11.10 William Tait: The myth of intuition
11.20-12.10 Sören Stenlund: On the notion of finite numbers

Lunch break

14.00-14.50 Michael Rathjen: Non-definite existence axioms and open problems
15.00-15.50 Peter Dybjer: Program testing and constructive validity

Coffee/tea

16.20-17.10 Peter Aczel: Predicate logic over a type setup
17.20-18.10 Giovanni Sambin: A minimalist foundation for mathematics

18.15- Reception (Linnésalen)

Wednesday, May 6

9.00-9.50 Jean-Yves Girard: Towards non-commutative foundations

Coffee/tea

10.20-11.10 Steve Awodey: Type theory and homotopy theory
11.20-12.10 Thierry Coquand: Forcing and type theory

Lunch break

14.00-14.50 Christine Paulin-Mohring: Reasoning on randomized programs in Coq
15.00-15.50 Anton Setzer: Coalgebras as types determined by their elimination rules

Coffee/tea

16.20-17.10 Peter Pagin: Assertion, truth and judgement
17.20-18.10 Sten Lindström: Church-Fitch's knowability paradox revisited

Thursday, May 7

9.00-9.50 Stewart Shapiro: Structures and logics: a case for relativism

Coffee/tea

10.20-11.10 Colin McLarty: What are the things of mathematics?
– Identity and existence in categorical foundations
11.20-12.10 Wilfried Sieg: Reductive structuralism

Lunch break

14.00-14.50 Jouko Väänänen: Second order logic, set theory and the foundations of mathematics
15.00-15.50 Mark van Atten: Different times: Kant and Brouwer on real numbers

Coffee/tea

16.20-17.10 Jan von Plato: Aristotle's deductive logic: a proof-theoretical study
17.20-18.10 Erik Palmgren: Formal topology and foundational problems

Friday, May 8

9.00-9.50	Per Martin-Löf: Logic: epistemological or ontological?

Coffee/tea

10.20-11.10	Juliet Floyd: Wittgenstein, Gödel and Turing
11.20-12.10	Jan Smith: Can Hume's analysis of causality tell us something about the rules of logic?

Lunch break

14.00-14.50	Aarne Ranta: Levels of abstraction in language and logic
15.00-15.50	Göran Sundholm: Three key-features of Martin-Löf's philosophy of logic

Coffee/tea

16.20-17.10	Dag Prawitz: Truths and proofs – some epistemic and ontological questions

19.00	Conference dinner in Orangeriet (separate registration required)

Abstracts

Peter Aczel: *Predicate logic over a type setup*

There are a variety of closely related notions aimed at capturing the abstract structure of type dependency in the syntax and semantics of dependent type theories. Examples of such notions are category with attributes, category with families, category with display maps, contextual category, comprehension category, and there are more.

The notion of a type setup is yet one more such notion, which differs from the others in taking a more syntactic approach in its explicit use of contexts, as finite lists of typed variable declarations. This makes it closer to the syntax of dependent type theories, while still abstracting away from the usual inductive structure of syntax and the corresponding recursive definition of substitution.

Predicate logic over a type setup is simply defined as a sorted predicate logic where the sorts are the types of the type setup and the sorted terms and substitution are also given by the type setup. Logic over a type setup generalises logic-enriched type theory which, in turn generalises dependently sorted logic, a dependent generalisation of many-sorted logic.

Many results of predicate logic generalise to predicate logic over a type setup. I will consider the disjunction and existence properties of intuitionistic predicate logic and end with a characterisation of the logic of the propositions-as-types interpretation of intuitionistic predicate logic.

Mark van Atten: *Different times: Kant and Brouwer on real numbers*

Kant held that under the concept of the square root of 2 falls only a geometrical magnitude, but not a number. In particular, he explicitly distinguished the square root of 2 from infinite converging sequences of rationals. Like Kant, Brouwer based his foundations of mathematics on the a priori intuition of time, and indeed he presented his position as fundamentally Kantian. Yet, unlike Kant, Brouwer did identify the square root of 2 with an infinite sequence. The question arises where this difference comes from. I will suggest that it has its origin in the difference in their views on the relation of time to intuition.

Steve Awodey: *Type theory and homotopy theory*

In recent research it has become clear that there are deep and fascinating connections between the intensional type theory of Per Martin-Löf and homotopy theory, via the modern approach to the latter in terms of Quillen model categories, as well as the theory of higher dimensional categories. This talk will survey some of these developments.

Thierry Coquand: *Forcing and type theory*

Forcing is an important tool in constructive mathematics since it is a general technique to give constructive meaning to some ideal elements. A typical example, that I will recall, is Joyal's constructive explanation of the algebraic closure of a field. (Even the construction of a splitting field of a polynomial requires this technique.) I will then explain how to adapt this technique to type theory, giving for instance a way to extend type theory with a decidable algebraic closure of a (decidable) field.

Peter Dybjer: *Program testing and constructive validity*

In this talk I will discuss the connection between program testing and Martin-Löf's meaning explanations for intuitionistic type theory. First I give a short overview of the historical development of the ideas behind the meaning explanations. Then I explain the connection with program testing. Finally, I will mention the possibility of pursuing the testing point of view for some other logical systems including impredicative ones.

Juliet Floyd: *Wittgenstein, Gödel and Turing*

In 1946, recalling his discussions with Turing in Cambridge before the war, Wittgenstein stressed that Turing's 'machines' are really "humans who calculate" (RPP I 1096). Was this intended to embrace or to reject Turing's model of human calculative activity? What form of anthropomorphism was (and is) at stake in regarding humans as machines, and in playing imitation games? The question becomes even more intriguing when we reflect that while Gödel held that it was Turing's "precise and unquestionably adequate" definition of the notion of a formal system that allowed his own incompleteness theorems to be proved rigorously for the first time, Gödel also held that Turing made a "philosophical error" in holding that human mental procedures cannot go beyond mechanical procedures. We shall contrast the viewpoints of Wittgenstein, Gödel and Turing, emphasizing the evolution of the logical systems of notation that each one of them provided, and discussing how each viewed the philosophical significance of logic and mathematics.

Jean-Yves Girard: *Towards non-commutative foundations*

Quantum physics, operator algebra and the non-commutative geometry of Connes deeply challenge old style foundations. Roughly speaking, the object is entangled, since non-commutative, whereas the subject appears as a commutative window, hence a set- theoretic reduction. Issues, partial results, working hypotheses, will be discussed in the talk.

Sten Lindström, *Church-Fitch's knowability paradox revisited*

According to a non-realist conception, the notion of truth is epistemically constrained: the anti-realist accepts one version or another of Dummett's *Knowability Principle*:

(K) *If a statement is true, then it must in principle be possible to know that it is true.*

There is, however, a well-known argument, due to Alonzo Church and Frederic Fitch, which seems to threaten the anti-realist position. Starting out from seemingly innocuous assumptions, Fitch (JSL 1963) claims to prove: *if there is some true proposition which nobody knows to be true, then there is a true proposition which nobody can know to be true.*

The Church-Fitch argument is simple. Suppose that q is a true proposition that is not known to be true. Consider then the proposition (p): q and it is not known that q. This is obviously a true proposition. And it cannot be known. For suppose that p were known to be true. Then the following proposition would be true: It is known that (q and it is not known that q). Since knowledge distributes over conjunction, it would then also be true that: it is known that q and it is known that it is not known that q. Since knowledge implies truth, it would then follow that it is known that q and it is not known that q. That is, the proposition (p) could not be known to be true.

Roughly speaking, we can envisage the following reactions to this argument:

- The argument is valid and constitutes a refutation of the anti-realist position.
- The argument is valid, but it does not constitute a threat to the anti-realist position.
- A detailed analysis of the argument shows it to be invalid.

In the talk I plan to discuss these three kinds of reactions to the Knowability argument of Church and Fitch.

Per Martin-Löf: *Logic: epistemological or ontological?*

What is logic? Is it the study of the process of inference or reasoning, called demonstration in mathematics, by means of which we justify our judgements? Or is it the study of the logical and set-theoretical concepts, like proposition, truth and consequence on the one hand, and set, element and function on the other, that make their appearance in the contents of our judgements? This is the fundamental question whether logic is in its essence epistemological or ontological. The answer is presumably that it is both, which is to say that, within logic, one can distinguish between two parts, or two layers, the one epistemological and the other ontological. But there remains the question of the order of priority between these two layers: Which comes first? Is epistemology prior to ontology, or is it the other way round? Bolzano, whose logic in four volumes, called Wissenschaftslehre, has the most clear architectonic structure of all logics that have so far been written, treated of the ontological notions of proposition, truth and logical consequence (Ableitbarkeit) in the first two volumes of his Wissenschaftslehre, relegating the epistemology to the third volume. Thus he let ontology take priority over epistemology. Although the line of demarcation between the two was drawn in exactly the right place by Bolzano, my own work on constructive type theory has forced me to the conclusion that the order of priority between ontology and epistemology is nevertheless the reverse of the order in which they are treated in the Wissenschaftslehre. The epistemological notions of judgement and inference have to be in place already when you begin to deal with propositions, truth and consequence, as well as with other purely ontological notions, like the set-theoretical ones.

Colin McLarty: *What are the things of mathematics?*
–Identity and existence in categorical foundations

Philosophical treatments of identity and existence in mathematics most often take the Zermelo-Frankel conception of extensionality as the norm for individuating objects, which a structuralist account must elude in some way. We will look at the issues with an axiomatic foundation in the category of categories where that kind of individuation is the exception from the start.

Peter Pagin: *Assertion, truth, and judgment*

There is an interesting connection between Martin-Löf's proposition/judgment distinction and a certain puzzle about speech acts. When a speaker asserts

(1) The moon reflects light from the sun

her assertion is in a sense *about* its own possible world, even though the *proposition* she asserts can be evaluated at many worlds. If we treat the world of utterance as an *index*, we get the content of

(2) In *w*, the moon reflects light from the sun

where *w* gets the world of utterance as value. As a result, the content is either the necessary proposition, if true, or the impossible proposition, if false. This reduces content to truth value.

An alternative is to separate the world parameter from the proposition asserted, so that the assertoric content is a distinct entity:

(3) *w*: The moon reflects light from the sun

This makes assertoric content look much like a *judgment* in Martin-Löf's sense, of the format

$$a : A$$

where *a* is a proof object, or truth-maker, and *A* is the proposition. Similarly, in (3), the world of utterance is also a truth-maker of the proposition.

How far does this analogy go?

Erik Palmgren: *Formal topology and foundational problems*

Per Martin-Löf together with Giovanni Sambin pioneered formal topology, the systematic development of point-free topology as based on inductive constructions. It was obtained as a generalization of Scott's domain theory using the method of inductively generating covers for the Cantor space presented in Martin-Löf's 1970 doctoral thesis. Formal topologies may be regarded as special cases of Grothendieck topologies, and thus representations of locales, but their importance derives from the fact that they makes it possible to deal with locale theory in a (generalized) predicative way. This makes them a prime candidate for the development of topology along constructive lines acceptable to the Bishop school of constructive mathematics.

A main difference from locale theory, as developed in an impredicative setting, is that the collection points of a given formal topology need not form a set. This is a difficulty. The same problem pertains to the collection of continuous maps between two formal topologies. There are however natural classes and categories of formal topologies where these collections are indeed sets.

Another problem is to relate formal topology to the established metric space topology as developed by e.g. Errett Bishop. Here the collections of points, compact subsets and continuous functions between spaces naturally form sets as the topologies are point-based.

We survey some recent results regarding these problems.

Christine Paulin-Mohring: *Reasoning on randomized programs in Coq*

We show how advanced features of type theory like type classes can be used for modeling distributions and reasoning on randomized programs with applications to the analysis of computational properties of cryptographic programs.

Jan von Plato: *Aristotle's deductive logic: a proof-theoretical study*

Aristotle's deductive logic, as presented in his book Prior Analytics, is a system of rules of proof. The structure of derivations by these rules is analyzed. It is shown that derivations can be so transformed that steps of indirect proof are applied at most once as a last rule, the only way in which Aristotle used indirect proof.

Dag Prawitz: *Truths and proofs – some epistemic and ontological questions*

Ontological and epistemological aspects of mathematics are in dispute already in connection with mathematical truth. Both a realist and a non-realist position need truth as an ontological concept, but they disagree as to how the concept is to be analysed. Realists hold that truth is conceptually independent of knowledge, while non-realists of a verificationist kind think that truth has to be analysed in terms of knowledge, more precisely, truths come out in their analysis as being in principle possible to prove or verify. This theme has played an important role in Per Martin-Löf's writings on the philosophy of mathematics. As his thinking has evolved he has taken quite different or opposite stands on questions raised in this connection. Confronting these positions with each other, I shall take up some issues concerning the nature of proofs.

Aarne Ranta: *Levels of abstraction in language and logic*

Both linguists and logicians make abstractions when studying language. Classical examples of linguistic abstractions are phonemes as opposed to concrete sounds, and lexemes as opposed to concrete words. Typical abstractions in logic, on the other hand, have to do with different notions of equality: logical equivalence, definitional equality, alpha convertibility. Constructive type theory is particularly rich in such distinctions of levels. The talk will take a type-theoretical perspective on some linguistic problems concerning equality and abstraction levels. This will shed light on questions such as what is literal quoting and whether it is possible to translate from one language to another without loss of meaning.

Michael Rathjen: *Non-definite existence axioms and open problems*

The most typical non-definite existence axiom is the axiom of choice which asserts the existence of a function which may not be definable. Unsurprisingly, the axiom of choice does not have a unambiguous status in constructive mathematics. While the full axiom of choice cannot be added to systems of extensional set theory without yielding constructively unacceptable cases, various restricted forms can be justified via the propositions-as-types interpretation in constructive type theory. There are other existence axioms of a non-definite nature which play an even more important role in constructive set theories. Pivotal examples of definite/non-definite couplings of axioms are replacement/collection and exponentiation/subset collection.

In the talk I intend to consider the role of non-definite axioms in the constructive context, to look at results about how they relate to their definite versions, and to list several open problems related to them.

Giovanni Sambin: *A minimalist foundation of mathematics*

The foundational system first introduced in Maietti-Sambin 2005 corresponds well to the general philosophical attitude which considers abstraction as a dynamic process (Sambin 2001). It is dynamic in that it does without any form of platonistic assumption: every notion (set, proposition, collection,...) is "open" and no universe of speech is necessary. It is minimalist since it is compatible with most other foundational proposals, basing on classical logic (axiomatic set theory, Feferman's explicit mathematics) or intuitionistic logic (Martin-Löf's constructive type theory, Lawvere-Tierney's topos theory, Myhill-Aczel's constructive set theory), as well as with systems for the formalization of mathematics on computers (such as Coq).

Technically, the other foundations are obtained by adding a combination of some basic principles (law of excluded middle, propositions-as-sets, axiom of choice, powerset axiom, extensional equality of functions, etc.). More positively, the minimalist foundation respects any kind of meaning attributed to mathematical abstractions (computational content, predicative definitions, deductions, geometrical intuition) without assuming it as the only ingredient.

The main conceptual novelty, which allows to assemble some principles which in a traditional system would be plainly contradictory, is the distinction of two levels of abstraction, an intensional one, which works as the underlying "machine", and an extensional one, in which one develops mathematics and which is obtained from the first by allowing extensional equality. The intensional level, called minimal type theory, is a type theory in which logic is not reduced to sets (via the propositions-as-sets interpretation of intuitionistic logic) and lacks any form of choice principles; even the so-called axiom of unique choice (common to other foundations) is not assumed. The extensional theory is then obtained simply by passing to quotients, that is, only the notion of equality is changed. As an expectation, this should be done in such a way that no computational content is lost; this important result has actually been proved recently (Maietti 2009). In practical terms, each piece of mathematics expressible at the extensional level is automatically formalizable at the intensional one, that is, in a "proof assistant". One could also start from this practical result and argue backwards: we claim that, assuming one does not ask mathematicians to change their nature, one should reach conclusions very similar to ours.

The minimalist foundation has already been tested in the production of mathematics; for example, the whole "Basic Picture" (Sambin 2010), a generalization and enrichment of constructive topology, relies on it. The resulting coexistence and distinction between real (effective) and ideal (infinitary) aspects sheds new light on some old specific issues, as Brouwer's notion of choice sequence and principle of continuity.

It hardly needs saying that work is in progress: adjustments and further applications are expected.

References:

M.E. Maietti, A minimalist two-level foundation for constructive mathematics, Ann. of Pure and Appl. Logic (2009), in press.

M.E. Maietti, G. Sambin, Toward a minimalist foundation for constructive mathematics. In: L. Crosilla and P. Schuster, eds., From Sets and Types to Topology and Analysis, Oxford Logic Guides 48, OUP (2005), 91--114.

G. Sambin, Steps towards a dynamic constructivism, in P. Gärdenfors et al. eds., Proceedings of the XI Int. Congress of Logic, Methodology and Philosophy of Science, Krakow, 1999, Kluwer (2001), 261 - 284.

G. Sambin, The Basic Picture: structures for constructive topology, Oxford U. P., to appear in 2010.

Anton Setzer: *Coalgebras as types determined by their elimination rules*

In this talk we will discuss the addition of coinductive data types or weakly final coalgebras to Martin-Löf Type Theory. It turns out that such types are determined by their elimination rules, and the introduction rules are derived from these, whereas inductive and inductive-recursive data types are given by their introduction rules, and the elimination rules are derived. This explains the seemingly impredicative nature of the introduction rules for coinductive types, which is similar to the impredicative nature of the elimination rules for inductive types.

We will introduce a PER model for this extension of type theory and will see how types introduced by their elimination rules are defined in such a model. We will discuss the implications for meaning explanations for a type theory with both inductive and coinductive types.

Stewart Shapiro: *Structures and logics: a case for relativism*

I begin with the structuralist-Hilbertian thesis that any coherent structure is a legitimate object of mathematical study. In most cases, consistency is a reasonable gloss on coherence. The next observation is the truism that consistency is a matter of logic. With weaker logics, more theories are consistent and thus coherent. There are a variety of structures that employ intuitionistic logic that become inconsistent if excluded middle is added. The philosophical literature on paradox has produced a number of other logics that, at least prima facie, seem to yield mathematically legitimate structures. The result, I suggest, is that logic is relative to structure. Ramifications of this concerning various slogans about logic, issues concerning the meaning of the logical terms, and general problems with relativism are broached along the way.

Wilfried Sieg: *Reductive structuralism*

The *Grundlagenstreit* of the 1920s between Brouwer and Hilbert is slowly fading into the background. To some extent it still colors our perspectives on the foundations of mathematics, but it is no longer viewed as pivotal. The roots of issues can be found in the radical transformation of mathematics during the second half of the 19th century, i.e., the development of abstract axiomatics, set theory and logic.

In the first part of my talk, I sketch the connections of Hilbert's early foundational work (on geometry and the arithmetic of real numbers) with Dedekind's. Then, in the second part, I describe the evolution towards Hilbert's finitist consistency program and the difficulties of articulating the epistemologically distinctive aspects of finitist mathematics. In the third part, an analysis of the informal ideas underlying this approach motivates the formulation of reductive structuralism.

My goal is to present an integrative outlook on philosophical work and to formulate focused problems for mathematical investigation.

Jan Smith: *Can Hume's analysis of causality tell us something about the rules of logic*?

Hume's crucial observation that there is no direct evidence for causality opens up for an evolutionary understanding of causality. I will also argue that in an evolutionary perspective, the rules of logic play a similar role as causality.

I will not rely on any result in biology, except that the gene-centered view of evolution has in general broadened the scope of evolutionary explanations. Rather, I will corroborate my claims by Kant's view on causality and Aristotle's on the rules of logic.

Sören Stenlund: *On the notion of finite numbers*

Two different senses of the word 'number' will be distinguished. On the one hand the usual notion of pure mathematics, i.e. the sense of number which is implicit in the notion of an arbitrary finite sequence and iteration. On the other hand, there is a sense of 'number' and 'multitude' that is connected with practices of counting concrete things and where the counting is a temporal procedure. It will be shown that the tendency to conflate these senses of number has been a source of problems in the discussion of the foundations of mathematics.

Göran Sundholm: *Three key-features of Martin-Löf's philosophy of logic*

(1) The Type Theory is not a metamathematical language but has content;

(2) The notion of judgement is restored to logical pride of place;

(3) The epistemological perspective is first, rather than the customary third, person.

William Tait: *The myth of intuition*

In the first part of the talk, I will argue that intuition for Kant was simply the locus of mathematical construction and computation. In the second part I will call into question the concepts of 'intuition of' and 'intuition that' in the foundations of mathematics. I will discuss the latter in particular in connection with the finitism of Hilbert and Bernays, concluding that the concept of intuition in their writings provides no basis at all for what should count as finitist definition or finitist proof.

Jouko Väänänen: *Second order logic, set theory, and foundations of mathematics*

I will compare second order logic and set theory as foundations of mathematics and argue that there is very little difference between the two.

Organization and Programme Committee:
Peter Dybjer, Sten Lindström, Erik Palmgren (Chair),
Dag Prawitz, Sören Stenlund, Viggo Stoltenberg-Hansen

Local Organization:
Anna Eriksson-Trenter, Anton Hedin, Tore Hållander,
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