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AN ASYMPTOTIC PROPERTY OF EQUILIBRIUM ON FUTURES MARKETS ARISING FROM SPECULATION *

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This paper presents an asymptotic property of a joint spot-futures market equilibrium established in Cheng–Magill (1982). As speculators diversify over a large number of markets, the equilibrium risk premium converges to an asymptotic premium, the behaviour of which depends solely on the stochastic dependence between the spot price and an index of average returns on other markets. Risk arising from the variability of the spot price itself is diversified away. The results are related to the arbitrage pricing theory of Ross (1976).

1. Partial equilibrium with spot and futures market

Consider a simple partial equilibrium model of the production and consumption activity for a single perishable commodity. The industry producing the commodity consists of $m > 1$ identical firms each of which produces y units of output by incurring a known cost $c(y)$. We say that the cost function c: $R^+ \to R^+$ satisfies a *monotonicity* condition if $c \in \mathcal{C}^1(R^+)$, $c(0) = 0$, $0 \leqq c'(0) \leqq \lambda < \infty$, $c'(\cdot)$ is strictly increasing on R^+ and $c'(y) \to \infty$ as $y \to \infty$.

Each firm must make its production decision in the spring – output appears in the fall. However in the spring firms are uncertain what consumer demand will be in the fall. To introduce such uncertainty, let $(\Omega, \mathcal{G}, \mathcal{P})$ denote a probability space and let $\phi(x, \omega)$ denote the demand price that consumers will pay for the total amount x when the state of nature is $\omega \in \Omega$. We say that the demand function ϕ: $R^+ \times \Omega \to R^+$

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satisfies a *monotonicity* condition if $\phi(\cdot, \omega)$ is continuous on R^+, $\forall \omega \in \Omega$, $\phi(x, \cdot)$ is measurable on Ω, $\forall x \in R^+$, $\phi(\cdot, \omega)$ is strictly decreasing on R^+, $\forall \omega \in \Omega$, $\underline{\lambda} < \phi(0, \omega) < \bar{\lambda} < \infty$, $\forall \omega \in \Omega$ and $\mathcal{P}(\omega \in \Omega | \phi(x, \omega) \neq \bar{\phi}_x) > 0$, $\forall x \in R^+$, where $\bar{\phi}_x = \int_\Omega \phi(x, \omega) \mathrm{d}\mathcal{P}(\omega)$.

Each firm has access to two competitive markets: a spot market which meets in the fall and a futures market which meets in the spring and in the fall. Each firm's output y is sold on the spot market at the random spot price $p(\omega)$. Each firm also decides on the number z of futures contracts that it wants to sell at the spring futures price q, each contract calling for the delivery of one unit of the commodity in the fall. To close out its position on the futures market each firm buys back z contracts in the fall at the fall futures price – which we assume by arbitrage coincides with the spot price $p(\omega)$. Each firm's profit is thus given by

$$\pi(y, z; \omega) = p(\omega) y - c(y) + z(q - p(\omega)).$$

Each producer is assumed to be risk-averse. The firm's output-hedging decision (y, z) is thus the solution of the following problem:

$$\sup_{(y,z) \in R^+ \times R} \int_\Omega u(\pi(y, z; \omega)) \mathrm{d}\mathcal{P}(\omega). \qquad (\mathcal{U})$$

We say that each producer satisfies a *risk-aversion* condition if the von Neumann–Morgenstern utility function u: $R \to R$ satisfies $u \in \mathcal{C}^2(R)$, $u'(\pi) > 0$, $u''(\pi) < 0$, $\pi \in R$.

The output-hedging activity of producers is confined to a single market. By contrast, *speculators* – the other agents that trade on the futures market – are not involved in production, but their trading activity is not confined to this futures market. Each of the $k \geqq 0$ speculators is viewed as an agent endowed with a substantial pool of investable capital spread over a broad array of other futures markets in addition to this one, whose concern is to maximise the expected utility of the average return obtained on all the markets collectively. To simplify the analysis, the trading activity of the speculator on the other futures markets is taken as exogenously given. Thus if the speculator buys ξ futures contracts (on the futures market under consideration) at the spring price q and sells them back at the fall price $p(\omega)$, his average profit on all markets is given by

$$\Pi(\xi; \omega) = (\xi/n)(p(\omega) - q) + A_n(\omega),$$

where $A_n(\omega) = (1/n)\Sigma_{i=2}^{n}\pi_i(\omega)$ denotes the average profit from his transactions on the $n-1$ other markets. The random variables π_i, $i = 2,\dots,n$ are assumed to have finite expectations and to be uniformly bounded from below.

Each speculator is assumed to be risk-averse. The speculator's trading decision ξ is thus the solution of the following problem:

$$\sup_{\xi \in R} \int_{\Omega} w(\Pi(\xi;\omega))\mathrm{d}\mathcal{P}(\omega). \qquad (\mathcal{US})$$

Let $\mathcal{L}_{\infty}(\Omega, \mathcal{I}, \mathcal{P})$ denote the space of real-valued $\mathcal{P}$-essentially bounded measurable functions defined on the probability space $(\Omega, \mathcal{I}, \mathcal{P})$ and let $\mathcal{L}_{\infty}^{+}(\Omega, \mathcal{I}, \mathcal{P}) = \{\eta \in \mathcal{L}_{\infty} | \eta(\omega) \geqq 0 \text{ a.s.}\}$. We define a spot-futures *market equilibrium*, when speculators trade on n markets, as a pair

$$\langle (y_n, z_n, \xi_n), (p_n, q_n) \rangle \in R^{3+} \times \mathcal{L}_{\infty}^{+}(\Omega, \mathcal{I}, \mathcal{P}) \times R^{+}$$

such that (y_n, z_n) solves $(\mathcal{UL})$ and ξ_n solves $(\mathcal{US})$ with $(p, q) = (p_n, q_n)$, $p_n(\omega) = \phi(my_n, \omega)$ a.s. (the spot market clears almost surely) and $mz_n = k\xi_n$ (the futures market clears). This is a *rational expectations equilibrium*.

The following result is established in Cheng–Magill (1982).

Proposition 1. Under the monotonicity condition on the cost and demand functions and the risk-aversion condition on producers and speculators, there exists a spot-futures market equilibrium.

The sequence of equilibria for $n = 1, 2,\dots$ has an important asymptotic property if a stability condition is satisfied by the sequence of average profits A_n obtained by speculators on the $n-1$ other markets.

2. Asymptotic property of equilibrium

Let $\Phi(x,\omega) = \int_0^x \phi(z, \omega)\mathrm{d}z$, $x \in R^{+}$. In the case where ϕ is *stochastically independent* of prices on other markets we define the (unique) solution x^* of the problem

$$\sup_{x \in R^{+}} \int_{\Omega} \left(\Phi(x,\omega) - mc\left(\frac{x}{m}\right)\right)\mathrm{d}\mathcal{P}(\omega)$$

as the *optimum output* with no prior information. We say that the sequence of average profits A_n is *asymptotically stable* if A_n converges in probability to an asymptotic average profit A. In Cheng–Magill (1982) we introduce the important notion of *positive (negative) stochastic dependence* which, loosely speaking, expresses for a pair of arbitrary random variables what the concept of *covariance* expresses for a pair of *normally distributed* random variables. Finally the *equilibrium risk premium* (also called the *bias* in the futures price) is defined by $\delta_n = E(p_n) - q_n$.

Proposition 2. Let the conditions of Proposition 1 be satisfied. If A_n is asymptotically stable, then

$$\delta_n \to \delta, \qquad x_n \to x \quad \text{as} \quad n \to \infty,$$

where (i) $\delta = 0$, $x = x^*$ if (ϕ, A) are independent, (ii) $\delta > 0$ (<0), $x < x^*$ $(>x^*)$ if (ϕ, A) are positively (negatively) dependent.

The proof depends crucially on the existence result of Proposition 1 and is based on properties of stochastic dependence and a risk-premium way of writing the first-order conditions for the optimal portfolio of each speculator.

Example. Let $u(\pi) = -\mathrm{e}^{-\alpha\pi}$, $\alpha > 0$, $w(\Pi) = -\mathrm{e}^{-\beta\Pi}$, $\beta > 0$ and let $(\phi_x, \pi_2, \ldots, \pi_n)$ be joint normally distributed random variables $\forall x \in R^+$. Let $V(A_n) = \sigma^2_{A_n}$, cov $(\phi_x, A_n) = \sigma_{\phi A_n}(x)$, $E(\phi_x) = \mu(x)$, $V(\phi_x) = \sigma^2(x)$. Then $E(p_n) = \mu(x_n)$, $q_n = c'(x_n/m)$, $\delta_n = \underline{\delta}(x_n) + \bar{\delta}(x_n)$ so that equilibrium output is determined by the condition

$$\mu(x_n) - c'(x_n/m) = \underline{\delta}(x_n) + \bar{\delta}(x_n), \quad \text{where}$$

$$\underline{\delta}(\xi) = \left(\frac{\alpha}{1 + \alpha k n/\beta m}\right)\left(\frac{\xi}{m}\right)\sigma^2(\xi), \quad \bar{\delta}(\xi) = \left(\frac{\beta}{1 + \beta m/\alpha k n}\right)\sigma_{\phi A_n}(\xi).$$

The risk premium δ_n is composed of two terms: $\underline{\delta}$ measures the premium required by producers and speculators to cover the risks arising from the variability of the spot price (σ^2) – the *idiosyncratic risk premium*. The second term $\bar{\delta}$ measures the premium required by speculators to cover the risks that arise from the covariance $\sigma_{\phi A_n}$ between ϕ and A_n – the *covariance risk premium*. Under the conditions of Proposition 2

$$\underline{\delta} \to 0, \qquad \bar{\delta} \to \beta\sigma_{\phi A} \quad \text{as} \quad n \to \infty.$$

The asymptotic risk premium is determined by the covariance risk alone – the idiosyncratic risk is diversified away. (Note that a similar result holds when $k \to \infty$.) These results are related to well-known results in the theory of capital asset pricing. In view of the large market argument ($n \to \infty$) on which the proof of Proposition 2 is based, the affinity is closest with the *arbitrage pricing theory* of Ross (1976) where the common random factor (or index) plays a role similar to that of A_n in our model.

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