Bifurcation of nontrivial periodic solutions for an impulsively controlled pest management model

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Abstract

In this paper, we investigate the existence of nontrivial periodic solutions for an integrated pest management model which is impulsively controlled by means of biological and chemical controls used in a periodic fashion. For this model, a nonlinear incidence rate is employed to describe the transmission of the disease which is induced through the use of the biological control, while the chemical control is used with the same periodicity as the biological control, although not at the same time. Our problem is treated by means of an operator theoretic approach, being reduced first to a fixed point problem. It is then shown that once a threshold condition is reached, the trivial periodic solution loses its stability and a nontrivial periodic solution appears via a supercritical bifurcation.

Key words: Nonlinear periodic solutions, bifurcation, impulsive controls, nonlinear infection rate, fixed point approach.
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1 Introduction

Although synthetic pesticides were first seen as a miraculous way of solving all pest-related issues, it has been quickly noticed that the heavy use of pesticides creates in the long run more problems that it solves. In some situations, chemicals become increasingly ineffective, as many pests quickly develop new generations which are resistant to various chemical agents. Also, when pesticides are used to control a given pest, its natural predators may be killed as well as a side effect, which may actually cause in the long run an increase in the size of the pest population, rather than the expected reduction. If the pest is living out of reach or just hiding, then pesticides may simply have no effect on the pest population. Finally, many pesticides are known to cause environmental problems and actually damage human health.

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An integrated pest management system (IPM) is often more effective and less damaging to the environment than using pesticides alone. This strategy involves the use of a wide array of controls (mechanical, biological, chemical and not only), the emphasis being on the control of the pest population, not on its eradication, as the later might be unfeasible or counterproductive. Generally, an IPM strategy is considered successful when the pest population stabilizes under the economic injury level (EIL), defined in Stern et al [16] as the amount of pest injury which will justify the cost of using controls or the lowest pest density which causes economic damage.

To use biological controls in an effective manner, detailed knowledge of the pest and of its natural enemies is needed. While some approaches to biological controls consist in the use of parasitoids or pathogens of the target pests, or in the periodic release of the natural predators of the given pest, another approach is to release pests which are infected in laboratories, with the purpose of creating and maintaining a disease in the target pest population, on the grounds that infective pests usually cause less environmental damage. This is the approach to biological control which we use in the present paper. Usually, biological controls are of much less environmental concern, lower cost and might be more effective if applied correctly. Also, they are self-regulating up to some extent.

Regarding the disease which is caused by the periodic release of infected pests, it has been observed (see Liu et al [11], Korobeinikov and Maini [8], Korobeinikov [7]) that the dependence on the size of the infected population \( I \) plays a more prominent role than the size of the susceptible pest population \( S \), as far as the incidence rate of the infection is concerned.

Consequently, an incidence rate of type \( g(I)S \) may be appropriate in many situations. See also Capasso and Serio [2], Hethcote and van den Driessche [5], Ruan and Wang [15], Xiao and Ruan [17], in which particular rates of this type are employed. In the following, we shall use a general incidence rate of type \( g(I)S \) to model disease transmission, under a few assumptions on the nonlinear force of infection \( g \).

As far as chemical controls are concerned, synthetic pesticides are used in IPM strategies only as a last resort, when deemed an absolute necessity, and are chosen to specifically target the pest to be controlled.

A central problem in IPM strategies is to choose the appropriate moment to use each type of control. To account for the fact that pesticides cannot be sprayed continuously, we use a model introduced in Georgescu and Moroşanu [4], in which both the biological and chemical controls are employed in an impulsive and periodic fashion, with the same period but not at the same time. The choice of using impulsive controls is, in our opinion, justified since for certain pesticides the effect follows shortly after application and also since the size of the infected pest population grows immediately after the release of the infective individuals. Therefore, such changes can be modeled as immediate jumps in the population sizes. In this regard, a general account of the theory of impulsive ordinary differential equations can be found in Bainov and Simeonov [1].

An unified approach to deal with the existence of nontrivial periodic solutions for a large class of two dimensional systems of differential equations which are impulsively perturbed in a periodic fashion by means of possibly nonlinear controls has been laid out in Lakmeche and Arino [9]. Their method consists in reformulating the problem as a fixed point problem for an operator defined \( ad-hoc \) which incorporates the effects of the impulsive perturbations, and solve the latter using the method of bifurcation theory; specifically, a certain projection method is employed. They also apply their general method to the study of the existence of nontrivial
periodic solutions for a concrete problem arising from the chemotherapeutic treatment of tumors. Their concrete model contains nonlinearities of logistic type and linear impulses and has been originally introduced by Panetta in [14].

Consequently, our paper employs the method introduced in [9] together with some of the notations therein, although our model is structurally different from Panetta’s, in the sense that it is not a competitive model, like the one in [14] (it is actually neither competitive nor cooperative). Notably, we obtain the bifurcation of nontrivial periodic solutions for general nontrivial infection rates and employ two distinct types of impulsive controls, corresponding to the use of a biological and a chemical control, respectively. See also Lakmeche and Arino [10], where the bifurcation of nontrivial periodic solutions for a Kolmogorov-like system arising from heterogeneous tumor therapy by several drugs with instantaneous effects administered one at a time is studied by the same method. The approach devised by Lakmeche and Arino is also employed, among others, by Lu, Chi and Chen in [12] for a predator-pest model subject to pulsed use of insecticides and by the same authors in [13] for a SIR epidemic model with horizontal and vertical transmission which is subject to pulse vaccination.

This paper is organized as follows: in Section 2, we formulate our impulsive control model and give its stability and persistence properties. In Section 3, we introduce a few definitions and notations and reformulate our problem as a fixed point problem. In Section 4, we study the onset of nontrivial periodic solutions by means of bifurcation theory. Our findings are then discussed in Section 5. Finally, some more technical computations necessary in the above are given in the Appendix.

2 The model and its stability properties

In the following, we consider the model which has been studied in [4] from the viewpoint of finding sufficient conditions for the global stability of the susceptible pest-eradication solution and for the persistence of the disease, respectively. We denote by $S$ the size of the susceptible pest population, by $I$ the size of the infective pest population and suppose that all pests are either susceptible or infective.

We may formulate the following impulsively controlled system

\[
\begin{align*}
I'(t) &= g(I(t))S(t) - wI(t), & t \neq (n + l - 1)T, t \neq nT; \\
S'(t) &= S(t)n(S(t)) - g(I(t))S(t), & t \neq (n + l - 1)T, t \neq nT; \\
\Delta I(t) &= -\delta_2 I(t), & t = (n + l - 1)T; \\
\Delta S(t) &= -\delta_1 S(t), & t = (n + l - 1)T; \\
\Delta I(t) &= \mu, & t = nT; \\
\Delta S(t) &= 0, & t = nT.
\end{align*}
\]

For the biological background and the assumptions which led to the formulation of (S), we refer the reader to [4]. Here, $T > 0$, $0 < l < 1$, $\Delta \varphi(t) = \varphi(t+) - \varphi(t)$ for $\varphi \in \{S, I\}$, $0 \leq \delta_1, \delta_2 < 1$, $n \in \mathbb{N}^*$, $w > 0$. We shall also denote $n(0) = r > 0$.

The functions $n$ and $g$ satisfy the following hypotheses indicated below.

\[
\text{(N) } n \text{ is decreasing on } [0, \infty), \lim_{S \to \infty} n(S) < -w, S \mapsto Sn(S) \text{ locally Lipschitz on } (0, \infty).\]
(G) \( g(0) = 0, g \) is increasing and globally Lipschitz on \([0, \infty)\).

Under these hypotheses, it has been shown in [4] that the initial value problem for the system (S) is biologically well-posed, in the sense that for any positive initial data \((I(0), S(0))\) there corresponds a positive solution \((I(t), S(t))\) which is globally defined, while if the initial data is strictly positive component-wise, the solution is also strictly positive component-wise. It has also been shown in [4, Lemma 3.3] that all solutions of (S) are bounded.

We now give some properties of the subsystem

\[
(SI) \begin{cases}
I'(t) = -wI(t), & t \neq nT, (n + l - 1)T; \\
\Delta I(t) = -\delta_2I(t), & t = (n + l - 1)T; \\
\Delta I(t) = \mu, & t = nT; \\
I(0+) = I_0,
\end{cases}
\]

which is used to describe the dynamics of the susceptible pest-eradication state. It has been seen in [4] that the system formed with the first three equations of (SI) has a periodic solution \(I^*\) to which all solutions of (SI) tend as \(t \to \infty\). More precisely, \(I^*\) is given by

\[
\left\{ \begin{array}{ll}
I^*(t) = e^{-wt}I^*(0+), & t \in (0, lT]; \\
I^*(t) = e^{-wt}I^*(0+)(1 - \delta_2), & t \in (lT, T];
\end{array} \right.
\]

where, by the \(T\)-periodicity requirement,

\[
I^*(0+) = \frac{\mu}{1 - e^{-wT}(1 - \delta_2)}.
\]

It has also been shown in [4, Theorems 4.1 and 5.1] that the susceptible pest-eradication periodic solution \((I^*, 0)\) is globally asymptotically stable provided that

\[
\int_0^T g(I^*(t)) dt > rT + \ln(1 - \delta_1),
\]

while if the opposite inequality is satisfied, then the susceptible pest-eradication solution, called also in the following the trivial periodic solution, loses its stability and the system (S) becomes uniformly persistent. In the following, we shall mainly study this loss of stability and prove that it is due to the onset of nontrivial periodic solutions obtained via a supercritical bifurcation.

3 Definition and notations. The fixed point problem

In the following, we shall denote by \(\Phi(t; X_0)\) the solution of the (unperturbed) system formed with the first two equations in (S) for the initial data \(X_0 = (x_0^1, x_0^2)\); also, \(\Phi = (\Phi_1, \Phi_2)\). We define \(I_1, I_2 : \mathbb{R}^2 \to \mathbb{R}^2\) by

\[
I_1(x_1, x_2) = ((1 - \delta_2)x_1, (1 - \delta_1)x_2), \quad I_2(x_1, x_2) = (x_1 + \mu, x_2)
\]

and \(F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}\) by

\[
F_1(x_1, x_2) = g(x_1)x_2 - wx_1, \quad F_2(x_1, x_2) = x_2n(x_2) - g(x_1)x_2,
\]
also, $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$,
\[
F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2)).
\]
First, we reduce the problem of finding a periodic solution of (S) to a fixed point problem. To this purpose, let us define $\Psi : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by
\[
\Psi(T, X_0) = I_2(\Phi((1 - l)T; I_1(\Phi(lT; X_0))));
\]
also
\[
\Psi(T, X_0) = (\Psi_1(T; X_0), \Psi_2(T; X_0)).
\]
Then $X$ is a periodic solution of period $T$ for (S) if and only if its initial data $X(0) = X_0$ is a fixed point for $\Psi$. Consequently, to study the existence of nontrivial periodic solutions for (S), we need to study the existence of nontrivial fixed points of $\Psi$.

First, we note that
\[
D_X \Psi(T, X) = D_X \Phi((1 - l)T; I_1(\Phi(lT; X))) \begin{pmatrix} 1 - \delta_2 & 0 \\ 0 & 1 - \delta_1 \end{pmatrix} D_X \Phi(lT; X).
\]
Let us denote
\[
X_0 = (x_0, 0) \text{ the starting point for the trivial periodic solution } (I^*, 0), \text{ where } x_0 = I^*(0+), I^*(0+) \text{ being given by } (2.1). \text{ We are interested in the bifurcation of nontrivial periodic solutions near } (I^*, 0). \text{ To this purpose, we need to find } D_X \Phi(t; X_0), \text{ which can be computed by (formally) deriving the first two equations in } (S) \text{ (see the Appendix). One then obtains that}
\[
D_X \Psi(T, X) = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix}
\]
with
\[
\begin{aligned}
 d_{11} &= (1 - \delta_2) e^{-wT}, \quad d_{11} \in (0, 1); \\
 d_{12} &= e^{-wT} \left[(1 - \delta_2) \int_0^{IT} g(I^*(s)) e^{(r+w)s - \int_0^s g(I^*(\tau))d\tau} ds \\
 & \quad + (1 - \delta_1) \int_{IT}^T g(I^*(s)) e^{(r+w)s - \int_0^s g(I^*(\tau))d\tau} ds \right]; \\
 d_{22} &= (1 - \delta_1) e^{rT - \int_0^T g(I^*(s))ds}, \quad d_{22} > 0.
\end{aligned}
\]
It is known that $(I^*, 0)$, the trivial periodic solution starting from $X_0$, is exponentially stable if and only if the spectral radius $\rho(D_X \Psi(T, X_0))$ is less than 1 (see Iooss [6]). From the above, it follows that the trivial periodic solution $(I^*, 0)$ is exponentially stable if and only if
\[
(1 - \delta_1) e^{rT - \int_0^T g(I^*(s))ds} < 1.
\]

4 The bifurcation of nontrivial periodic solutions

We now study the bifurcation of nontrivial periodic solutions near $(I^*, 0)$. To this purpose, let us denote
\[
\tau = T + \tau, \quad X = X_0 + \overline{X}.
\]
To find a nontrivial periodic solution of period $\tau$ with initial data $X$, we need to solve the fixed point problem $X = \Psi(T + \tau, X_0 + X)$. Let us define

$$N(\tau, X) = X_0 + X - \Psi(T + \tau, X_0 + X); \quad N(\tau, X) = (N_1(\tau, X), N_2(\tau, X)).$$

Using the newly defined function $N$, it then remains to solve the equation $N(\tau, X) = 0$. Let us denote

$$D_X N(0, (0, 0)) = \begin{pmatrix} a'_0 & b'_0 \\ c'_0 & d'_0 \end{pmatrix}.$$

Since $D_X N(0, (0, 0)) = I_2 - D_X \Psi(T, X_0)$, it follows that

$$a'_0 = 1 - d_{11}, \quad b'_0 = -d_{12}, \quad c'_0 = -d_{21}, \quad d'_0 = 1 - d_{22}$$

and consequently

$$a'_0 = 1 - (1 - \delta_2) e^{-w T}$$
$$b'_0 = -e^{-w T} \left[ (1 - \delta_2) \int_0^T g(I^*(s)) e^{(r+w)s-f_0^* g(I^*(\tau))} d\tau ds \right]$$
$$c'_0 = 0;$$
$$d'_0 = 1 - (1 - \delta_1) e^{r T-f_0^* g(I^*(s))} ds.$$

A necessary condition for the bifurcation of nontrivial periodic solutions near $(I^*, 0)$ is

$$\det [D_X N(0, (0, 0))] = 0$$

and since $D_X N(0, (0, 0))$ is upper triangular and $a'_0 = 1 - (1 - \delta_2) e^{-w T} \neq 0$, it consequently follows that $d'_0 = 0$, that is,

$$d'_0 = 1 - (1 - \delta_1) e^{r T-f_0^* g(I^*(s))} ds.$$

It is seen that

$$\dim(\text{Ker} [D_X N(0, (0, 0))]) = 1,$$

and a basis in $\text{Ker} [D_X N(0, (0, 0))]$ is $(-\frac{b'_0}{a'_0}, 1)$. Then the equation $N(\tau, X) = 0$ is equivalent to

$$\begin{cases} 
N_1(\tau, \alpha Y_0 + z E_0) = 0; \\
N_2(\tau, \alpha Y_0 + z E_0) = 0,
\end{cases}$$

where

$$E_0 = (1, 0), \quad Y_0 = (-\frac{b'_0}{a'_0}, 1).$$
and $\overline{X} = \alpha Y_0 + zE_0$ represents the direct sum decomposition of $\overline{X}$ using the projections onto $\text{Ker}[D_X N(0,(0,0))]$ and $\text{Im}[D_X N(0,(0,0))]$. See [3, Section 2.4] for details.

Let us denote

\begin{align}
(4.6) \quad f_1(\bar{\tau}, \alpha, z) &= N_1(\bar{\tau}, \alpha Y_0 + zE_0); \\
(4.7) \quad f_2(\bar{\tau}, \alpha, z) &= N_2(\bar{\tau}, \alpha Y_0 + zE_0).
\end{align}

First, we see that

$$\frac{\partial f_1}{\partial z}(0,0,0) = \frac{\partial N_1}{\partial x_1}(0,(0,0)) = a_0' \neq 0.$$ 

By the implicit function theorem, one may locally solve the equation $f_1(\bar{\tau}, \alpha, z) = 0$ near $(0,0,0)$ with respect to $z$ as a function of $\bar{\tau}$ and $\alpha$ and find $z = z(\bar{\tau}, \alpha)$ such that $z(0,0) = 0$ and

$$f_1(\bar{\tau}, \alpha, z(\bar{\tau}, \alpha)) = N_1(\bar{\tau}, \alpha Y_0 + z(\bar{\tau}, \alpha)E_0) = 0.$$ 

Moreover, the first order partial derivatives $\frac{\partial z}{\partial \alpha}(0,0)$ and $\frac{\partial z}{\partial \tau}(0,0)$ are given by

\begin{align}
\left\{
\begin{array}{l}
\frac{\partial z}{\partial \alpha}(0,0) = 0 \\
\frac{\partial z}{\partial \tau}(0,0) = -\frac{w}{a_0'} I^*(T)
\end{array}
\right.
\end{align}

(see the Appendix).

It now remains to study the solvability of the equation

\begin{align}
(4.8) \quad f_2(\bar{\tau}, \alpha, z(\bar{\tau}, \alpha)) = 0,
\end{align}

that is,

\begin{align}
(4.9) \quad N_2(\bar{\tau}, \alpha Y_0 + z(\bar{\tau}, \alpha)E_0) = 0.
\end{align}

The equation (4.9) is called the determining equation and the number of its solutions equals the number of periodic solutions of (S). We now proceed to solving (4.9) (or, equivalently, (4.8)).

Let us denote

\begin{align}
(4.10) \quad f(\bar{\tau}, \alpha) &= f_2(\bar{\tau}, \alpha, z(\bar{\tau}, \alpha)).
\end{align}

First, it is easy to see that

$$f(0,0) = N(0,(0,0)) = 0.$$ 

To determine the number of solutions of (4.9), we first determine the Taylor expansion of $f$ around $(0,0)$. To this goal, we compute the first order partial derivatives $\frac{\partial f}{\partial \tau}(0,0)$ and $\frac{\partial f}{\partial \alpha}(0,0)$ and observe that

$$\frac{\partial f}{\partial \tau}(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = 0.$$ 

For the proof of this fact, see the Appendix.
It now becomes necessary to compute the second order partial derivatives $\frac{\partial^2 f}{\partial \alpha^2}(0, 0)$, $\frac{\partial^2 f}{\partial \alpha \partial \tau}(0, 0)$, $\frac{\partial^2 f}{\partial \tau^2}(0, 0)$. It is seen that

$$A = \frac{\partial^2 f}{\partial \alpha^2}(0, 0) = 0$$
$$B = \frac{\partial^2 f}{\partial \alpha \partial \tau}(0, 0) < 0$$
$$C = \frac{\partial^2 f}{\partial \tau^2}(0, 0) > 0.$$

We need now find a nontrivial solution of the equation $f(\tau, \alpha) = 0$ near $(0, 0)$. By expanding $f$ into a second order Taylor series, one obtains that

$$f(\tau, \alpha) = B\alpha \tau + C\alpha^2 + o(\tau, \alpha)(\tau^2 + \alpha^2).$$

By denoting $\tau = k\alpha$ ($k = k(\alpha)$), it is seen that

$$Bk + C\frac{k^2}{2} + o(\alpha, k\alpha)(1 + k^2) = 0,$$

equation which is solvable with respect to $k$ as a function of $\alpha$, since $B < 0$ and $C > 0$. Moreover, $k \approx -\frac{2B}{C}$, that is, $k$ is positive.

From the above, one sees that there is a supercritical bifurcation of a nontrivial periodic solution near a period $T$ which satisfies the sufficient condition for bifurcation given in (4.5). Note that, as it appears via a supercritical bifurcation, the nontrivial periodic solution is stable. More precisely, one obtains the following result, in which $X_0, Y_0, E_0, z, \tau$ are as indicated above.

**Theorem 4.1.** Suppose that the impulsive period $T$ satisfies condition (4.5). Then there is $\epsilon > 0$ such that for all $0 < \alpha < \epsilon$ there is a stable positive nontrivial periodic solution of (S) with period $T + \tau(\alpha)$ which starts in $X_0 + \alpha Y_0 + z(\tau(\alpha), \alpha)E_0$.

5 Conclusion

In this paper, the existence of nontrivial periodic solutions for an impulsively controlled integrated pest management model is investigated using an operator theoretic approach. To limit the damaging potential of the pest population, biological controls, consisting in the release of infective pests, and chemical controls, consisting in pesticide spraying, are used in a periodic fashion, with the same period but not in the same time. It is assumed that the infective pest population neither damages the crops nor reproduces and a nonlinear incidence rate is employed to describe the transmission of the disease.

In concrete terms, a nontrivial periodic solution corresponds to the apparition of a persistent susceptible pest population, while a nontrivial periodic solution with small amplitude, below the economic injury level, indicates that the pest management strategy is still successful.

The problem of finding such nontrivial periodic solutions is reduced to a fixed point problem, which is treated using the methods of bifurcation theory. It is shown that once a threshold condition is reached, then the trivial periodic solution loses its stability and a nontrivial periodic solution appears via a supercritical bifurcation.
We shall briefly comment upon the threshold condition (4.5), which may be reformulated as

\[ \int_0^T g(I^*(s))ds - \ln(1 - \delta_1) = rT. \]

Let us suppose that \((I(t), S(t))\) approaches the trivial solution \((I^*, 0)\). Then, as the incidence rate of the infection is of the form \(g(I)S\), the integral \(\int_0^T g(I^*(t))dt\) approximates the (per-susceptible) loss of susceptible pests in a period due to their movement in the infective class, while since the production of newborn susceptible pests is given by \(Sn(S)\) and \(n(0) = r\), \(rT\) approximates the total (per-susceptible) gain of susceptible pests in a period. A correction term \(-\ln(1 - \delta_1)\) should also be added to account for the loss of susceptible pests due to pesticide spraying. Then the threshold condition represents just the fact that the total (per-susceptible) loss of susceptible pests in a period balances the total (per-susceptible) gain of newborn susceptible pests in a period.

In this regard, it has been shown in [4] that \((I^*, 0)\) is globally asymptotically stable provided that

\[ \int_0^T g(I^*(s))ds - \ln(1 - \delta_1) > rT, \]

while if the opposite inequality is satisfied, then \((I^*, 0)\) loses its stability and \((S)\) becomes uniformly persistent.

In the case in which \(g\) is a linear force of infection, \(g(I) = \beta I\), then the threshold condition can be reformulated as

\[ \frac{1}{T} \int_0^T I^*(s)ds = \frac{r + (1/T) \ln(1 - \delta_1)}{\beta}. \]

By defining \(I_C = \frac{r + (1/T) \ln(1 - \delta_1)}{\beta}\) as an “epidemic threshold”, it is then seen from the above and Theorem 4.1 that nontrivial periodic solutions \((I, S)\) appear when the average of the susceptible pest-eradication periodic solution over a period reaches the epidemic threshold \(I_C\). As mentioned above, if the average of \(I^*\) is greater than \(I_C\), then the susceptible pest-eradication periodic solution is globally stable, while if the average of \(I^*\) is less than \(I_C\), then the system \((S)\) is uniformly persistent.

Let us also define, for a general \(g\),

\[ R_0^S = \frac{rT}{\int_0^T g(I^*(s))ds - \ln(1 - \delta_1)} \]

as being a “basic reproduction number”-like quantity with respect to the susceptible pest population. Note that this is a “mirror image” of what usually a basic reproduction number means, since the survival of the susceptible pest population is usually unquestioned, the main problem being whether or not the infection becomes endemic. In the usual situation, the alternative endings are, roughly, an infection-free state and an endemic state, in which the infective pest population persists, at a certain level, alongside the susceptible pest population.

Here, the situation is somewhat different. The long-term survival of the infective pest population is unquestionable, due to the pulsed supply of infectives at \(t = nT\) and what is at stake is the survival of the susceptible pest population, the alternative endings being a susceptible-free state and an endemic state.
With this notation, the threshold condition can simply be rewritten as $R_0^S = 1$. If $R_0^S < 1$, then the newborn susceptibles are not produced fast enough and the system tends to the susceptible pest-eradication periodic solution, while if $R_0^S > 1$, then the system becomes uniformly persistent. See [4] for details.

6 Appendix

6.1 The first order partial derivatives of $\Phi_1$, $\Phi_2$

By (formally) deriving

$$\frac{d}{dt} (\Phi(t; X_0)) = F(\Phi(t; X_0)),$$

one obtains

$$\frac{d}{dt} \left[ D_X \Phi(t; X_0) \right] = D_X F(\Phi(t; X_0)) D_X \Phi(t; X_0).$$

Also, it is clear that

$$\Phi(t; X_0) = (\Phi_1(t; X_0), 0).$$

We then deduce

$$\frac{d}{dt} \left( \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_1}{\partial x_2} \right) (t; X_0) = \begin{pmatrix} -w & g(\Phi_1(t; X_0)) \\ 0 & r - g(\Phi_1(t; X_0)) \end{pmatrix} \left( \frac{\partial \Phi_1}{\partial x_1} \frac{\partial \Phi_1}{\partial x_2} \right) (t; X_0),$$

the initial condition being

$$D_X \Phi(0; X_0) = I_2.$$ 

Here, $I_2$ is the identity matrix in $M_2(\mathbb{R})$. It follows that

$$\frac{d}{dt} \left( \frac{\partial \Phi_2}{\partial x_1} (t; X_0) \right) = (r - g(\Phi_1(t; X_0))) \frac{\partial \Phi_2}{\partial x_1} (t; X_0)$$

and then

$$\frac{\partial \Phi_2}{\partial x_1} (t; X_0) = e^{\int_0^t (r - g(\Phi_1(s; X_0))) ds} \frac{\partial \Phi_2}{\partial x_1} (0; X_0),$$

which implies, using (6.1), that

$$\frac{\partial \Phi_2}{\partial x_1} (t; X_0) = 0 \quad \text{for } t \geq 0.$$ 

One then gets

$$\begin{cases} 
\frac{d}{dt} \left( \frac{\partial \Phi_1}{\partial x_1} (t; X_0) \right) = -w \frac{\partial \Phi_1}{\partial x_1} (t; X_0) \\
\frac{d}{dt} \left( \frac{\partial \Phi_1}{\partial x_2} (t; X_0) \right) = -w \frac{\partial \Phi_1}{\partial x_2} (t; X_0) + g(\Phi_1(t; X_0)) \frac{\partial \Phi_2}{\partial x_2} (t; X_0) \\
\frac{d}{dt} \left( \frac{\partial \Phi_2}{\partial x_1} (t; X_0) \right) = (r - g(\Phi_1(t; X_0))) \frac{\partial \Phi_2}{\partial x_1} (t; X_0),
\end{cases}$$

(6.3)
from which we deduce, using (6.1), that
\[
\begin{align*}
\frac{\partial \Phi_1}{\partial x_1}(t; X_0) &= e^{-wt} \\
\frac{\partial \Phi_1}{\partial x_2}(t; X_0) &= e^{-wt} \int_0^t g(\Phi(1; X_0)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
\frac{\partial \Phi_2}{\partial x_2}(t; X_0) &= e^{rt-f_0^T g(\Phi(1; X_0))} ds.
\end{align*}
\]
Also, from (6.2), it follows that
\[
D_X \Psi(T, X_0) = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix},
\]
with \(d_{11}, d_{12}, d_{22}\) being given by
\[
\begin{align*}
(6.4) & \quad d_{11} = (1-\delta_2) \frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(I; X_0))) \frac{\partial \Phi_1}{\partial x_1}(IT; X_0) \\
(6.5) & \quad d_{12} = (1-\delta_2) \frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(I; X_0))) \frac{\partial \Phi_1}{\partial x_2}(IT; X_0) \\
& \quad + (1-\delta_1) \frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(I; X_0))) \frac{\partial \Phi_2}{\partial x_2}(IT; X_0) \\
(6.6) & \quad d_{22} = (1-\delta_1) \frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(I; X_0))) \frac{\partial \Phi_2}{\partial x_2}(IT; X_0).
\end{align*}
\]
Consequently,
\[
\begin{align*}
(6.7) & \quad d_{11} = (1-\delta_2)e^{-wT}, \\
(6.8) & \quad d_{12} = (1-\delta_2)e^{-w(1-l)T}e^{-wT} \int_0^{IT} g(I^*(s)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
& \quad + (1-\delta_1)e^{-w(1-l)T} \int_0^{(1-l)T} g(I^*(s); I_1(\Phi(I; X_0))) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
& \quad \cdot e^{rT-f_0^T g(\Phi(1; X_0))} ds \\
& \quad = (1-\delta_2)e^{-wT} \int_0^T g(I^*(s)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
& \quad + (1-\delta_1)e^{-w(1-l)T} \int_0^{(1-l)T} g(I^*(s+lT)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
& \quad \cdot e^{rT-f_0^T g(I^*(s))} ds \\
& \quad = e^{-wT} \left[ (1-\delta_2) \int_0^T g(I^*(s)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \\
& \quad + (1-\delta_1) \int_0^{(1-l)T} g(I^*(s)) e^{(r+w)s-f_0^T g(\Phi(1; X_0))} dr ds \right]; \\
(6.9) & \quad d_{22} = (1-\delta_1) e^{r(1-l)T-f_0^T \int_0^{(1-l)T} g(\Phi_1(s; I_1(\Phi(I; X_0)))) ds} e^{rT-f_0^T g(\Phi(1; X_0))} ds.
\[ (1 - \delta_1) e^{rT - f_0^{(1-l)}} g(I^*(s+lt))ds - f_0^{(1-l)} g(I^*(s))ds \]

\[ = (1 - \delta_1) e^{rT - f_0^T} g(I^*(s))ds \]

### 6.2 The partial derivatives of \( z \) at \( (0, 0) \)

From the implicit function theorem, it follows that

\[ \frac{\partial N_1}{\partial x_1}(0, (0, 0)) \left( -\frac{b'_0}{a'_0} \right) + \frac{\partial N_1}{\partial x_2}(0, (0, 0)) + \frac{\partial N_1}{\partial x_1}(0, (0, 0)) \frac{\partial z}{\partial \alpha}(0, 0) = 0 \]

and consequently

\[ a'_0 \left( -\frac{b'_0}{a'_0} \right) + b'_0 + a'_0 \frac{\partial z}{\partial \alpha}(0, 0) = 0, \]

from which we obtain that

\[ \frac{\partial z}{\partial \alpha}(0, 0) = 0. \]

The computations required for finding \( \frac{\partial z}{\partial \alpha}(0, 0) \) are somewhat more complicated, as \( \frac{\partial N}{\partial \alpha}(0, (0, 0)) \) is not known beforehand, unlike \( \frac{\partial N}{\partial x_2}(0, (0, 0)) \) and \( \frac{\partial N}{\partial x_2}(0, (0, 0)) \). Again, by the implicit function theorem, it follows from (4.6) that

\[ \frac{\partial z}{\partial \tau}(0, 0) = \frac{\partial \Phi_1}{\partial \tau}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - l) \]

\[ + \frac{\partial \Phi_1}{\partial x_1}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \tau}(lT; X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1}(lT; X_0) \frac{\partial z}{\partial \tau}(0, 0) \right) \]

\[ + \frac{\partial \Phi_1}{\partial x_2}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_1) \left( \frac{\partial \Phi_2}{\partial \tau}(lT; X_0) \cdot l + \frac{\partial \Phi_2}{\partial x_1}(lT; X_0) \frac{\partial z}{\partial \tau}(0, 0) \right). \]

Since

(6.10) \[ \frac{\partial \Phi_2}{\partial x_1}(lT; X_0) = 0, \]

(6.11) \[ \frac{\partial \Phi_2}{\partial \tau}(lT; X_0) = 0, \]

it follows that

\[ \frac{\partial z}{\partial \tau}(0, 0) = \frac{\partial \Phi_1}{\partial \tau}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - l) \]

\[ + \frac{\partial \Phi_1}{\partial x_1}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \tau}(lT; X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1}(lT; X_0) \frac{\partial z}{\partial \tau}(0, 0) \right) \]

and consequently

\[ \frac{\partial z}{\partial \tau}(0, 0) \left( 1 - \frac{\partial \Phi_1}{\partial x_1}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_2) \frac{\partial \Phi_1}{\partial x_1}(lT; X_0) \right) \]

\[ = \frac{\partial \Phi_1}{\partial \tau}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - l) + \frac{\partial \Phi_1}{\partial x_1}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_2) \frac{\partial \Phi_1}{\partial \tau}(lT; X_0) \cdot l \]
From (4.1), it follows that
\[
\frac{\partial z}{\partial \tau}(0,0) = \frac{1}{\alpha_0} \left[ \frac{\partial \Phi_1}{\partial \tau}((1-l)T; I_1(\Phi(lT; X_0)))(1-l) + \frac{\partial \Phi_1}{\partial x_1}((1-l)T; I_1(\Phi(lT; X_0)))(1-\delta_2) \frac{\partial \Phi_1}{\partial \tau}(lT; X_0) \cdot I \right].
\]

Consequently, one may obtain that
\[
\frac{\partial z}{\partial \tau}(0,0) = \frac{1}{\alpha_0} \left[ -w I^*(T)(1-l) + (1-\delta_2)e^{-w(l-1)T}(-w I^*(lT)) \cdot I \right]
\[
= -\frac{w}{\alpha_0} \left[ I^*(T)(1-l) + e^{-w(l-1)T} I^*(lT) \cdot I \right]
\[
= -\frac{w}{\alpha_0} I^*(T)
\]

6.3 The first order partial derivatives of \( f \) at \((0,0)\)

It is easy to see that
\[
\frac{\partial f}{\partial \alpha}(\tau, \alpha) = \frac{\partial}{\partial \alpha} \left[ \alpha - \Psi_2(T + \tau, X_0 + \alpha Y_0 + z(\tau, \alpha)E_0) \right]
\[
= 1 - \frac{\partial}{\partial \alpha} \left[ \Phi_2((1-l)(T + \tau); I_1(\Phi(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0))) \right]
\[
= 1 - \frac{\partial \Phi_2}{\partial x_1}((1-l)(T + \tau); I_1(\Phi(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0)))
\]
\[
\cdot (1-\delta_2) \left( \frac{\partial \Phi_1}{\partial x_1}(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0) \left( -\frac{b'_0}{\alpha_0} + \frac{\partial z}{\partial \alpha}(\tau, \alpha) \right) \right.
\]
\[
+ \frac{\partial \Phi_1}{\partial x_2}(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0) \right)
\]
\[
- \frac{\partial \Phi_2}{\partial x_2}((1-l)(T + \tau); I_1(\Phi(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0)))
\]
\[
\cdot (1-\delta_1) \left( \frac{\partial \Phi_2}{\partial x_1}(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0) \left( -\frac{b'_0}{\alpha_0} + \frac{\partial z}{\partial \alpha}(\tau, \alpha) \right) \right.
\]
\[
+ \frac{\partial \Phi_2}{\partial x_2}(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha)E_0) \right). \]

It then follows that
\[
\frac{\partial f}{\partial \alpha}(0,0)
\]
\[
= 1 - \frac{\partial \Phi_2}{\partial x_1}((1-l)T; I_1(\Phi(lT; X_0)))(1-\delta_2) \left( \frac{\partial \Phi_1}{\partial x_1}(lT; X_0) \left( -\frac{b'_0}{\alpha_0} + \frac{\partial z}{\partial \alpha}(0,0) \right) + \frac{\partial \Phi_1}{\partial x_2}(lT; X_0) \right)
\]
\[
- \frac{\partial \Phi_2}{\partial x_2}((1-l)T; I_1(\Phi(lT; X_0)))(1-\delta_1) \left( \frac{\partial \Phi_2}{\partial x_1}(lT; X_0) \left( -\frac{b'_0}{\alpha_0} + \frac{\partial z}{\partial \alpha}(0,0) \right) + \frac{\partial \Phi_2}{\partial x_2}(lT; X_0) \right). \]
From (6.10) and
\[(6.12) \quad \frac{\partial \Phi_2}{\partial x_1}((1 - l)T; I_1(\Phi(lT; X_0))) = 0,\]
it is seen that
\[
\frac{\partial f}{\partial \alpha}(0, 0) = 1 - \frac{\partial \Phi_2}{\partial x_2}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - \delta_1) \frac{\partial \Phi_2}{\partial x_2}(lT; X_0)
\]
\[= d_0 - \delta_1 \frac{\partial \Phi_2}{\partial x_2}(lT; X_0) \]
\[= 0.
\]
It is also seen that
\[
\frac{\partial f}{\partial \varphi}(\varphi, \alpha) = \frac{\partial}{\partial \varphi} \left[ \alpha - \Psi_2(T + \varphi, X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0) \right]
\]
\[= - \frac{\partial}{\partial \varphi} \left[ \Phi_2((1 - l)(T + \varphi); I_1(\Phi(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0))) \right]
\]
\[= - \frac{\partial \Phi_2}{\partial x_1}((1 - l)(T + \varphi); I_1(\Phi(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0)))(1 - l)
\]
\[\cdot (1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \varphi}(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0) \cdot l 
\]
\[+ \frac{\partial \Phi_1}{\partial x_1}(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0) \frac{\partial z}{\partial \varphi}(\varphi, \alpha) \right) 
\]
\[= - \frac{\partial \Phi_2}{\partial x_2}((1 - l)(T + \varphi); I_1(\Phi(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0)) 
\]
\[\cdot (1 - \delta_1) \left( \frac{\partial \Phi_2}{\partial \varphi}(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0) \cdot l 
\]
\[+ \frac{\partial \Phi_2}{\partial x_1}(l(T + \varphi); X_0 + \alpha Y_0 + z(\varphi, \alpha)E_0) \frac{\partial z}{\partial \varphi}(\varphi, \alpha) \right). 
\]
It then follows that
\[
\frac{\partial f}{\partial \varphi}(0, 0) = - \frac{\partial \Phi_2}{\partial \varphi}((1 - l)T; I_1(\Phi(lT; X_0)))(1 - l)
\]
\[\cdot (1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \varphi}(lT; X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1}(lT; X_0) \frac{\partial z}{\partial \varphi}(0, 0) \right) 
\]
\[= - \frac{\partial \Phi_2}{\partial x_2}((1 - l)T; I_1(\Phi(lT; X_0)) 
\]
\[\cdot (1 - \delta_1) \left( \frac{\partial \Phi_2}{\partial \varphi}(lT; X_0) \cdot l + \frac{\partial \Phi_2}{\partial x_1}(lT; X_0) \frac{\partial z}{\partial \varphi}(0, 0) \right). 
\]
From (6.10),(6.11),(6.12) and
\[(6.13) \quad \frac{\partial \Phi_2}{\partial \varphi}((1 - l)T; I_1(\Phi(lT; X_0))) = 0,
\]
it consequently follows that
\[ \frac{\partial f}{\partial \tau}(0, 0) = 0. \]

### 6.4 The second order partial derivatives of \( \Phi_2 \)

Again, by formally deriving
\[ \frac{d}{dt}(\Phi(t; X_0)) = F(\Phi(t; X_0)), \]
one may obtain \( \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) \), \( \frac{\partial^2 \Phi_2}{\partial x_2^2}(t; X_0) \), \( \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2}(t; X_0) \) as the solutions of certain initial value problems. One sees that
\[ \frac{d}{dt} \left( \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) \right) = (r - g(\Phi_1(t; X_0))) \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) - \frac{\partial \Phi_1}{\partial x_1}(t; X_0) \frac{\partial \Phi_2}{\partial x_1}(t; X_0) \]
and since
\[ \frac{\partial \Phi_2}{\partial x_1}(t; X_0) = 0 \quad \text{for } t \geq 0, \]
it follows that
\[ \frac{d}{dt} \left( \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) \right) = (r - g(\Phi_1(t; X_0))) \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) \]
and consequently
\[ \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) = \frac{e^{rt} - \int_0^t g(\Phi_1(s; X_0)) ds}{\partial x_1^2}(0; X_0). \]

Since \( \frac{\partial^2 \Phi_2}{\partial x_1^2}(0; X_0) = 0 \), this implies that
\[ \frac{\partial^2 \Phi_2}{\partial x_1^2}(t; X_0) = 0 \quad \text{for } t \geq 0. \]

Also,
\[ \frac{d}{dt} \left( \frac{\partial^2 \Phi_2}{\partial x_2^2}(t; X_0) \right) \]
\[ = (r - g(\Phi_1(t; X_0))) \frac{\partial^2 \Phi_2}{\partial x_2^2}(t; X_0) - \frac{\partial \Phi_1}{\partial x_2}(t; X_0) \frac{\partial \Phi_2}{\partial x_2}(t; X_0) \]
and since
\[ \frac{\partial \Phi_2}{\partial x_2}(t; X_0) = 0, \]
one may deduce that
\[ \frac{\partial^2 \Phi_2}{\partial x_2^2}(0; X_0) = 0, \]

\[ (6.14) \frac{\partial^2 \Phi_2}{\partial x_2^2}(t; X_0) \]
\[ = -e^{rt} \int_0^t g(\Phi_1(s; X_0)) ds \int_0^s \frac{\partial \Phi_1}{\partial x_2}(s; X_0) \frac{\partial \Phi_2}{\partial x_2}(s; X_0) e^{-(r\tau - \int_0^\tau g(\Phi_1(\tau; X_0)) d\tau)} d\tau ds \]
\[ = -e^{rt} \int_0^t g(\Phi_1(s; X_0)) ds \int_0^s \frac{\partial \Phi_1}{\partial x_2}(s; X_0) ds. \]
Similarly, 
\[ \frac{d}{dt} \left( \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} (t; X_0) \right) \]
\[ = (r - g(\Phi_1(t; X_0))) \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} (t; X_0) - g'(\Phi_1(t; X_0)) \frac{\partial \Phi_1}{\partial x_1} (t; X_0) \frac{\partial \Phi_2}{\partial x_2} (t; X_0) \]
and since 
\[ \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} (0; X_0) = 0, \]
one obtains that 
\[ \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} (t; X_0) = 0, \]

(6.15) \[ \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} (t; X_0) \]
\[ = -e^{rt - \int_0^t g(\Phi_1(s; X_0)) ds} \int_0^t g'(\Phi_1(s; X_0)) \frac{\partial \Phi_1}{\partial x_1} (s; X_0) \frac{\partial \Phi_2}{\partial x_2} (s; X_0) e^{-(t - s) \int_0^s g(\Phi_1(\tau; X_0)) d\tau} ds \]
\[ = -e^{rt - \int_0^t g(\Phi_1(s; X_0)) ds} \int_0^t g'(\Phi_1(s; X_0)) \frac{\partial \Phi_1}{\partial x_1} (s; X_0) ds. \]

6.5 The second order partial derivatives of \( f \)

One remarks that 
\[ \frac{\partial^2 \Phi_2}{\partial \tau \partial x_1} ((1 - l)T; I_1(\Phi(lT; X_0)) = 0 \]
(6.16) 
\[ \frac{\partial^2 \Phi_2}{\partial x_1^2} ((1 - l)T; I_1(\Phi(lT; X_0)) = 0 \]
(6.17) 
\[ \frac{\partial^2 \Phi_2}{\partial x_2^2} (lT; X_0) = 0. \]
(6.18)

By (6.16)-(6.18), combined with (6.10)-(6.13), it follows that 
\[ \frac{\partial^2 f}{\partial \tau^2} (0, 0) = -\frac{\partial^2 \Phi_2}{\partial \tau^2} ((1 - l)T; I_1(\Phi(lT; X_0)))(1 - l)^2. \]

Since 
\[ \frac{\partial^2 \Phi}{\partial \tau^2} ((1 - l)T; I_1(\Phi(lT; X_0)) = 0, \]
(6.19) 
it is then concluded that 
\[ \frac{\partial^2 f}{\partial \tau^2} (0, 0) = 0. \]
We then compute $\frac{\partial^2 f}{\partial \alpha^2}(0,0)$. By (6.10) and (6.12), it follows that

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) = - \frac{\partial}{\partial \alpha} \left[ \frac{\partial \Phi_2}{\partial x_1} ((1-l)(T + \tau); I_1(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0)) \right] \bigg|_{(\tau, \alpha) = (0,0)}$$

$$\cdot (1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial x_1} (lt; X_0) \left( - \frac{b'_{\alpha}}{a'_0} + \frac{\partial z}{\partial \alpha}(0,0) \right) + \frac{\partial \Phi_1}{\partial x_2} (lt; X_0) \right)$$

$$- \frac{\partial}{\partial \alpha} \left[ \frac{\partial \Phi_2}{\partial x_2} ((1-l)(T + \tau); I_1(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0)) \right] \bigg|_{(\tau, \alpha) = (0,0)}$$

$$\cdot (1 - \delta_1) \left( \frac{\partial \Phi_2}{\partial x_1} (lt; X_0) \left( - \frac{b'_{\alpha}}{a'_0} + \frac{\partial z}{\partial \alpha}(0,0) \right) + \frac{\partial \Phi_2}{\partial x_2} (lt; X_0) \right)$$

$$- \frac{\partial \Phi_2}{\partial x_2} ((1-l)T; I_1(l(T + \tau); X_0)) \cdot \frac{\partial}{\partial \alpha} \left[ (1 - \delta_1) \left( \frac{\partial \Phi_2}{\partial x_1} (lt(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0) \left( - \frac{b'_{\alpha}}{a'_0} + \frac{\partial z}{\partial \alpha}(\tau, \alpha) \right) \right) \right.$$}

$$+ \frac{\partial \Phi_2}{\partial x_2} (lt(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0) \bigg|_{(\tau, \alpha) = (0,0)} \right).$$

Using again (6.17) and (6.2), it follows that

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) = -2 \frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2} ((1-l)T; I_1(l(T + \tau); X_0))(1 - \delta_1)(1 - \delta_2)$$

$$\cdot \left( \frac{\partial \Phi_1}{\partial x_1} (lt; X_0) \left( - \frac{b'_{\alpha}}{a'_0} \right) + \frac{\partial \Phi_1}{\partial x_2} (lt; X_0) \right) \frac{\partial \Phi_2}{\partial x_2} (lt; X_0)$$

$$- \frac{\partial^2 \Phi_2}{\partial x_2^2} ((1-l)T; I_1(l(T + \tau); X_0))(1 - \delta_1)^2 \left( \frac{\partial \Phi_2}{\partial x_2} (lt; X_0) \right)^2$$

$$- \frac{\partial \Phi_2}{\partial x_2} ((1-l)T; I_1(l(T + \tau); X_0))(1 - \delta_1)$$

$$\cdot \left[ 2 \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1} (lt; X_0) \left( - \frac{b'_{\alpha}}{a'_0} \right) + \frac{\partial^2 \Phi_2}{\partial x_2^2} (lt; X_0) \right].$$

Consequently, from (6.14),(6.15),(4.1) and (4.2) one easily gets that

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) > 0.$$

From (6.10),(6.11) and (6.12), one may see that

$$\frac{\partial^2 f}{\partial \alpha \partial \tau}(0,0) = - \frac{\partial}{\partial \alpha} \left[ \frac{\partial \Phi_2}{\partial \tau} ((1-l)(T + \tau); I_1(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0)) \right] \bigg|_{(\tau, \alpha) = (0,0)}$$

$$\cdot (1 - l)$$

$$- \frac{\partial}{\partial \alpha} \left[ \frac{\partial \Phi_2}{\partial x_1} ((1-l)(T + \tau); I_1(l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0)) \right] \bigg|_{(\tau, \alpha) = (0,0)}$$

$$\cdot (1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \tau} (lt; X_0) \cdot \frac{\partial \Phi_1}{\partial x_1} (lt; X_0) \frac{\partial z}{\partial \tau}(0,0) \right)$$
Using again (6.16) and (6.18), one sees that
\[ \left. \left( \partial \Phi_2 \right) \partial x_1 \right|_{(\tau, \alpha) = (0, 0)} \]

\[ = -(1 - \delta_1) \left( \partial \Phi_2 \right) \partial x_2 \partial \tau \]

\[ \cdot (1 - \delta_1) \frac{\partial \Phi_2}{\partial x_2} (l(T + \tau); X_0 + \alpha Y_0 + z(\tau, \alpha) E_0) \cdot l - \left. \left( \partial \Phi_2 \right) \partial x_1 \right|_{(\tau, \alpha) = (0, 0)} \cdot l + \frac{\partial \Phi_2}{\partial x_1} (l(T; X_0) \frac{\partial z}{\partial \tau} (0, 0)) \]

Using again (6.16) and (6.18), one sees that
\[ \frac{\partial^2 f}{\partial \alpha \partial \tau} (0, 0) = - \frac{\partial^2 \Phi_2}{\partial x_2 \partial \tau} ((1 - l)T; I_1(\Phi(lT; X_0))) (1 - \delta_1) \frac{\partial \Phi_2}{\partial x_2} (l(T; X_0) (1 - l) - \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1} ((1 - l)T; I_1(\Phi(lT; X_0))) (1 - \delta_1) \frac{\partial \Phi_2}{\partial x_2} (l(T; X_0)) \cdot (1 - \delta_2) \left( \frac{\partial \Phi_1}{\partial \tau} (l(T; X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1} (l(T; X_0) \frac{\partial z}{\partial \tau} (0, 0)) \right) - \frac{\partial \Phi_2}{\partial x_2} ((1 - l)T; I_1(\Phi(lT; X_0))) \cdot (1 - \delta_1) \left( \frac{\partial^2 \Phi_2}{\partial x_2 \partial \tau} ((l(T; X_0)) \cdot l + \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1} (l(T; X_0) \frac{\partial z}{\partial \tau} (0, 0)) \right). \]

We now determine the sign of \( \frac{\partial^2 f}{\partial \alpha \partial \tau} (0, 0) \). It is seen that
\[ = e^{r(1 - l)T - J_0 (1 - l)T} \Phi_1 (s; I_1(\Phi(lT; X_0))) ds \cdot \left( \int_0^{(1 - l)T} g' (\Phi_1(s; I_1(\Phi(lT; X_0))) \right) e^{-w s} ds \]

\[ \cdot (1 - \delta_1) e^{rT - J_0 T} \Phi_1 (s; X_0) ds \]

\[ = e^{rT - J_0 T} g(I^*(s + lT) ds - \int_0^{(1 - l)T} g'(I^*(s)) ds (1 - \delta_1) \left( \int_0^{(1 - l)T} g'(I^*(s + lT)) e^{-w s} ds \right) \]

\[ = e^{rT - J_0 T} g(I^*(s)) ds (1 - \delta_1) \left( \int_0^{(1 - l)T} g'(I^*(s + lT)) e^{-w s} ds \right). \]

Since
\[ \int_0^T g(I^*(s)) ds = rT + \ln(1 - \delta_1), \]

it follows that
\[ - \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1} ((1 - l)T; I_1(\Phi(lT; X_0))) (1 - \delta_1) \frac{\partial \Phi_2}{\partial x_2} (l(T; X_0) \]

\[ = \int_0^{(1 - l)T} g'(I^*(s + lT)) e^{-w s} ds. \]
Similarly,

\[-\frac{\partial^2 \Phi_2}{\partial x_2 \partial \tau} ((1-l)T; I_1(\Phi(lT; X_0))))(1-\delta_1) \frac{\partial \Phi_2}{\partial x_2} ((lT; X_0) (1-l) \]

\[= -(r - g(\Phi_1((1-l)T; I_1(\Phi(lT; X_0)))) \frac{\partial \Phi_2}{\partial x_2} ((lT; I_1(\Phi(lT; X_0))) \]

\[\cdot (1-\delta_1) \frac{\partial \Phi_2}{\partial x_2} (lT; X_0) (1-l) \]

\[= -(r - g(I^*(T))) (1-d'_0) (1-l) \]

\[= -(r - g(I^*(T))) (1-l). \]

Also,

\[(1-\delta_2) \left( \frac{\partial \Phi_1}{\partial \tau} (lT; X_0) \cdot l + \frac{\partial \Phi_1}{\partial x_1} (lT; X_0) \frac{\partial z}{\partial \tau} (0, 0) \right) \]

\[= (1-\delta_2) \left( -w I^*(lT) \cdot l + e^{-wlT} \left( \left( -\frac{1}{a'_0} \right) w I^*(T) \right) \right) \]

\[= -w (1-\delta_2) e^{-wlT} \left( I^*(0+) \cdot l + \frac{1}{a'_0} I^*(T) \right). \]

It is seen that

\[-\frac{\partial \Phi_2}{\partial x_2} ((1-l)T; I_1(\Phi(lT; X_0))))(1-\delta_1) \left[ \frac{\partial^2 \Phi_2}{\partial x_2 \partial \tau} (lT; X_0) \cdot l + \frac{\partial \Phi_2}{\partial x_2} (lT; X_0) \frac{\partial z}{\partial \tau} (0, 0) \right] \]

\[= -\frac{\partial \Phi_2}{\partial x_2} ((1-l)T; I_1(\Phi(lT; X_0))))(1-\delta_1) \]

\[\left[ (r - g(\Phi_1(lT; X_0))) \frac{\partial \Phi_2}{\partial x_2} (lT; X_0) \cdot l - \left( \frac{\partial \Phi_2}{\partial x_2} (lT; X_0) \int_0^{lT} g'(l^*(s)) e^{-ws} ds \right) \frac{\partial z}{\partial \tau} (0, 0) \right]. \]

Since \(d'_0 = 0\), it follows that

\[-\frac{\partial \Phi_2}{\partial x_2} ((1-l)T; I_1(\Phi(lT; X_0))))(1-\delta_1) \left[ \frac{\partial^2 \Phi_2}{\partial x_2 \partial \tau} (lT; X_0) \cdot l + \frac{\partial \Phi_2}{\partial x_2} (lT; X_0) \frac{\partial z}{\partial \tau} (0, 0) \right] \]

\[= -(r - g(I^*(lT))) \cdot l + \left( \int_0^{lT} g'(I^*(s)) e^{-ws} ds \right) \left( -\frac{1}{a'_0} w I^*(T) \right) \]

\[= - \left[ (r - g(I^*(lT))) \cdot l + \frac{w}{a'_0} \left( \int_0^{lT} g'(I^*(s)) e^{-ws} ds \right) I^*(T) \right]. \]

It is consequently deduced that

\[\frac{\partial^2 f}{\partial \alpha \partial \tau} (0, 0) = -(r - g(I^*(T))) (1-l) \]

\[+ \left( \int_0^{(1-l)T} g'(I^*(s + lT)) e^{-ws} ds \right) \left( -w (1-\delta_2) e^{-wlT} \left( I^*(0+) \cdot l + \frac{1}{a'_0} I^*(T) \right) \right) \]

\[= - \left[ (r - g(I^*(lT))) \cdot l + \frac{w}{a'_0} \left( \int_0^{lT} g'(I^*(s)) e^{-ws} ds \right) I^*(T) \right]. \]
\[
= - [r - l g(I^*(T)) \cdot (1 - l) g(I^*(T))]
- w \left( \int_0^{1-l} g'(I^*(s + lT)) e^{-w(s+lT)} ds \right) (1 - \delta_2) \left( I^*(0+) \cdot l + \frac{1}{a_0} I^*(T) \right)
- \frac{w}{a_0} \left( \int_0^{lT} g'(I^*(s)) e^{-w s} ds \right) I^*(T),
\]

which implies

\[\frac{\partial^2 f}{\partial \alpha \partial \tau} (0, 0) = - [r - l g(I^*(T)) \cdot (1 - l) g(I^*(T))]
- w \left( \int_0^{lT} g'(I^*(s)) e^{-w s} ds \right) (1 - \delta_2) \left( I^*(0+) \cdot l + \frac{1}{a_0} I^*(T) \right)
- \frac{w}{a_0} \left( \int_0^{lT} g'(I^*(s)) e^{-w s} ds \right) I^*(T).\]

We note that

\[r T - \int_0^T g(I^*(s)) ds = - \ln(1 - \delta_1) > 0\]

and also, since \(I^*\) is decreasing on \((0, T)\),

\[\int_0^T g(I^*(s)) ds = \int_0^{lT} g(I^*(s)) ds + \int_{lT}^T g(I^*(s)) ds > lT g(I^*(lT)) + (1 - l) T g(I^*(T)).\]

Consequently, the first term in the right-hand side of (6.20) is negative. Since \(g\) is increasing and \(I^*\) is positive, the other terms are negative as well and consequently

\[\frac{\partial^2 f}{\partial \alpha \partial \tau} (0, 0) < 0.\]

References


