Sparse and stable Markowitz portfolios

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Portfolio optimization

- N securities with (stationary) returns $r_{i\,t}$ at time t.
- $\mathbf{r}_t$ = $N \times 1$ vector of returns at time t.
- Vector of expected returns : $\boldsymbol{\mu} = \mathbf{E}[\mathbf{r}_t]$
  Covariance matrix of returns:

$$\mathbf{E}[(\mathbf{r}_t - \boldsymbol{\mu})(\mathbf{r}_t - \boldsymbol{\mu})^\top] = \boldsymbol{C}$$

- A portfolio is defined by a $N \times 1$ vector of weights w_i summing to one (unit of capital): $\mathbf{w}^\top \mathbf{1}_N = 1$, where $\mathbf{1}_N$ denotes the $N \times 1$ vector of ones.
- Expected return of the portfolio: $\mathbf{w}^\top \boldsymbol{\mu}$
  Variance of the portfolio: $\mathbf{w}^\top \boldsymbol{C}\,\mathbf{w}$

Markowitz portfolios

- Find a portfolio $\widetilde{\mathbf{w}}$ which has minimal variance for a given expected return $\rho = \mathbf{w}^\top \boldsymbol{\mu}$, i.e.

$$\begin{aligned} \widetilde{\mathbf{w}} \quad &= \quad \arg\min_{\mathbf{w}} \mathbf{w}^\top \boldsymbol{C}\, \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^\top \boldsymbol{\mu} = \rho \\ & \mathbf{w}^\top \mathbf{1}_N = 1 \end{aligned}$$

- Since $\boldsymbol{C} = \mathbf{E}[\mathbf{r}_t \mathbf{r}_t^\top] - \boldsymbol{\mu}\boldsymbol{\mu}^\top$, this is equivalent to

$$\begin{aligned} \widetilde{\mathbf{w}} \quad &= \quad \arg\min_{\mathbf{w}} \mathbf{E}\left[|\rho - \mathbf{w}^\top \mathbf{r}_t|^2 \right] \\ \text{s. t.} \quad & \mathbf{w}^\top \boldsymbol{\mu} = \rho \\ & \mathbf{w}^\top \mathbf{1}_N = 1 \end{aligned}$$

Our proposal

- For empirical implementation, replace expectations with sample averages and solve the following regression problem

$$\begin{aligned} \widehat{\mathbf{w}} &= \arg\min_{\mathbf{w}} \frac{1}{T} \|\rho \mathbf{1}_T - \boldsymbol{R}\mathbf{w}\|_2^2 \\ \text{s. t.} \quad & \mathbf{w}^\top \widehat{\boldsymbol{\mu}} = \rho \\ & \mathbf{w}^\top \mathbf{1}_N = 1, \end{aligned}$$

  where $\widehat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{r}_t$ and $\boldsymbol{R}$ is the $T \times N$ matrix of the available returns.

- Add a L_1-norm penalty to ensure sparsity and stability.

Our sparse and stable portfolios

- Find the weights

$$\mathbf{w}^{[\tau]} = \arg\min_{\mathbf{w}} \left[||\rho \mathbf{1}_T - \boldsymbol{R}\mathbf{w}||_2^2 + \tau ||\mathbf{w}||_1 \right] \quad (1)$$

$$\text{s. t.} \quad \mathbf{w}^\top \widehat{\boldsymbol{\mu}} = \rho \quad (2)$$

$$\mathbf{w}^\top \mathbf{1}_N = 1. \quad (3)$$

  where $||\mathbf{w}||_1 = \sum_i |w_i|$ and τ is a positive parameter tuning the balance between the two terms.

- "Lasso" regression (Tibshirani 1996), but with extra constraints.

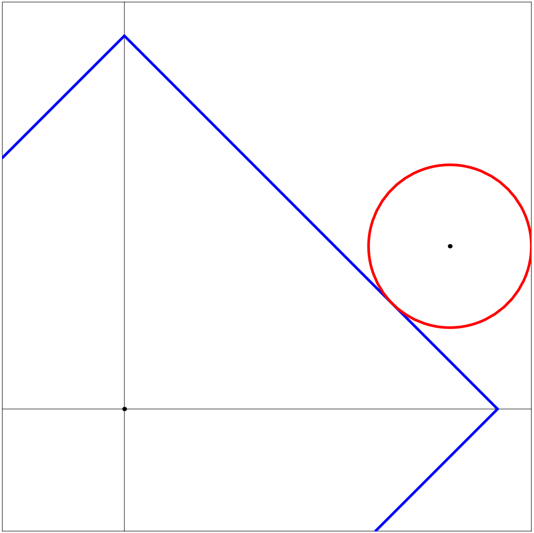
Why stability?

- We suspect ill-conditioning due to high collinearity is a major problem for practical implementations of the Markowitz framework and may also account for the following fact.
- It was recently shown that many portfolio constructions proposed in the literature fail to outperform the naive equally-weighted ("$1/N$") portfolio.
(DeMiguel, Garlappi and Uppal 2007)
- The L_1-norm penalty is known to regularize (stabilize) the problem.
(Daubechies, Defrise and De Mol 2004)

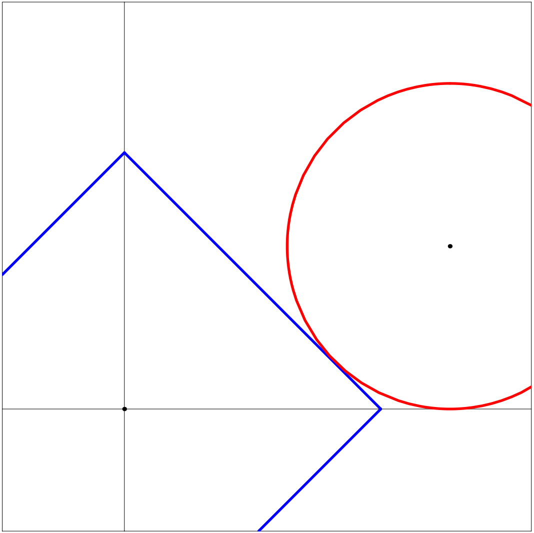
Sparsity

- The L_1-norm penalty enforces sparsity of the portfolio, i.e. favors the presence of many zero weights ($\leftrightarrow$ few active assets).
- This allows for variable (asset) selection.
- Why?

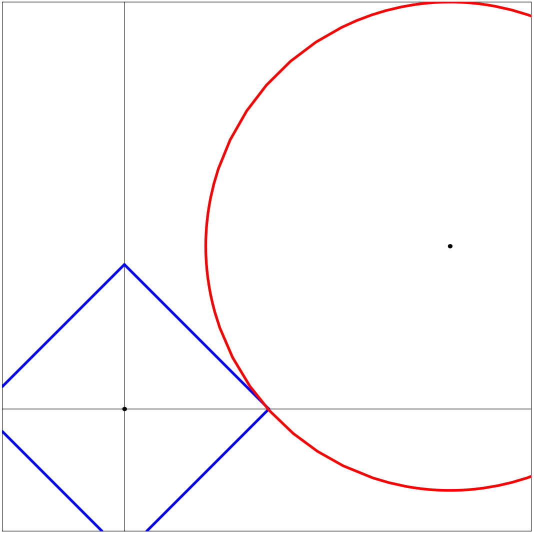
Lasso regression and sparsity



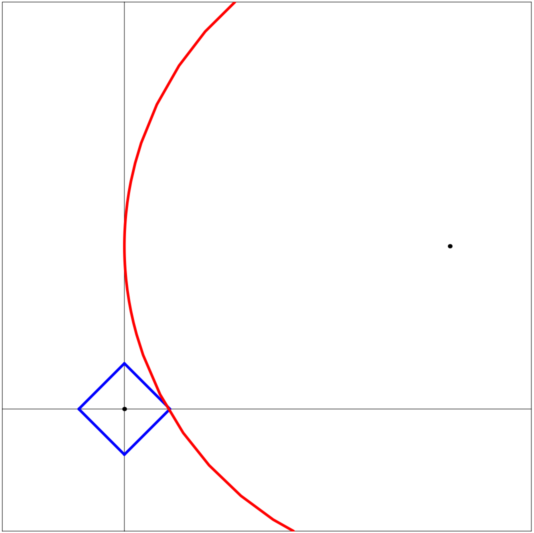
Lasso regression and sparsity



Lasso regression and sparsity



Lasso regression and sparsity



Why sparsity?

- Difficulty of managing several hundreds of assets.
- The traditional Markowitz framework does not take into account transaction and monitoring costs.
- The L_1-norm penalty allows to cope with the transaction costs, modelling linear transaction costs and/or serving as a proxy to the L_0-norm ($\leftrightarrow$ sparsity, to keep control on the fixed costs)

Constrained LARS Algorithm

- The recursive LARS/homotopy algorithm allows to compute efficiently the Lasso regression solutions for all values of τ, starting from the largest ones.
  (Osborne et al. 2000, Efron et al. 2004)

- We devised a modification of LARS able to enforce the linear constraints.

- Varying τ allows to tune the number of active positions.

Special case: No-short portfolios

- Notice $\sum_i w_i = 1 \implies \sum_i |w_i| \geq 1$.
- Limit case for τ large: $\sum_i |w_i| = 1 \iff w_i \geq 0$ for all i (no short positions).
- Positive portfolios known for their good performances (Jagannathan and Ma 2003) .
- But the fact that no-short portfolios are sparse seems to have gone unnoticed !

Empirical application

- We used as assets the Fama and French 48 industry portfolios (FF48) and 100 portfolios formed on size and book-to-market (FF100).
- We constructed our portfolios in June of each year from 1976 to 2006 using 5 years of historical (monthly) returns and a target return equal to the historical return of the equally-weighted portfolio.
- Performance is evaluated by out-of-sample monthly mean return m, standard deviation σ and Sharpe ratio $S = m/\sigma$.
- Benchmark (tough!) is the evenly-weighted portfolio.

Empirical results

Performance of sparse portfolio with no short-selling (FF48)

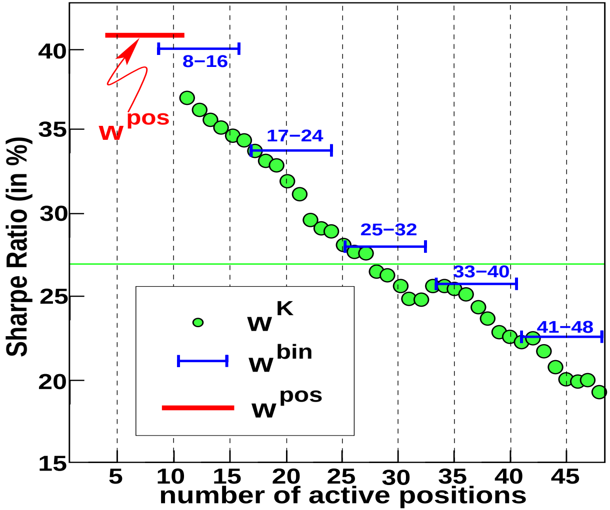
Evaluation period	$w_i \geqslant 0$ for all i			Equal weight		
	m	σ	S	m	σ	S
06/76-06/06	17	41	41	17	61	27
07/76-06/81	23	48	49	29	66	44
07/81-06/86	23	41	57	18	58	31
07/86-06/91	9	45	20	5	72	7
07/91-06/96	16	26	62	18	41	44
07/96-06/01	16	40	40	11	67	17
07/01-06/06	13	43	30	18	60	30

Empirical results

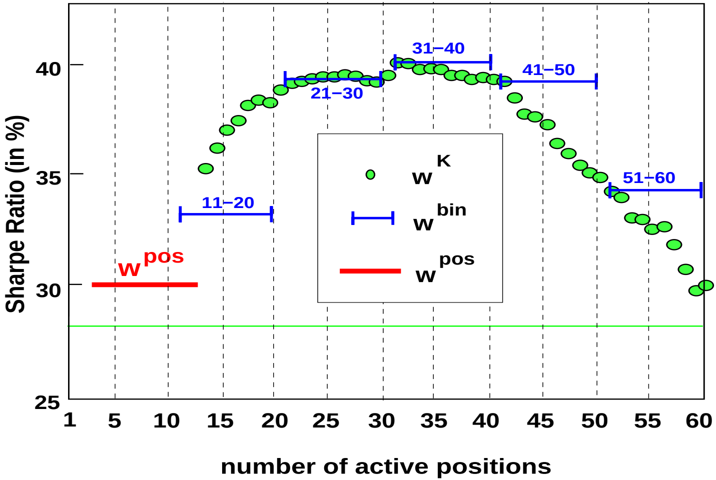
Performance of sparse portfolio with no short-selling (FF100)

Evaluation period	$w_i \geqslant 0$ for all i			Equal weight		
	m	σ	S	m	σ	S
06/76-06/06	16	53	30	17	59	28
07/76-06/81	12	59	21	23	61	38
07/81-06/86	24	49	49	20	53	38
07/86-06/91	10	65	15	9	71	13
07/91-06/96	19	31	61	18	34	53
07/96-06/01	18	52	35	16	63	26
07/01-06/06	11	55	21	12	64	19

Empirical results FF48



Empirical results FF100



Generalizations

- Weighted L_1-norm penalty: $\sum_i s_i |w_i|$.
- Index tracking (replace target return ρ by index y_t).
- Portfolio Adjustment (Rebalancing).

Full Paper

- Sparse and stable Markowitz portfolios

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  Working paper (July 2007), ECORE DP2007/61, CEPR DP6474,
  Working Paper European Central Bank No 936 (2008)

  Downloadable from http://arxiv.org/abs/0708.0046

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