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$rg\alpha$-Closed Sets and $rg\alpha$-Open Sets in Topological Spaces

A. Vadivel and K. Vairamanickam

Department of Mathematics, Annamalai University
Annamalainagar-608 002, India
avmaths@gmail.com
kvaufeat@gmail.com

Abstract

In this paper, we introduce $rg\alpha$-closed sets and $rg\alpha$-open sets and some of its basic properties.

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1 Introduction

N. Levine [14] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Balachandran, Sundaram and Maki [5], Bhattacharyya and Lahiri [6], Arockiarani [2], Dunham [11], Gnanambal [12], Malghan [18], Palaniappan and Rao [23], Park [24] Arya and Gupta [3] and Devi [8], Benchalli and wali [29] have worked on generalized closed sets, their generalizations and related concepts in general topology. In this paper, we define and study the properties of regular generalized α-closed sets (briefly $rg\alpha$-closed). Moreover, in this paper we define $rg\alpha$-open sets and obtained some of its basic properties as results.

Throughtout the paper X and Y denote the topological spaces (X, τ) and (Y, σ) respectively and on which no seperation axioms are assumed unless otherwise explicitely stated. For any subset A of a space (X, τ), the closure of A, interior of A, semi-interior of A, semi-closure of A, w-interior of A, w-closure of A, gpr-interior of A, gpr-closure of A, α-closure of A, α-interior of A and the complement of A are denoted by $cl\,(A)$ or τ-$cl\,(A)$, $int\,(A)$ or

τ-*int* (A), *sint* (A), *scl* (A), *w-int* (A), *w-cl* (A), *gpr-int* (A), *gpr-cl* (A), α-*int* (A), α-*cl* (A) and A^C or $X - A$ respectively. (X, τ) will be replaced by X if there is no chance of confusion.

Let us recall the following definitions as pre requesties.

Definition 1.1. *A subset A of a space X is called*

1) a preopen set [20] if $A \subseteq intcl\,(A)$ and a preclosed set if $clint\,(A) \subseteq A$.

2) a semiopen set [13] if $A \subseteq clint\,(A)$ and a semiclosed set if $intcl\,(A) \subseteq A$.

3) a α-open set [22] if $A \subseteq intclint\,(A)$ and a α-closed set if $clintcl\,(A) \subseteq A$.

4) a semi-preopen set [1] if $A \subseteq clintcl\,(A)$ and a semi-preclosed set if $intclint\,(A) \subseteq A$.

5) a regular open set [28] if $A = intcl\,(A)$ and a regular closed set if $A = clint\,(A)$.

The intersection of all semiclosed (resp-semiopen) subsets of X containing A is called the semi-closure (resp. semi-kernal) of$A$ and is denoted by $scl(A)$(resp. $sker(A)$). Also the intersection of all preclosed (resp. semi-preclosed and α-closed) subsets of X containing A is called pre-closure (resp. semi-pre closure and α-closure) of A and is denoted by $pcl(A)$(resp. $spcl(A)$ and α-$cl\,(A)$).

Definition 1.2. *A subset A of a space X is called*

1) generalized closed set (briefly, g-closed)[14] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

2) semi-generalized closed set (briefly, sg-closed)[6] if $scl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

3) generalized semiclosed set (briefly, gs-closed)[4] if $scl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

4) generalized α-closed set (briefly, gα-closed)[16] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.

5) α-generalized closed set (briefly, αg-closed)[15] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

6) generalized semi-preclosed set (briefly, gsp-closed)[9] if $spcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

7) regular generalized closed set (briefly, rg-closed)[23] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

8) generalized preclosed set (briefly, gp-closed)[17] if $pcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

9) generalized preregular closed set (briefly, gpr-closed)[12] if $pcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

10) weakely generalized closed set (briefly, wg-closed)[21] if $clint\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

11) strongly generalized semi-closed set [25] (briefly, g^*-closed) if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.

12) π-generalized closed set (briefly, πg-closed)[10] if $cl\,(A) \subseteq U$ whenever

$A \subseteq U$ and U is π-open in X.
13) weakely closed set (briefly, w-closed)[27] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.
14) mildly generalized closed set (briefly, mildly g-closed)[24] if $clint(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.
15) semi weakly generalized closed set (briefly, swg-closed) if $clint(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.
16) regular weakly generalized closed set (briefly, rwg-closed) if $clint(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.
17) A subset A of a space X is called regular semiopen[7] if there is a regular open U such $U \subset A \subset cl(U)$. The family of all regular semiopen sets of X is denoted by $RSO(X)$.

The complements of the above mentioned closed sets are their respective open sets.

2 $rg\alpha$-closed sets and their basic properties.

We introduce the following definition

Definition 2.1. *A subset A of a space X is called regular α-open set (briefly, $r\alpha$-open) if there is a regular open set U such that $U \subset A \subset \alpha cl(U)$.*

The family of all regular α-open sets of X is denoted by $R\alpha O(X)$.

Definition 2.2. *A subset A of a space X is called a regular generalized α-closed set (briefly, $rg\alpha$-closed) if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is regular α-open in X. We denote the set of all $rg\alpha$-closed sets in X by $RG\alpha C(X)$.*

First we prove that the class of $rg\alpha$-closed sets has properly lies between the class of $g\alpha$-closed sets and the class of regular generalized closed sets.

Theorem 2.1. *Every $g\alpha$-closed set in X is $rg\alpha$-closed set in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular open sets are regular α-open. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 2.1. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{a, d, e\}$ is $rg\alpha$-closed set but not $g\alpha$-closed set in X.*

Theorem 2.2. *Every w-closed set in X is $rg\alpha$-closed set in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular α-open set is semiopen and closed sets are α-closed. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 2.2. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{b\}$ is $rg\alpha$-closed set but not w-closed set in X.*

Theorem 2.3. *Every rw-closed set in X is $rg\alpha$-closed set in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that closed sets are α-closed. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 2.3. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{b\}$ is $rg\alpha$-closed set but not rw-closed set in X.*

Theorem 2.4. *Every $rg\alpha$-closed set in X is rg-closed set in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular open sets are regular α-open. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 2.4. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{a, b\}$ is rg-closed set but not $rg\alpha$-closed set in X.*

Corollary 2.1. *Every closed set is $rg\alpha$-closed but not conversely.*

Proof. Follows from sheik John [26] and theorem 2.3.

Corollary 2.2. *Every regular closed set is $rg\alpha$-closed but not conversely.*

Proof. Follows from stone [19] and corollary 2.1.

Corollary 2.3. *Every $rg\alpha$-closed set is a gpr-closed but not conversely.*

Proof. Follows from Gnanmbal [12] and theorem 2.4.

Corollary 2.4. *Every π-closed set is a $rg\alpha$-closed set but not conversely.*

Proof. Follows from [10] and corollary 2.1.

Theorem 2.5. *Every $rg\alpha$-closed set in X is rwg-closed set in X, but not conversely.*

Proof. The proof follows from the definitions and the fact that every regular open sets are regular α-open. ∎

The converse of the above theorem need not be true, as seen from the following example.

Example 2.5. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{a, b\}$ is rwg-closed set but not $rg\alpha$-closed set in X.*

Remark 2.1. *From the above discussions and known results we have the following implications*
In the following diagram, by
$A \rightarrow B$ we mean A implies B but not conversely and
$A \nleftrightarrow B$ means A and B are independent of each other.

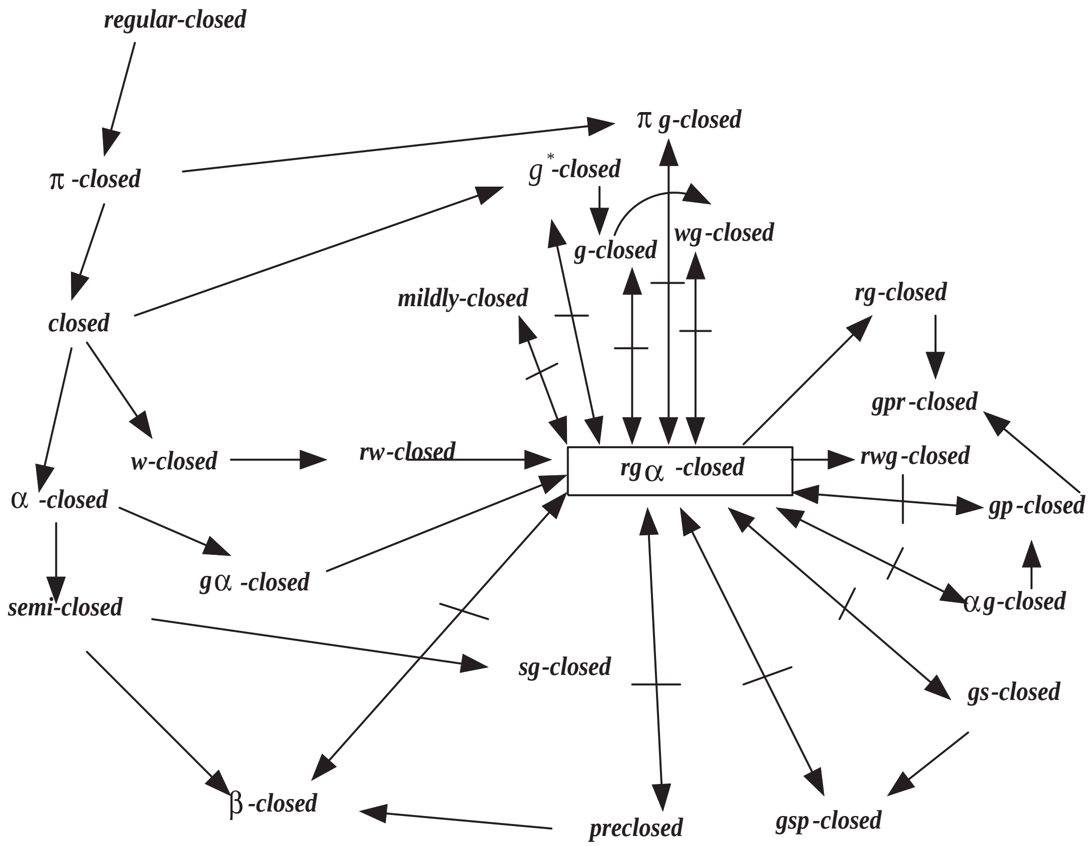

fig-1

Theorem 2.6. *The union of two $rg\alpha$-closed subsets of X is also $rg\alpha$-closed subset of X.*

Proof. Assume that A and B are $rg\alpha$-closed set in X. Let U be regular α-open in X such that $A \cup B \subset U$. Then $A \subset U$ and $B \subset U$. Since A and B are $rg\alpha$-closed, $\alpha cl(A) \subset U$ and $\alpha cl(B) \subset U$. Hence $\alpha cl\,(A \cup B) = (\alpha cl(A)) \cup (\alpha cl(B)) \subset U$. That is $\alpha cl\,(A \cup B) \subset U$. Therefore $A \cup B$ is $rg\alpha$-closed set in X. ■

Remark 2.2. *The intersection of two $rg\alpha$-closed sets in X is generally not $rg\alpha$-closed set in X.*

Example 2.6. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. If $A = \{a, b, c\}$ and $B = \{a, d, e\}$ Then A and B are $rg\alpha$-closed sets in X, but $A \cap B = \{a\}$ is not a $rg\alpha$-closed set in X.*

Theorem 2.7. *If a subset A of X is $rg\alpha$-closed set in X. Then $\alpha cl(A)\backslash A$ does not contain any nonempty regular α-open set in X.*

proof. Suppose that A is $rg\alpha$-closed set in X. We prove the result by contradiction. Let U be a regular α-open set such that $\alpha cl(A)\backslash A \supset U$ and $U \neq \phi$. Now $U \subset \alpha cl\,(A)\,\backslash A$. Therefore $U \subset X\backslash A$ which implies $A \subset X\backslash U$. Since U is regular α-open set, $X\backslash U$ is also regular α-open in X. Since A is $rg\alpha$-closed set in X, by definition we have $\alpha cl(A) \subset X\backslash U$. So $U \subset X\backslash \alpha cl\,(A)$. Also $U \subset \alpha cl\,(A)$. Therefore $U \subset (\alpha cl(A) \cap (X\backslash \alpha cl(A))) = \phi$. This shows that, $U = \phi$ which is contradiction. Hence $\alpha cl(A)\backslash A$ does not contains any non-empty regular α-open set in X. ■

The converse of the above theorem need not be true seen from following example.

Example 2.7. *If $\alpha cl(A)\backslash A$ contains no non-empty $rg\alpha$-open subset in X, Then A need not be $rg\alpha$-closed set. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$ and $A = \{a, b\}$. Then $\alpha cl(A)\backslash A = \{a, b, c\} \backslash \{a, b\} = \{c\}$ does not contain non-empty regular α-open set in X, but A is not a $rg\alpha$-closed set in X.*

Corollary 2.5. *If a subset A of X is $rg\alpha$-closed set in X, then $\alpha cl(A)\backslash A$ does not contain any regular open set in X, but not conversely.*

Proof. Follows from thorem 2.7. and the fact that every regular open set is regular α-open.

Corollary 2.6. *If a subset A of X is $rg\alpha$-closed set in X, then $\alpha cl(A)\backslash A$ does not contain any non-empty regular closed set in X, but not conversely.*

Proof. Follows from thorem 2.7. and the fact that every regular open set is regular α-open.

Theorem 2.8. *For an element $x \in X$, the set $X\backslash \{x\}$ is $rg\alpha$-closed or regular α-open.*

proof. Suppose $X\backslash \{x\}$ is not regular α-open set. Then X is the only regular α-open set containing $X\backslash \{x\}$. This implies $\alpha cl(X\backslash \{x\}) \subset X$. Hence $X\backslash \{x\}$ is $rg\alpha$-closed set in X. ■

Theorem 2.9. *If A is regular open and $rg\alpha$-closed then A is regular closed and hence α-clopen.*

proof. Suppose A is regular open and $rg\alpha$ -closed. As every regular open is regular α-open and $A \subset A$, we have $\alpha cl(A) \subset A$. Also $A \subset \alpha cl A$. Therefore $\alpha cl A = A$. That is A is α-closed. Since A is regular open, A is α-open. Now $cl(int(A)) = cl(A) = A$. Therefore A is a regular closed and α-clopen. ■

Theorem 2.10. *If A is $rg\alpha$-closed subset of X such that $A \subset B \subset \alpha cl(A)$. Then B is $rg\alpha$-closed set in X.*

Proof. If A is $rg\alpha$-closed subset of X such that $A \subset B \subset \alpha cl(A)$. Let U be a regular α-open set of X such that $B \subset U$. Then $A \subset U$. Since A is a $rg\alpha$-closed we have $\alpha cl(A) \subset U$. Now $\alpha cl(B) \subset \alpha cl\left(\alpha cl(A)\right) = \alpha cl(A) \subset U$. Therefore B is $rg\alpha$-closed set in X. ■

Remark 2.3. *The converse of the theorem 2.10. need not be true in general. Consider the topological space (X, τ), where $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Let $A = \{b\}$ and $B = \{b, c\}$. Then A and B are $rg\alpha$-closed sets in (X, τ), but $A \subset B$ is not subset in $\alpha cl(A)$.*

Theorem 2.11. *Let A be a $rg\alpha$-closed in (X, τ). Then A is α-closed if and only if $\alpha cl(A)\backslash A$ is a regular α-open.*

Proof. Suppose A is a α-closed in X. Then $\alpha cl(A) = A$ and so $\alpha cl(A)\backslash A = \phi$, which is regular α-open in X. Conversely, suppose $\alpha cl(A)\backslash A$ is a regular α-open set in X. Since A is $rg\alpha$-closed, by theorem 2.7. $\alpha cl(A)\backslash A$ does not contain any nonempty regular α-open in X. Then $\alpha cl(A)\backslash A = \phi$, hence A is α-closed set in X. ■

Theorem 2.12. *If A is regular open and rg-closed, then A is $rg\alpha$-closed set in X.*

Proof. Let U be any regular α-open set in X such that $A \subset U$. Since A is regular open and rg-closed, we have $\alpha cl(A) \subset A$. Then $\alpha cl(A) \subset A \subset U$. Hence A is $rg\alpha$-closed set in X. ■

Theorem 2.13. *If a subset A of topological space X is both regualr α-open and $rg\alpha$-closed, then it is α-closed.*

Proof. Suppose a subset A of topological space X is both regualr α-open and $rg\alpha$-closed. Now $A \subset A$. Then $\alpha cl(A) \subset A$. Hence A is α-closed. ■

Corollary 2.7. *Let A be regular α-open and $rg\alpha$-closed subset in X. Suppose that F is α-closed set in X. Then $A \cap F$ is an $rg\alpha$-closed set in X.*

Proof. Let A be a regular α-open and $rg\alpha$-closed subset in X and F be closed. By theorem 2.13., A is α-closed. So $A \cap F$ is a α-closed and hence $A \cap F$ is $rg\alpha$-closed set in X.

Theorem 2.14. *If A is an open and S is α-open in topological space X, then $A \cap S$ is α-open in X.*

Theorem 2.15. *If A is both open and g-closed set in X, then it is $rg\alpha$-closed set in X.*

Proof. Let A be an open and g-closed set in X. Let $A \subset U$ and let U be a regular α-open set in X. Now $A \subset A$. By hypothesis $\alpha cl(A) \subset A$. That is $\alpha cl(A) \subset U$. Thus A is $rg\alpha$-closed in X. ■

Remark 2.4. *If A is both open and $rg\alpha$-closed in X, then A need not be g-closed, in general, as seen from the following example.*

Example 2.8. *Consider $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, d, e\}$ is an open and $rg\alpha$-closed set, but not g-closed.*

Theorem 2.16. *In a topological space X, if $R\alpha O(X) = \{X, \phi\}$, then every subset of X is a $rg\alpha$-closed set.*

Proof. Let X be a topological space and $R\alpha O(X) = \{X, \phi\}$. Let A be any subset of X. Suppose $A = \phi$. Then ϕ is $rg\alpha$-closed set in X. Suppose $A \neq \phi$. Then X is the only regular α-open set containing A and so $\alpha cl(A) \subset X$. Hence A is $rg\alpha$-closed set in X. ■

The converse of the theorem 2.16. need not be true in general as seen from the following example.

Example 2.9. *Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then every subset of (X, τ) is $rg\alpha$-closed set in X, But $R\alpha O(X, \tau) = \{X, \phi, \{a, b\}, \{c, d\}\}$.*

Theorem 2.17. *In a topological space X, $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$ iff every subset of X is a $rg\alpha$-closed set.*

Proof. Suppose that $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. Let A be any subset of X such that $A \subset U$, where U is a regular α-open. Then $U \in R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. That is $U \in \{F \subset X : F^c \in \tau\}$. Thus $U$is a α-closed set. Then $\alpha cl(U) = U$. Also $\alpha cl(A) \subset \alpha cl(U) = U$. Hence A is $rg\alpha$-closed set in X.

Conversely, suppose that every subset of (X, τ) is $rg\alpha$-closed. Let $U \in R\alpha O(X, \tau)$. Since $U \subset U$and $U$is $rg\alpha$-closed, we have $\alpha cl(U) \subset U$. Thus $\alpha cl(U) = U$ and $U \in \{F \subset X : F^c \in \tau\}$. Therefore $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. ■

Definition 2.3. *The intersection of all regular α-open subsets of (X, τ) containing A is called the regular α-kernal of A and is denoted by $r\alpha ker(A)$.*

Lemma 2.1. *Let X be a topological space and A be a subset of X. If A is a regular α-open in X, then $r\alpha ker\,(A) = A$ but not conversely.*

Proof. Follows from definition. 2.3. ■

Lemma 2.2. *For any subset A of X, $\alpha ker\,(A) \subset r\alpha ker\,(A)$.*

Proof. Follows from the implication $R\alpha O\,(X) \subset \alpha O\,(X)$. ■

Lemma 2.3. *For any subset A of X, $A \subset r\alpha ker\,(A)$.*

Proof. Follows from the definition. 2.3. ■

3 $rg\alpha$-open sets and $rg\alpha$-neighbourhoods.

In this section, we introduce and study $rg\alpha$-open sets in topological spaces and obtain some of their properties. Also, we introduce $rg\alpha$-neighbourhood (shortly $rg\alpha$-nbhd in topological spaces by using the notion of $rg\alpha$-open sets. We prove that every nbhd of x in X is $rg\alpha$-nbhd of x but not conversely.

Definition 3.1. *A subset A in X is called regular generalized α-open (briefly, $rg\alpha$-open) in X if A^c is $rg\alpha$-closed in X. We denote the family of all $rg\alpha$-open sets in X by $RG\alpha O\,(X)$.*

Theorem 3.1. *If a subset A of a space X is w-open then it is $rg\alpha$-open but not conversely.*

Proof. Let A be a w-open set in a space X. Then A^c is w-closed set. By theorem 2.2. A^c is $rg\alpha$-closed. Therefore A is $rg\alpha$-open set in X. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.1. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{b\}$ is $rg\alpha$-open but not w-open.*

Corollary 3.1. *Every open set is $rg\alpha$-open set but not conversely.*

Proof. Follows from sheik john [26] and theorem 3.1.

Corollary 3.2. *Every regular open set is $rg\alpha$-open set but not conversely.*

Proof. Follows from stone [28] and theorem 3.1.

Theorem 3.2. *If a subset A of a space X is $rg\alpha$-open, then it is rg-open set in X.*

Proof. Let A be $rg\alpha$-open set in space X. Then A^c is $rg\alpha$-closed set in X. By theorem 2.4., A^c is rg-closed set in X. Therefore A is rg-open set in space X. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.2. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is rg-open but not $rg\alpha$-open set in X.*

Theorem 3.3. *If a subset A of a space X is $rg\alpha$-open, then it is gpr-open set in X, but not conversely.*

Proof. Let A be $rg\alpha$-open set in a space X. Then A^c is $rg\alpha$-closed set in X. By corollary 2.3. A^c is gpr-closed in X. Therefore A is gpr-open set in X. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.3. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is gpr-open but not $rg\alpha$-open.*

Theorem 3.4. *If a subset A of a topological space X is $rg\alpha$-open, then it is rwg-open set in X, but not conversely.*

Proof. Let A be $rg\alpha$-open set in a space X. Then A^c is $rg\alpha$-closed set in X. By theorem 2.5. A^c is rwg-closed in X. Therefore A is rwg-open subset in X. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.4. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is rwg-open set in X, but not $rg\alpha$-open.*

Theorem 3.5. *If A and B are $rg\alpha$-open sets in a space X. Then $A \cap B$ is also $rg\alpha$-open set in X.*

Proof. If A and B are $rg\alpha$-open sets in a space X. Then A^c and B^c are $rg\alpha$-closed sets in a space X. By theorem 2.6. $A^c \cup B^c$ is also $rg\alpha$-closed set in X. That is $A^c \cup B^c = (A \cap B)^c$ is a $rg\alpha$-closed set in X. Therefore $A \cap B$ $rg\alpha$-open set in X. ■

Remark 3.1. *The union of two $rg\alpha$-open sets in X is generally not a $rg\alpha$-open set in X.*

Example 3.5. *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. If $A = \{a\}$ and $B = \{c\}$, then A and B are $rg\alpha$-open sets in X, but $A \cup B = \{a, c\}$ is not a $rg\alpha$-open set in X.*

Theorem 3.6. *If a set A is $rg\alpha$-open in a space X, then $G = X$, whenever G is regular α-open and $int(A) \cup A^c \subset G$.*

Proof. Suppose that A is $rg\alpha$-open in X. Let G be regular α-open and $int(A) \cup A^c \subset G$. This implies $G^c \subset (int(A) \cup A^c)^c = (int(A))^c \cap A$. That is $G^c \subset (int(A))^c - A^c$, Thus $G^c \subset cl(A)^c - A^c$, Since $(int(A))^c = cl(A^c)$. Now G^c is also regular α-open and A^c is $rg\alpha$-closed, by theorem 2.7., it follows that $G^c = \phi$. Hence $G = X$. ■

The converse of the above theorem is not true in general as seen from the following example.

Example 3.6. *Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $R\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{b, c, d\}\}$. Take $A = \{b, c, d\}$. Then A is not $rg\alpha$-open. However $int(A) \cup A^c = \{b, c\} \cup \{a\} = \{a, b, c\}$. So for some regular α-open G, we have $int(A) \cup A^c = \{a, b, c\} \subset G$ gives $G = X$, but A is not $rg\alpha$-open.*

Theorem 3.7. *Every singleton point set in a space is either $rg\alpha$-open or $r\alpha$-open.*

Proof. Let X be a topological space. Let $x \in X$. To prove $\{x\}$ is either $rg\alpha$-open or $r\alpha$-open. That is to prove $X - \{x\}$ is either $rg\alpha$-closed or $r\alpha$-open, which follows from theorem 2.8. ■

Analogus to a neighbourhood in space X, we define $rg\alpha$-neighbourhood in a space X as follows.

Definition 3.2. *Let X be a topological space and let $x \in X$. A subset N of X is said to be a $rg\alpha$-nbhd of x iff there exists a $rg\alpha$-open set G such that $x \in G \subset N$.*

Definition 3.3. *A subset N of space X, is called a $rg\alpha$-nbhd of $A \subset X$ iff there exists a $rg\alpha$-open set G such that $A \subset G \subset N$.*

Remark 3.2. *The $rg\alpha$-nbhd N of $x \in X$ need not be a $rg\alpha$-open in X.*

Example 3.7. *Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. Note that $\{a, c\}$ is not a $rg\alpha$-open set, but it is a $rg\alpha$-nbhd of $\{a\}$. Since $\{a\}$ is a $rg\alpha$-open set such that $a \in \{a\} \subset \{a, c\}$.*

Theorem 3.8. *Every nbhd N of $x \in X$ is a $rg\alpha$-nbhd of X.*

Proof. Let N be a nbhd of point $x \in X$. To prove that N is a $rg\alpha$-nbhd of x. By definition of nbhd, there exists an open set G such that $x \in G \subset N$. As every open set is $rg\alpha$-open set G such that $x \in G \subset N$. Hence N is $rg\alpha$-nbhd of x. ■

Remark 3.3. *In general, a $rg\alpha$-nbhd N of $x \in X$ need not be a nbhd of x in X, as seen from the following example.*

Example 3.8.
Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. The set $\{a, c\}$ is $rg\alpha$-nbhd of the point c, since the $rg\alpha$-open sets $\{c\}$ is such that $c \in \{c\} \subset \{a, c\}$. However, the set $\{a, c\}$ is not a nbhd of the point c, since no open set G exists such that $c \in G \subset \{a, c\}$.

Theorem 3.9. *If a subset N of a space X is $rg\alpha$-open, then N is a $rg\alpha$-nbhd of each of its points.*

Proof. Suppose N is $rg\alpha$-open. Let $x \in N$. We claim that N is $rg\alpha$-nbhd of x. For N is a $rg\alpha$-open set such that $x \in N \subset N$. Since x is an arbitrary point of N, it follows that N is a $rg\alpha$-nbhd of each of its points. ■

Remark 3.4. *The converse of the above theorem is not true in general as seen from the following example.*

Example 3.9.
Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. The set $\{a, d\}$ is a $rg\alpha$-nbhd of the point a, since the $rg\alpha$-open set $\{a\}$ is such that $a \in \{a\} \subset \{a, d\}$. Also the set $\{a, d\}$ is a $rg\alpha$-nbhd of the point $\{d\}$, Since the $rg\alpha$-open set $\{d\}$ is such that $d \in \{d\} \subset \{a, d\}$. That is $\{a, d\}$ is a $rg\alpha$-nbhd of each of its points. However the set $\{a, d\}$ is not a $rg\alpha$-open set in X.

Theorem 3.10. *Let X be a topological space. If F is a $rg\alpha$-closed subset of X, and $x \in F^c$. Prove that there exists a $rg\alpha$-nbhd N of x such that $N \cap F = \phi$.*

Proof. Let F be $rg\alpha$-closed subset of X and $x \in F^c$. Then F^c is $rg\alpha$-open set of X. So by theorem 3.9. F^c contains a $rg\alpha$-nbhd of each of its points. Hence there exists a $rg\alpha$-nbhd N of x such that $N \subset F^c$. That is $N \cap F = \phi$. ■

Definition 3.4. *Let x be a point in a space X. The set of all $rg\alpha$-nbhd of x is called the $rg\alpha$-nbhd system at x, and is denoted by $rg\alpha\text{-}N(x)$.*

Theorem 3.11. *Let X be a topological space and for each $x \in X$, Let $rg\alpha\text{-}N(x)$ be the collection of all $rg\alpha$-nbhds of x. Then we have the following results.*
(i) $\forall x \in X$, $rg\alpha\text{-}N(x) \neq \phi$.
(ii) $N \in rg\alpha\text{-}N(x) \Rightarrow x \in N$.
(iii) $N \in rg\alpha\text{-}N(x)$, $M \supset N \Rightarrow M \in rg\alpha\text{-}N(x)$.
(iv) $N \in rg\alpha\text{-}N(x), M \in rg\alpha\text{-}N(x) \Rightarrow N \cap M \in rg\alpha\text{-}N(x)$.
(v) $N \in rg\alpha\text{-}N(x) \Rightarrow$ there exists $M \in rg\alpha\text{-}N(x)$ such that $M \subset N$ and $M \in rg\alpha\text{-}N(y)$ for every $y \in M$.

Proof. (i) Since X is a $rg\alpha$-open set, it is a $rg\alpha$-nbhd of every $x \in X$. Hence there exists at least one $rg\alpha$-nbhd (namely - X) for each $x \in X$. Hence $rg\alpha\text{-}N(x) \neq \phi$ for every $x \in X$.
(ii) If $N \in rg\alpha\text{-}N(x)$, then N is a $rg\alpha$-nbhd of x. So by definition of $rg\alpha$-nbhd, $x \in N$.
(iii) Let $N \in rg\alpha\text{-}N(x)$ and $M \supset N$. Then there is a $rg\alpha$-open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is $rg\alpha$-nbhd of x. Hence $M \in rg\alpha\text{-}N(x)$.
(iv) Let $N \in rg\alpha\text{-}N(x)$ and $M \in rg\alpha\text{-}N(x)$. Then by definition of $rg\alpha$-nbhd there exists $rg\alpha$-open sets G_1 and G_2 such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$. Hence $x \in G_1 \cap G_2 \subset N \cap M \rightarrow (1)$. Since $G_1 \cap G_2$ is a $rg\alpha$-open set, (being the intersection of two $rg\alpha$-open sets), it follows from (1) that $N \cap M$ is a $rg\alpha$-nbhd of x. Hence $N \cap M \in rg\alpha\text{-}N(x)$.
(v) If $N \in rg\alpha\text{-}N(x)$, then there exists a $rg\alpha$-open set M such that $x \in M \subset N$. Since M is a $rg\alpha$-open set, it is $rg\alpha$-nbhd of each of its points. Therefore $M \in rg\alpha\text{-}N(y)$ for every $y \in M$.

Theorem 3.12. *Let X be a nonempty set, and for each $x \in X$, let $rg\alpha\text{-}N(x)$ be a nonempty collection of subsets of X satisfying following conditions.*
(i) $N \in rg\alpha\text{-}N(x) \Rightarrow x \in N$
(ii) $N \in rg\alpha\text{-}N(x), M \in rg\alpha\text{-}N(x) \Rightarrow N \cap M \in rg\alpha\text{-}N(x)$.
Let τ consists of the empty set and all those non-empty subsets of G of X having the property that $x \in G$ implies that there exists an $N \in rg\alpha\text{-}N(x)$ such that $x \in N \subset G$, Then τ is a topology for X.

Proof. (i) $\phi \in \tau$ by definition. We now show that $x \in \tau$. Let x be any arbitrary element of X. Since $rg\alpha\text{-}N(x)$ is nonempty, there is an $N \in rg\alpha\text{-}N(x)$ and so $x \in N$ by (i). Since N is a subset of X, we have $x \in N \subset X$.

Hence $X \in \tau$.
(ii) Let $G_1 \in \tau$ and $G_2 \in \tau$. If $x \in G_1 \cap G_2$ then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau$ and $G_2 \in \tau$, there exists $N \in rg\alpha\text{-}N\,(x)$ and $M \in rg\alpha\text{-}N\,(x)$, such that $x \in N \subset G_1$ and $x \in M \subset G_2$. Then $x \in N \cap M \subset G_1 \cap G_2$. But $N \cap M \in rg\alpha\text{-}N\,(x)$ by (2). Hence $G_1 \cap G_2 \in \tau$.
(iii) Let $G_\lambda \in \tau$ for every $\lambda \in \wedge$. If $x \in \cup\{G_\lambda : \lambda \in \wedge\}$, then $x \in G_{\lambda x}$ for some $\lambda_x \in \wedge$. Since $G_{\lambda x} \in \tau$, there exists an $N \in rg\alpha\text{-}N\,(x)$ such that $x \in N \subset G_{\lambda x}$ and consequently $x \in N \subset \cup\{G_\lambda : \lambda \in \wedge\}$. Hence $\cup\{G_\lambda : \lambda \in \wedge\} \in \tau$. It follows that τ is topology for X. ■

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