Clock Jitter Error in Multi-bit Continuous-Time Sigma-Delta Modulators with Non-Return-to-Zero Feedback Waveform

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ABSTRACT

This paper presents a detailed study of the clock jitter error in multi-bit continuous-time $\Sigma\Delta$ modulators with non-return-to-zero feedback waveform. It is demonstrated that jitter-induced noise power can be separated into two main components: one that depends on the modulator loop filter transfer function and the other dependent on input signal parameters, i.e. amplitude and frequency. The latter component, not considered in previous approaches, allows us accurately to predict the resolution loss caused by jitter, showing effects not taken into account previously in literature despite the fact that they are especially critical in broadband telecom applications. Moreover, the use of state-space formulation makes the analysis quite general and applicable to either cascade or single-loop architectures. Closed-form expressions are derived for in-band error power and signal-to-noise ratio that can be used to optimize modulator performance in terms of jitter insensitivity. Time-domain simulations of several modulator topologies (both single-loop and cascade) intended for VDSL application demonstrate the validity of the presented approach.

Keywords: Analog-to-digital conversion, continuous-time $\Sigma\Delta$ modulation, clock jitter.
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I. INTRODUCTION

Nowadays, the increasing demand for ever faster Analog-to-Digital Converters (ADCs) in broadband communication systems has boosted the interest in Continuous-Time (CT) Sigma-Delta Modulators (ΣΔMs). These modulators have intrinsic anti-aliasing filtering and the potential to operate at higher sampling rates and with lower power consumption than their Discrete-Time (DT) counterparts [1][2]. However, CT ΣΔMs are more sensitive than DT ΣΔMs to some circuit non idealities. Particularly, their performance in high-speed applications is largely impacted by time uncertainties in the clock-signal edges, commonly referred to as clock jitter [1]. Clock jitter occurs at those points within the modulator architecture where signals are transformed from the CT-domain to the DT-domain and vice versa, that is, at the sample-and-hold and reconstruction Digital-to-Analog Converter (DAC), respectively. The error introduced by the sample-and-hold is attenuated within the signal band by the modulator noise shaping effect, and can hence be neglected. However, the jitter error at the DAC occurs without attenuation at the input node thereby limiting modulator accuracy.

Jitter analysis of high-speed CT ΣΔMs has attracted significant interest in technical literature [1]-[7]. Most studies have addressed only the case of modulators with single-bit quantizers and Return-to-Zero (RZ) DACs whereas a much smaller number of studies have been focused on modulators employing multi-bit quantizers and Non-Return-to-Zero (NRZ) DAC waveforms. This may be due to the fact that analysis of these multi-bit NRZ waveforms is mathematically more complex than their RZ counterparts. As a consequence, designers willing to use these waveforms are obliged to rely on inexact semi-empirical estimations based on simulation results [1][5][8]. To the best of the authors’ knowledge, only the work in [7] addresses the case of NRZ DACs. However, in [7] some effects are not considered which, as shown in this paper, might become critical in medium-to-high-frequency applications ¹.

Despite the lack of theoretical coverage, multibit quantizers and NRZ DACs are appealing

¹ Basically, jitter error contains two terms: one associated with the input signal and the other associated with the modulator loop transfer function. For the sake of simplicity, [7] addresses only the latter.
for practical usage because they have low sensitivity to jitter. Many of the recently reported prototypes for medium-to-high resolutions (11-14 bit) and large signal bandwidths (1-15 MHz) employ either multi-bit quantizers, or NRZ DACs, or both [9][10]. This paper is basically aimed at bridging the gap between theory and practice by providing an analytical coverage of jitter error in multi-bit CT ΣΔMs with NRZ DACs. As compared to [7] both the effect of the modulator loop filter transfer function and that of the input signal are covered herein.

We use state-space formulation [11] to derive closed-form relations between jitter error, sampling frequency, modulator specifications (resolution and signal bandwidth), circuit topology (loop filter transfer function and number of bits of the internal quantizer) and input signal parameters (amplitude and frequency). This state-space formulation is general, architecture-independent, and can be applied to any kind of CT ΣΔMs either cascade or single-loop.

The study described in this paper highlights effects undisclosed by previous approaches; effects that are significant for medium-to-high-frequency applications. It is demonstrated that, as the number of bits of the internal quantizer increases, the signal-dependent term dominates over the modulator-dependent term. As a consequence, lowering the sampling frequency might increase the impact of jitter instead of decreasing it as has been assumed up to now. In fact, it is found that for a given set of modulator specifications, there is an optimum value for the sampling frequency that minimizes the in-band noise power dominated by jitter.

This paper is organized as follows. Section II describes a model for the jitter effect on CT ΣΔMs, with emphasis on those architectures using NRZ DACs. Section III applies the state-space formulation to derive closed-form expressions for the in-band noise power and the signal-to-noise ratio dominated by the jitter error. Section IV compares those expressions with previous approaches. Finally, Section V shows several time-domain simulations of different modulators using both single-loop and cascade topologies in order to validate the theoretical predictions given by the presented approach.

II. MODELING CLOCK JITTER ERROR IN CT ΣΔMs WITH NRZ DACs

Fig.1 shows the conceptual block diagram of a single-loop CT ΣΔM. The loop filter is CT and a CT-to-DT transformation is performed at the sampler placed after this filter. An inverse
transformation, DT-to-CT, is effected at the DAC to obtain the CT feedback signal \( y(t) \) from the modulator output \( y(n) \). Jitter occurs at the points where these two transformations take place. On the one hand, the error introduced at the sampler is subject to the same processing (assuming linear models) as the quantization noise, and hence pushed out-of-band by the modulator noise-shaping effect. As a consequence its contribution within the signal band can be neglected. On the other, the error associated with the DAC directly adds to the input signal and therefore makes a significant contribution within the signal band.

Let us focus on the DAC jitter error. Fig. 2 illustrates how this error occurs in the time domain for RZ and NRZ multi-bit DAC feedback waveforms. Assume in both cases that the DAC output is a current, \( I_{\text{DAC}} \), and that the time uncertainty caused by jitter is represented by \( \Delta T(n) \). This time uncertainty produces a corresponding uncertainty in the total charge associated with the \( n \)-th clock period,

\[
\Delta Q(n) \equiv \Delta I_{\text{DAC}}(n) \cdot \Delta T(n)
\]

where \( \Delta I_{\text{DAC}}(n) \) stands for the amplitude of the signal transition at the edge of the period. Note in Fig. 2 that, since the output signal \( I_{\text{DAC}} \) in a RZ DAC goes to zero at each clock cycle, signal transition amplitudes are larger for RZ, \( \Delta I_{\text{DAC}}(n) \big|_{\text{RZ}} \equiv I_{\text{DAC}}(n) \), than for NRZ DACs,

\[\text{Fig. 1. Conceptual block diagram of a single-loop CT \( \Sigma \Delta M \).}\]

\[\text{Fig. 2. Feedback DAC pulse shaping. (a) RZ DAC. (b) NRZ DAC.}\]

\[\text{1} \quad \text{In DT modulators the sampler is at the input and all signals in the loop are DT.}\]

\[\text{2} \quad \text{In order to simplify the notation, } y(nT_s) \text{ is written as } y(n) \text{ with } T_s \text{ being the sampling period.}\]
\[ \Delta I_{\text{DAC}(n)}\rbrack_{\text{NRZ}} = I_{\text{DAC}(n)} - I_{\text{DAC}(n-1)} \]. Also, as the number of bits in the quantizer increases, \(|\Delta Q(n)|\) decreases for NRZ DACs but remain practically unchanged for RZ DACs.

A DAC output waveform with jitter can be represented as the sum of an unjittered output waveform and a stream of pulses with amplitude \(\Delta I_{\text{DAC}(n)}\) and width \(\Delta T(n)\) – often referred to as jitter error sequence [1]. In the case of a NRZ DAC, the jitter error sequence can be related to the modulator output signal by [12]:

\[ \varepsilon(n) = [y(n) - y(n-1)] \frac{\Delta T(n)}{T_s} \]

where \(T_s\) is the sampling period. Assuming that \(\Delta T(n)\) corresponds to a Gaussian random process with zero mean and standard deviation \(\sigma_{\Delta T}\), the power of the jitter error signal can be expressed as:

\[ P_{\varepsilon} = E\{\varepsilon(n)^2\} = \frac{\sigma_{\Delta T}^2}{T_s^2} E\{[y(n) - y(n-1)]^2\} \]

where \(E\{.\}\) stands for the mathematical operator expectation [13].

The following definition will be used:

\[ y(n) = x(n) + q(n) \]

that is, the output \(y(n)\) can be mathematically expressed as the combination of the input signal \(x(n)\) and an error signal \(q(n)\). Using this definition, \(E\{[y(n) - y(n-1)]^2\}\) is expanded as follows:

\[ E\{[y(n) - y(n-1)]^2\} = E\{[x(n) - x(n-1)]^2\} + E\{[q(n) - q(n-1)]^2\} - 2E\{[x(n) - x(n-1)][q(n) - q(n-1)]\} \]

Assuming that \(x(n)\) and \(q(n)\) are uncorrelated, (5) can be simplified to:

\[ E\{[y(n) - y(n-1)]^2\} \approx E\{[x(n) - x(n-1)]^2\} + E\{[q(n) - q(n-1)]^2\} = E\{\Delta x^2\} + E\{\Delta q^2\} \]

where \(\Delta x_n \equiv x(n) - x(n-1)\) and \(\Delta q_n \equiv q(n) - q(n-1)\).

The equation above contains two terms. The first one depends on the input signal. Con-
Considering a sinewave input signal of amplitude \( A \) and angular frequency \( \omega_i = 2\pi f_i \), this term \( \Delta x_n \) can be simplified as:

\[
\Delta x_n \approx \frac{dx(t)}{dt} \bigg|_{t = nT_s} \cdot T_s = A \omega_i T_s \cos \left( \omega_i (n-1) T_s \right)
\]

and hence,

\[
E\{ (\Delta x_n)^2 \} \approx T_s^2 A^2 \omega_i^2 E\{ [\cos (\omega_i (n-1) T_s)]^2 \} = \frac{T_s^2 A^2 \omega_i^2}{2}
\]

and

Regarding the second term in (6), the following approximated expression is derived in Appendix I,

\[
E\{ (\Delta q_n)^2 \} \approx E \left\{ \left( Z^{-1} [(1 - z^{-1})N_{TF}(e(z))] \right)^2 \right\} = \frac{X_{FS}^2}{12 \pi (2^B - 1)} \int_0^{\pi} \left| (1 - e^{-j\omega}) N_{TF}(e^{-j\omega}) \right|^2 d\omega
\]

where \( X_{FS} \) is the full-scale, \( B \) is the number of bits of the internal quantizer, \( N_{TF}(z) \) is the Noise Transfer Function and \( e(z) \) is the Z-transform of the quantization error.

From (3), (8) and (9), we obtained:

\[
P_e \approx \left( \frac{\sigma_{\Delta T}}{T_s} \right)^2 \cdot \left( \frac{A^2 \omega_i^2}{2f_s} + \frac{X_{FS}^2}{12 (2^B - 1)} \right) \int_0^{\pi} \left| (1 - e^{-j\omega}) N_{TF}(e^{-j\omega}) \right|^2 d\omega
\]

where \( f_s \equiv 1/T_s \) is the sampling frequency.

Depending on the actual modulator topology, the integration in (9) and hence the calculation of the second term in (10) may become too involved. Instead, \( E\{ (\Delta q_n)^2 \} \) can be calculated analytically by using the state-space formulation shown in the next section.

### III. STATE-SPACE FORMULATION

Let us consider the conceptual block diagram in Fig.1. Using the impulse-invariant transformation [14], the DT equivalent of the loop filter transfer function can be written as:

\[
G(z) = \frac{n_{(L-1)}z^{(L-1)} + \ldots + n_1z + n_0}{d_Lz^L + d_{(L-1)}z^{(L-1)} + \ldots + d_1z + d_0}
\]
where \( n_i \) and \( d_i \) are function of the coefficients of the original CT modulator loop filter, \( G(s) \). Therefore, \( N_{TF}(z) \) can be derived from (11) as:

\[
N_{TF}(z) = \frac{1}{1 + G(z)} = \frac{d_N z^L + d_{N-1} z^{L-1} + \ldots + d_1 z + d_0}{d_L z^L + (d_{L-1} + n_{L-1}) z^{L-1} + \ldots + (d_1 + n_1) z + (d_0 + n_0)} = \\
= \frac{d_L + d_{L-1} z^{-1} + \ldots + d_1 z^{-(L-1)} + d_0 z^{-L}}{d_L + (d_{L-1} + n_{L-1}) z^{-1} + \ldots + (d_1 + n_1) z^{-(L-1)} + (d_0 + n_0) z^{-L}}
\]

(12)

Fig. 3 shows the transpose of the Direct II implementation of (12) [13]. This block diagram does not have any direct correspondence with the physical implementation of the modulator and is only used to represent the relationship between the input \( e(n) \), output \( q(n) \) and state variables of \( N_{TF}(z) \). The so-called state-space representation of \( N_{TF}(z) \) can be implemented by the block diagram in Fig. 4, which is described by the following finite difference equations [13]:

\[
\begin{align*}
\overline{v}(n+1)_0 &= \overline{F}_0 \cdot \overline{v}(n)_0 + \overline{p}_0 \cdot e(n) \\
q(n) &= \overline{g}_0^T \cdot \overline{v}(n)_0 + e(n)
\end{align*}
\]

(13)

where \( \overline{F}_0 \) is the state matrix, \( \overline{v}(n)_0 \) is the \( L \times 1 \) state vector, \( \overline{p}_0 \) and \( \overline{g}_0 \) are \( L \times 1 \) vectors,
respectively given by:

\[
\begin{bmatrix}
  0 & 0 & 0 & \cdots & 0 & \frac{n_0 + d_0}{d_L} \\
  1 & 0 & 0 & \cdots & 0 & \frac{n_1 + d_1}{d_L} \\
  0 & 1 & 0 & \cdots & 0 & \frac{n_2 + d_2}{d_L} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & 0 & \cdots & 1 & \frac{n_{L-1} + d_{L-1}}{d_L}
\end{bmatrix}
\]

\[
\overline{F_0} = \begin{bmatrix}
  \frac{n_0}{d_L} \\
  \frac{n_1}{d_L} \\
  \frac{n_2}{d_L} \\
  \vdots \\
  \frac{n_{L-1}}{d_L}
\end{bmatrix}
\]

\[
\overline{p_0} = \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_{L-1} \end{bmatrix}, \quad \overline{g_0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]  

(14)

Equation system (13) can be solved recursively to find the relation between the initial state \(\overline{v(0)}_0\), previous inputs \(e(k)\), current input \(e(n)\) and output \(q(n)\) of the system \[13\]. This gives:

\[
q(n) = \overline{g_0}^T \cdot \overline{F_0}^n \cdot \overline{v(0)}_0 + \sum_{k=0}^{n-1} \overline{g_0}^T \cdot \overline{F_0}^{n-1-k} \cdot \overline{p_0} \cdot e(k) + e(n)
\]  

(15)

A. Expectation value of \((\Delta q_n)^2\)

To compute the power of the jitter error from (2) it is necessary to derive the mathematical expectation of \((\Delta q_n)^2\), given by:

\[
E\{ (\Delta q_n)^2 \} = E\{ (q(n) - q(n-1))^2 \} = 2 \cdot [E\{ q(n)^2 \} - E\{ q(n) \cdot q(n-1) \}] 
\]  

(16)

where the fact that \(E\{ q(n)^2 \} = E\{ q(n-1)^2 \}\) has been taken into account.

Assuming in (15) that the initial state, \(\overline{v(0)}_0\), is zero, \(E\{ q(n)^2 \}\) and \(E\{ q(n) \cdot q(n-1) \}\) can be written respectively as:
demonstrated in Appendix II, if the system in Fig.4 is stable and is diagonalized, expressions in (19) and (20) can be re-written as:

\[ E\{q(n)q(n-1)\} = E\{e(n)^2\} \left( \sum_{k=0}^{n-2} \left[ \overline{g_0}^T \cdot \overline{F_0}^{n-2-j} \cdot \overline{p_0} \right] E\{e(k)e(n)\} \right) \]

(18)

Considering that \( E\{e(k)e(j)\} = 0 \) for \( k \neq j \), (17) and (18) are respectively simplified into:

\[ E\{q(n)^2\} = E\{e(n)^2\} \left( 1 + \sum_{k=0}^{n-1} \left[ \overline{g_0}^T \cdot \overline{F_0}^{n-1-k} \cdot \overline{p_0} \right]^2 \right) \]

(19)

\[ E\{q(n)q(n-1)\} = E\{e(n)^2\} \left( \sum_{k=0}^{n-2} \overline{g_0}^T \overline{F_0}^{n-2-j} \overline{p_0} E\{e(k)e(n)\} \right) \]

(20)

Note that \( E\{q(n)^2\} \) and \( E\{q(n) \cdot q(n-1)\} \) depend on the time instant, \( n \). However, as demonstrated in Appendix II, if the system in Fig.4 is stable and \( \overline{F_0} \) is diagonalized, expressions in (19) and (20) can be re-written as:

\[ E\{q(n)^2\} = E\{e(n)^2\} \left( 1 - \sum_{k=1}^{L} \sum_{j=1}^{L} g_k p_k g_j p_j \frac{\lambda_k^{-1} \lambda_j^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \right) \]

(21)

\[ E\{q(n)q(n-1)\} = E\{e(n)^2\} \left( - \sum_{k=1}^{L} \sum_{j=1}^{L} g_k p_k g_j p_j \frac{\lambda_k^{-1} \lambda_j^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \right) \]

(22)
where \( L \) is the order of \( N_{TF} \), \( \lambda_i \) are the eigenvalues of \( \bar{F}_0 \) and \( g_i \) and \( p_j \) are respectively the elements of \( \bar{g}^T = \bar{g}_0^T \cdot \bar{T} \) and \( \bar{p} = \bar{T}^{-1} \cdot \bar{p}_0 \), with \( \bar{T} \) being the matrix of the eigenvectors of \( \bar{F}_0 \).

From (16), (21) and (22), the following relation is obtained:

\[
E\{ (\Delta q_n)^2 \} = 2E\{ e(n)^2 \} \left( 1 - g^T \cdot \bar{p} + \sum_{k=1}^{L} \sum_{j=1}^{L} g_k \bar{p}_k g_j \bar{p}_j \frac{\lambda_k^{-1} - \lambda_j^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \right) \quad (23)
\]

Taking into account that:

\[
E\{ e(n)^2 \} = \frac{X_{FS}^2}{12 \cdot (2^B - 1)^2} \quad (24)
\]

expression (23) can be rewritten as:

\[
E\{ (\Delta q_n)^2 \} = \frac{X_{FS}^2}{6(2^B - 1)^2} \cdot \psi(\bar{g}, \bar{p}, \bar{\lambda}, L) \quad (25)
\]

where,

\[
\psi(\bar{g}, \bar{p}, \bar{\lambda}, L) = 1 - g^T \cdot \bar{p} + \sum_{k=1}^{L} \sum_{j=1}^{L} g_k \bar{p}_k g_j \bar{p}_j \frac{\lambda_k^{-1} - \lambda_j^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \quad (26)
\]

The above expression gives a direct method to calculate \( E\{ (\Delta q_n)^2 \} \), and hence \( P_e \) from the eigenvalues of a \( L \times L \) matrix, which involves the summation of \( L^2 \) terms, normally with \( L < 5 \).

**B. In-band noise power and signal-to-noise ratio**

Replacing (9) with (25) in (10), the jitter noise power is given by:

\[
P_e = \frac{\sigma_{\Delta T}^2}{T_s} \left[ \frac{T_s^2 \lambda_i^2 \omega_0^2}{2} + \frac{X_{FS}^2}{6(2^B - 1)^2} \psi(\bar{g}, \bar{p}, \bar{\lambda}, L) \right] \quad (27)
\]

Taking into account that \( \Delta T(n) \) is a Gaussian random process, and that \( \Delta y_n = [y(n) - y(n-1)] \) is not correlated with \( \Delta T(n) \), the jitter error, \( \varepsilon \), given by (1) can be treated as an independent random signal. Once the error power is computed using (27), it can be assumed to be a white
noise source at the input of the modulator. Therefore, the in-band noise power can be written as:

\[ P_{\varepsilon_{\text{band}}} = (\sigma_{\Delta T})^2 \cdot B_w \cdot \left[ \frac{A^2 \omega_i^2}{f_s} + \frac{X_{FS} f_s}{3(2^B - 1)} \psi(\bar{g}, \bar{p}, \bar{\kappa}, L) \right] \] (28)

where \( B_w \) is the signal bandwidth.

The in-band noise power sets the accuracy which the modulator can attain. This accuracy can be alternatively formulated in terms of the Signal-to-Noise Ratio (SNR),

\[ SNR_{\text{jitter}} = \frac{A^2}{2 P_{\varepsilon_{\text{band}}}} \] (29)

Depending upon whether the in-band noise power is dominated only by the jitter terms or also includes the quantization error noise power one obtains, respectively:

\[ SNR_{\text{jitter}} = 10 \log \left( \frac{A^2}{2 B_w (\sigma_{\Delta T})^2 \left[ \frac{A^2 \omega_i^2}{f_s} + \frac{X_{FS} f_s}{3(2^B - 1)} \psi(\bar{g}, \bar{p}, \bar{\kappa}, L) \right]} \right) \] (30)

\[ SNR = 10 \log \left( \frac{A^2}{3f_s (2^B - 1)^2 \int_0^{B_w} [N_{TF}(f)]^2 \, df + 2 B_w (\sigma_{\Delta T})^2 \left[ \frac{A^2 \omega_i^2}{f_s} + \frac{X_{FS} f_s}{3(2^B - 1)} \psi(\bar{g}, \bar{p}, \bar{\kappa}, L) \right]} \right) \] (31)

where \( SNR_{\text{jitter}} \) corresponds to the case dominated by jitter, and \( SNR \) corresponds to the more general case where an uncorrelated quantization noise term \( P_{Q_{\text{band}}} \) is added to the jitter term to obtain the total in-band noise \( P_{\text{in-band}} = P_{\varepsilon_{\text{band}}} + P_{Q_{\text{band}}} \).

Note in (28) and (31) that the primary time uncertainty error \((\sigma_{\Delta T})^2\) is multiplied by two different factors which added together define the hereinafter called Jitter Amplification Factor, namely:

- the Signal-Dependent Term, SDT, that depends on the input signal parameters \( A \) and \( \omega_i \), and sampling frequency.

\[ P_{Q_{\text{band}}} = \frac{X_{FS}}{6f_s (2^B - 1)^2} \int_{-f_s}^{f_s} [N_{TF}(f)]^2 \, df \]

\[ PQ_{\text{band}} = X_{FS} \int_{-f_s}^{f_s} [N_{TF}(f)]^2 \, df \]
\[ SDT = \frac{A^2 \omega_i^2}{f_s} \]  

(32)

- and the **Modulator-Dependent Term**, \( MDT \), that depends on the modulator topology parameters \( B, X_{FS} \) and \( \psi(g, p, \lambda, L) \), and sampling frequency.

\[ MDT = \frac{X_{FS}^2 f_s}{3(2^g - 1)^2} \psi(g, \bar{p}, \bar{\lambda}, L) \]  

(33)

Note also that the latter term increases with \( f_s \) while the former decreases with \( f_s \). This suggests that a value can be found that minimizes the jitter amplification factor.

Consider for illustration the 3rd-order single-loop CT \( \Sigma \Delta \) depicted at conceptual level in Fig.5. Several modulators with different numbers of bits in the internal quantizer (2, 4 and 6b) have been studied. Table I shows the values of the loop-filter coefficients as well as the position of the pole and Table II shows the values of \( g_i, p_i \) and \( \lambda_i \) obtained from the state-space formulation. Fig.6 plots the jitter amplification factor, together with its two constitutive terms, \( SDT \) (common to all modulators) and \( MDT \), as a function of \( f_s \) for \( f_i = B_w = 20 \text{MHz} \).

Several conclusions can be drawn:

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value, 2b</th>
<th>Value, 4b</th>
<th>Value, 6b</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_0 )</td>
<td>( 7.4 \cdot 10^{24} )</td>
<td>( 2.4 \cdot 10^{25} )</td>
<td>( 3.2 \cdot 10^{25} )</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>( 5.3 \cdot 10^{16} )</td>
<td>( 9.7 \cdot 10^{16} )</td>
<td>( 1.8 \cdot 10^{17} )</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>( 3.3 \cdot 10^8 )</td>
<td>( 4.2 \cdot 10^8 )</td>
<td>( 5.0 \cdot 10^8 )</td>
</tr>
<tr>
<td>( \omega_p )</td>
<td>( 9.7 \cdot 10^7 )</td>
<td>( 9.7 \cdot 10^7 )</td>
<td>( 9.7 \cdot 10^7 )</td>
</tr>
</tbody>
</table>

![Fig. 5. Conceptual block diagram of a 5-bit 3rd-order single-loop CT \( \Sigma \Delta \).](image)
• For the modulator with a 2 bit internal quantizer, jitter sensitivity is completely dominated by the modulator-dependent term. In this case, lowering the sampling frequency (while keeping constant $B_w$) yields an improvement in the in-band noise power.

• In the case of the modulator with a 4-bit internal quantizer, even though in the higher

### TABLE II: Values of $g_i$, $p_i$, and $\lambda_i$ for the modulator in Fig.5

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value, 2b Mod.</th>
<th>Value, 4b Mod.</th>
<th>Value, 6b Mod.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.031</td>
<td>-0.222</td>
<td>0.183</td>
</tr>
<tr>
<td></td>
<td>0.457+j0.497</td>
<td>0.123+j0.815</td>
<td>-0.635+j0.377</td>
</tr>
<tr>
<td></td>
<td>0.457-j0.497</td>
<td>0.123-j0.815</td>
<td>-0.635-j0.377</td>
</tr>
<tr>
<td>$p$</td>
<td>-3.048</td>
<td>-2.807</td>
<td>-1.195</td>
</tr>
<tr>
<td></td>
<td>0.146-j0.64</td>
<td>-0.352-j1.37</td>
<td>-1.883-j7.541</td>
</tr>
<tr>
<td></td>
<td>0.146+j0.64</td>
<td>-0.352+j1.37</td>
<td>-1.883+j7.541</td>
</tr>
<tr>
<td>$\tilde{g}$</td>
<td>0.7</td>
<td>0.811</td>
<td>0.586</td>
</tr>
<tr>
<td></td>
<td>0.82</td>
<td>0.765</td>
<td>0.856</td>
</tr>
<tr>
<td></td>
<td>0.82</td>
<td>0.765</td>
<td>0.856</td>
</tr>
</tbody>
</table>

Fig. 6. Jitter components vs. $f_s$ for the modulator in Fig.5. Note that the right vertical scale applies to the modulator with 2 bit quantization and the left scale to the rest of the cases.
range of sampling frequency jitter-induced noise is still dominated by the modulator-dependent term, in the lower range the term dependent on the input signal becomes dominant. This means that improvement in the in-band noise power would be possible by lowering the sampling frequency down to a certain value below which noise power starts to raise again. In any case, dependency of jitter-induced noise on sampling frequency is reduced significantly with respect to the previous case.

- In the case of a 6-bit internal quantizer, the jitter amplification factor is dominated by the input signal-dependent term in most of the range under study.
- Note also that for the cases with 4 and 6 bit internal quantization, there is an optimum value of $f_s$ that minimizes the in-band jitter noise power and hence, maximizes $SNR$. Generally speaking, whenever jitter is the main limiting factor of a multi-bit CT $\Sigma\Delta M$ with NRZ DAC, obtaining the minimum value of the jitter amplification factor provides a convenient criterion for deciding loop filter parameters, sampling frequency and the number of bits in the internal quantizer for given specifications.

IV. COMPARISON WITH PREVIOUS APPROACHES

The main interest of the analysis carried out in this paper lies in the availability of simple, yet accurate, analytical expressions for the jitter-induced in-band noise power. These expressions help designers to gain an insight into the operation of multi-bit CT $\Sigma\Delta M$s with NRZ DACs. Among other things, this improved insight helps designers to better understand the impact of the input signal on the jitter error balance. As was shown in Fig.6, different design regions can be defined based on the relative importance of the terms (32) and (33). It was also illustrated how, under certain conditions, jitter-induced noise could increase with decreasing sampling frequency. This therefore limits the bandwidth that can be effectively achieved in practice. However, in spite of the practical relevance of such a limit, this effect has not been addressed in previous analysis.

Coverage provided by previous analyses, especially those in [3] and [7], are appropriate whenever the modulator term dominates the jitter amplification factor. This is generally the case with single bit quantization where $E\{(\Delta q_n)^2\}$ dominates due to the much greater value of the quantization step $\Delta$. Let us consider for illustration purposes the single-bit 2nd-order CT $\Sigma\Delta M$ of Fig. 7 with $f_i = B_w$, an input amplitude of 80% of the reference level, $f_s = 40$MHz and an
oversampling ratio \( M = f_s / (2 \cdot B_w) = 64 \).

In this case, the modulator dependent term is more than one thousand times higher than the signal-dependent term. In other words, jitter sensitivity in this modulator is completely dominated by \( E \{ (\Delta q_n)^2 \} \), making jitter-induced noise power practically independent of the input signal. Under these conditions, the equations derived in this paper provide similar predictions to the expressions reported in [3] for single bit quantization, which is recalled here for comparison,

\[
SNR_{\text{jitter}} = 10 \log \left( \frac{1}{16M\sigma_{\Delta q}^2 B_w^2} \right) \tag{34}
\]

and the analysis presented in [7]. This is illustrated in Fig.8. On the one hand, the curves at the top right-hand side depict the predictions made by the simplified expressions (30) and (34) respectively, showing good agreement. On the other hand, the bottom curves depict the predictions made by the previously shown equation (31) and by the analysis in [7], as well as the estimations made by simulations. The agreement is quite good. Fig.8 also shows that within the

Fig. 7. Block diagram of a single-bit 2nd-order CT \( \Sigma \Delta M \).

Fig. 8. Comparison of theory and simulations for the modulator of Fig.7.
region where the resolution is dominated by jitter the agreement between simplified and complete \( SNR \) estimations is reasonable.

As the signal-dependent term becomes first significant and eventually dominant, previous analyses fail to produce proper predictions. In the next section, it will be shown that when the oversampling ratio is reduced and the number of bits increased (as happens in broadband applications where the combination of high input bandwidth and technology limitations imposes a low oversampling ratio) behaviour in terms of jitter sensitivity is not well described by previous approaches, while expressions derived in this study closely match modulator performance.

At the limit when the \( SNR_{jitter} \) is fully dominated by the signal-dependent term and the modulator-dependent term is negligible, the maximum resolution is calculated from (30) as:

\[
SNR_{\text{MAX}} = 10 \log \left( \frac{f_s}{8\pi^2 \sigma_{\Delta T}^2 B_w^3} \right) = 10 \log \left( \frac{M}{4\pi^2 \sigma_{\Delta T}^2 B_w^2} \right) \tag{35}
\]

where we have assumed that \( f_i = B_w \). This is a limit valid both for single loop and cascade CT \( \Sigma \Delta \)Ms. This is also valid for any type of loop filter, since (35) is independent of the particular loop filter.

Note that exactly the same expression is obtained for DT \( \Sigma \Delta \)Ms [15]. In these modulators, this limit is established by the input sampler. In CT \( \Sigma \Delta \)Ms it is introduced by the feedback DAC. In fact, (35) reveals a limit that applies to any ADC, not only to those based on \( \Sigma \Delta \) modulation, and that is that no converter, when clocked with an imperfect signal, can perform better than an ideal sampler clocked with the same signal.

In summary, the analysis in this paper provides a more detailed coverage of practical design choices with application in broadband while at the same time discovering practical limits not contemplated by previous analysis and/or simplified expressions.

V. VALIDATION AND APPLICATION TO VDSL

Fig.9 shows two CT \( \Sigma \Delta \)Ms used as illustrative case studies in this section. Fig.9(a) is a 3rd-order single-loop and Fig.9(b) is a cascade 2-1 topology. Both architectures employ feed-forward stabilization and incorporate a feedback coefficient \( k_r \) to move one of the poles to an

\[5\] In these cascade architectures, jitter-induced noise introduced in the first stage is not affected by later stages since they process only quantization errors.
optimum position. In this section the results of the analysis in this paper are compared to those obtained through detailed time-domain behavioral simulation using SIMSIDES, a SIMULINK-based simulator for $\Sigma\Delta$Ms [16].

The modulators in Fig.9 have been synthesized following the methodology described in [17] for VDSL application with $B_w = 20MHz$. Three different cases are considered:

- CT $\Sigma$ΔM1: Fig.9(a) with $f_s = 400MHz$ and $B = 2$
- CT $\Sigma$ΔM2: Fig.9(a) with $f_s = 160MHz$ and $B = 5$
- CT $\Sigma$ΔM3: Fig.9(b) with $f_s = 160MHz$ and $B_1 = B_2 = 5$

where $B_1$ and $B_2$ are the number of bits in the quantizer in the first and second-stage of Fig.9(b) respectively. Table III shows the values of the loop-filter coefficients, $k_i$, and the position of the poles for the three cases mentioned above. An optimum distribution of $NTF$ zeros has been adopted as described in [18]. Table IV shows the values of $\overline{F_0}^T$, $\overline{g_0}^T$ and $\overline{p_0}^T$ also for these three cases.

Table V compares the values of $E\{(\Delta q_n)^2\}$ extracted from simulations with those calcul-
lated using (25). Note that the simulated $E\{(\Delta q_n)^2\}$ are independent of the input signal and clock jitter, as predicted by (25). Fig.10 shows several simulated output spectra of cases CT $\Sigma\Delta M_1$ (Fig.9(a)) and CT $\Sigma\Delta M_2$ (Fig.9(b)) corresponding to different values of $f_i$ and $f_c$.

### TABLE III: Loop-filter coefficients of CT $\Sigma\Delta M$s in Fig.9

<table>
<thead>
<tr>
<th></th>
<th>CT $\Sigma\Delta M_1$</th>
<th>CT $\Sigma\Delta M_2$</th>
<th>CT $\Sigma\Delta M_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{in}$</td>
<td>1.5</td>
<td>2</td>
<td>1.6</td>
</tr>
<tr>
<td>$k_{fb}$</td>
<td>-1.5</td>
<td>-2</td>
<td>-1.6</td>
</tr>
<tr>
<td>$k_g1$</td>
<td>1</td>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>$k_r$</td>
<td>-0.1</td>
<td>-0.37</td>
<td>-0.24</td>
</tr>
<tr>
<td>$k_{g2}$</td>
<td>0.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$k_{g3}$</td>
<td>0.5</td>
<td>1.2</td>
<td>-</td>
</tr>
<tr>
<td>$k_{ff1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$k_{ff2}$</td>
<td>0.5</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$k_{in2}$</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$k_{fb2}$</td>
<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>Poles</td>
<td>$\omega_1 = 0$</td>
<td>$\omega_2 = \sqrt{\frac{3}{5}} \cdot 2\pi \cdot B_w$</td>
<td></td>
</tr>
</tbody>
</table>

![Fig. 10. Effect of jitter error on the output spectra of (a) CT $\Sigma\Delta M_1$ and (b) CT $\Sigma\Delta M_2$.](image-url)
\( \sigma_{\Delta T} = 25 \text{ ps} \). Note that in Fig. 10(a), the in-band noise power does not depend on \( f_i \), in agreement with what is predicted by [7]. However, as \( B \) increases from \( B = 2 \) to \( B = 5 \), the modulator-dependent term in (28) decreases and hence, the in-band noise becomes dominated by the signal-dependent term as illustrated in Fig. 10(b).

This effect is better shown in Fig. 11, where the SNR-peak of the modulators in Fig. 9 is plotted vs. \( \sigma_{\Delta T} \) for several values of \( f_i \), showing simulation results and theoretical predictions. For comparison purposes, predictions given by [7] are also included. Note that simulated and theoretical data match very well when the combined effect of signal- and modulator-dependent jitter noise is taken into account.

Note that in Fig. 11(a), due to the fact that \( f_s \) and \( \Delta \) are large, the jitter amplification factor is dominated by the modulator-dependent term, meaning that the SNR is practically independ-

| TABLE IV: Values of \( \overrightarrow{F}_0 \), \( \overrightarrow{g}_0 \) and \( \overrightarrow{p}_0 \) for the CT \( \Sigma \Delta M \)s in Fig. 9 |
|---------------------------------|---------------------------------|
| \( \overrightarrow{F}_0 \) before diagonalization | \( \overrightarrow{F}_0 \) after diagonalization |
| \( \overrightarrow{g}_0 \) before diagonalization | \( \overrightarrow{g}_0 \) after diagonalization |
| \( \overrightarrow{p}_0 \) before diagonalization | \( \overrightarrow{p}_0 \) after diagonalization |

\[
\begin{align*}
\overrightarrow{F}_0 & = \begin{bmatrix} 0 & 0 & -0.208 \\ 1 & 1 & -0.198 \\ 0 & 1 & 1.001 \end{bmatrix} & \overrightarrow{F}_0 & = \begin{bmatrix} -0.328 & 0 & 0 \\ 0 & 0.664 & -j \cdot 0.438 \\ 0 & 0 & 0.664 - j \cdot 0.438 \end{bmatrix} \\
\overrightarrow{g}_0 & = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \overrightarrow{g}_0 & = \begin{bmatrix} 0.562 & 0.853 & 0.853 \end{bmatrix} \\
\overrightarrow{p}_0 & = \begin{bmatrix} -1.208 & 2.744 & -1.94 \end{bmatrix} & \overrightarrow{p}_0 & = \begin{bmatrix} 3.503 & 0.016 - j \cdot 0.184 & 0.016 + j \cdot 0.184 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\overrightarrow{F}_0 & = \begin{bmatrix} 0 & 0 & -0.112 \\ 1 & 1 & -4.675e-3 \\ 0 & 1 & -0.423 \end{bmatrix} & \overrightarrow{F}_0 & = \begin{bmatrix} 0.122 + j \cdot 0.39 & 0 & 0 \\ 0 & 0.122 - j \cdot 0.39 & 0 \\ 0 & 0 & -0.667 \end{bmatrix} \\
\overrightarrow{g}_0 & = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \overrightarrow{g}_0 & = \begin{bmatrix} 0.81 & 0.81 & 0.959 \end{bmatrix} \\
\overrightarrow{p}_0 & = \begin{bmatrix} -1.112 & 2.636 & -3.064 \end{bmatrix} & \overrightarrow{p}_0 & = \begin{bmatrix} 1.476 + j \cdot 1.722e-3 & 1.476 - j \cdot 1.722e-3 & -5.69 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\overrightarrow{F}_0 & = \begin{bmatrix} 0 & -0.705 \\ 1 & -1.087 \end{bmatrix} & \overrightarrow{F}_0 & = \begin{bmatrix} -0.543 + j \cdot 0.64 & 0 \\ 0 & -0.543 - j \cdot 0.64 \end{bmatrix} \\
\overrightarrow{g}_0 & = \begin{bmatrix} 0 & 1 \end{bmatrix} & \overrightarrow{g}_0 & = \begin{bmatrix} 0.766 & 0.766 \end{bmatrix} \\
\overrightarrow{p}_0 & = \begin{bmatrix} 0.295 & -2.728 \end{bmatrix} & \overrightarrow{p}_0 & = \begin{bmatrix} -1.781 - j \cdot 1.813 & -1.781 + j \cdot 1.813 \end{bmatrix}
\end{align*}
\]
ent of the input signal. However, when \( B \) is increased and \( f_s \) is decreased in order to improve jitter insensitivity, the resolution improves for low input frequencies, whereas for higher input frequencies this improvement is considerably reduced as shown in Fig.11(b). The same behavior is obtained in Fig.11(c), where jitter sensitivity is dominated by the first stage. It is important to note that this dependence on input signal, even though it might not be important in low bandwidth applications, can become dominant in broadband communications where the input frequency is high and designers resort to multi-bit implementations with low oversampling ratios.

<table>
<thead>
<tr>
<th>Modulator</th>
<th>( f_i ) (MHz)</th>
<th>( A ) (V)</th>
<th>Clock jitter (ps)</th>
<th>Simulated (10^{-2} \text{ V}^2)</th>
<th>Theory (10^{-2} \text{ V}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>2.197</td>
<td>0.3</td>
<td>10</td>
<td>12.2</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>8.79</td>
<td>0.3</td>
<td>10</td>
<td>12.1</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>17.58</td>
<td>0.3</td>
<td>10</td>
<td>12.2</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>8.79</td>
<td>0.1</td>
<td>10</td>
<td>11.2</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>8.79</td>
<td>0.2</td>
<td>10</td>
<td>14.1</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>8.79</td>
<td>0.3</td>
<td>20</td>
<td>12.2</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M1} )</td>
<td>8.79</td>
<td>0.3</td>
<td>30</td>
<td>12.2</td>
<td>16</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>2.197</td>
<td>0.3</td>
<td>10</td>
<td>1.21</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>8.79</td>
<td>0.3</td>
<td>10</td>
<td>1.23</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>17.58</td>
<td>0.3</td>
<td>10</td>
<td>1.50</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
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<td>0.1</td>
<td>10</td>
<td>1.25</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>8.79</td>
<td>0.2</td>
<td>10</td>
<td>1.22</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>8.79</td>
<td>0.3</td>
<td>20</td>
<td>1.23</td>
<td>1.2</td>
</tr>
<tr>
<td>CT ( \Sigma \Delta \text{M2} )</td>
<td>8.79</td>
<td>0.3</td>
<td>30</td>
<td>1.23</td>
<td>1.2</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>2.197</td>
<td>0.3</td>
<td>10</td>
<td>0.81</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>8.79</td>
<td>0.3</td>
<td>10</td>
<td>0.80</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>17.58</td>
<td>0.3</td>
<td>10</td>
<td>0.80</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>8.79</td>
<td>0.1</td>
<td>10</td>
<td>0.79</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>8.79</td>
<td>0.2</td>
<td>10</td>
<td>0.80</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>8.79</td>
<td>0.3</td>
<td>20</td>
<td>0.80</td>
<td>0.87</td>
</tr>
<tr>
<td>First stage CT ( \Sigma \Delta \text{M3} )</td>
<td>8.79</td>
<td>0.3</td>
<td>30</td>
<td>0.80</td>
<td>0.87</td>
</tr>
</tbody>
</table>
Fig. 11. $SNR$ vs. $\sigma_{\Delta T}$ for different values of $f_i$:

(a) CT $\Sigma\Delta M1$. (b) CT $\Sigma\Delta M2$. (c) CT $\Sigma\Delta M3$. 
CONCLUSIONS

The effect of clock jitter error on multi-bit CT $\Sigma\Delta$Ms with NRZ DAC has been analyzed. Based on the use of state-space formulation, easy-to-compute closed-form expressions have been derived for the in-band noise power and signal-to-noise ratio. It has been demonstrated that jitter-induced noise can be mathematically separated into two components: one depending on signal parameters and the other one depending on the modulator loop filter. Their combined effect – not predicted by previous approaches – has been confirmed by time-domain simulations of several CT $\Sigma\Delta$Ms intended for VDSL application.

ACKNOWLEDGMENT

This work has been supported by the Spanish Ministry of Science and Education (with support from the European Regional Development Fund) under contract TEC2004-01752/MIC.

APPENDIX I: CALCULATION OF $E\{\Delta q(n)^2\}$

It is proved herein that a good approximation of $E\{\Delta q(n)^2\}$ can be calculated taking into account only quantization error. This approximation simplifies enormously any further analysis but needs justification since it is not necessarily applicable to every feedback system. It is shown that in the particular case of $\Sigma\Delta$ Converters, it is a valid approximation and leads to accurate predictions. According to (2), the effect of clock jitter can be modelled as shown in Fig.12.

For simplicity\(^6\), the input to the modulator $x(t)$ is approximated by an ideally sampled version $x(n)$. With this assumption, $G(s)$ can be replaced by $G(z)$ which is the discrete-time equivalent calculated through the impulse invariant transformation [14]. The equation governing the modulator is:

\[ e(n) = x(t) + G(s) \left( \frac{\Delta T(n)}{T_s} \right) + y(n) \]

\[ z^{-1} - G(s) \]

DAC

\[ e(n) \]

\[ x(t) \]

\[ G(s) \]

\[ + \]

\[ y(n) \]

\[ z^{-1} \]

\[ \frac{\Delta T(n)}{T_s} \]

\[ + \]

\[ DAC \]

\[ G(s) \]

\[ + \]

\[ y(n) \]

Fig. 12. Clock jitter model used in the mathematical derivations.

\(^6\) The exact continuous-time signal can be used together with inverse Laplace and Z transforms making the analysis much more complicated. Since it is demonstrated above that input signal has no significant effect on the calculation of $E\{[q(n)-q(n-1)]^2\}$, the exact analysis does not contribute anything and the approximation makes the analysis much simpler.
Taking into account that \( y(z) = x(z) + q(z) \), the following equation is easily derived:

\[
q(z) - \left[ \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})q(z) \right] \frac{G(z)}{1 + G(z)} = \left[ \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})x(z) \right] \frac{G(z) - x(z)}{1 + G(z)} + \frac{e(z)}{1 + G(z)}
\]  \( (37) \)

In order to determine \( E\{ [q(n) - q(n - 1)]^2 \} \), the equation in (37) is multiplied by \( z^{-1} \) and subtracted from itself, giving:

\[
(1 - z^{-1})q(z) - \left[ \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})q(z) \right] \frac{(1 - z^{-1})G(z)}{1 + G(z)} =
\]

\[
= \frac{\frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})x(z)}{1 + G(z)} \frac{(1 - z^{-1})G(z)}{1 + G(z)} - \frac{(1 - z^{-1})x(z)}{1 + G(z)} + \frac{(1 - z^{-1})e(z)}{1 + G(z)}
\]  \( (38) \)

The expected value of both sides is calculated, starting with the term to the left of the equal sign:

\[
E\left\{ \left. Z^{-1} \left[ (1 - z^{-1})q(z) - \left[ \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})q(z) \right] \frac{(1 - z^{-1})G(z)}{1 + G(z)}\right] \right|^2 \right\} =
\]

\[
= E\{ [q(n) - q(n - 1)]^2 \} + E\left\{ \left. Z^{-1} \left[ \left( \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})q(z) \right) \frac{(1 - z^{-1})G(z)}{1 + G(z)} \right] \right]^2 \right\} - \]

\[
-2E\{ [q(n) - q(n - 1)]Z^{-1} \left[ \left. \frac{\Delta T(z)}{T_s} \otimes (1 - z^{-1})q(z) \right] \frac{(1 - z^{-1})G(z)}{1 + G(z)} \right] \}
\]  \( (39) \)

Since \( \Delta T(n) \) is a random signal with zero mean, it can be proven that the last term in (39) is zero. Considering that

\[
\left| \frac{G(z)}{1 + G(z)} \right| = |S_{TF}(z)| \leq 1
\]  \( (40) \)

an upper bound for the second-last term in (39) can be estimated:
Therefore, the term to the left of the equal sign in (38) can be approximated by:

\[
E \left\{ Z^{-1} \left[ \left( \frac{\Delta T(z)}{T_s} \right) \otimes (1 - z^{-1})q(z) \left( \frac{1-z^{-1}}{1+G(z)} \right) \right]^2 \right\} \leq \\
\leq E \left\{ Z^{-1} \left[ \left( \frac{\Delta T(z)}{T_s} \right) \otimes (1 - z^{-1})q(z) \right] (1-z^{-1}) \right\}^2 = \\
e \left\{ \frac{\Delta T(n)}{T_s} \right\} \left\{ [q(n) - q(n-1)] - \frac{\Delta T(n-1)}{T_s} [q(n-1) - q(n-2)] \right\} \right\}^2 = \\
= 2E \left\{ \left( \frac{\Delta T(n)}{T_s} \right)^2 \right\} E \{ [q(n) - q(n-1)]^2 \} \leq E \{ [q(n) - q(n-1)]^2 \} \leq
\]

(41)

The term on the right side of (38) can be approximated following a similar procedure. First, taking into account that \( x(z) \) is restricted to the signal band where \( G(z) \rightarrow \infty \), the following approximation can be applied:

\[
\frac{1-z^{-1}}{1+G(z)} x(z) \cong 0
\]

(43)

leading to the following approximation:

\[
E \left\{ Z^{-1} \left[ \left( \frac{\Delta T(z)}{T_s} \right) \otimes (1 - z^{-1})x(z) \right] (1-z^{-1})G(z) \right\} ^2 \\
= E \left\{ Z^{-1} \left[ \left( \frac{\Delta T(z)}{T_s} \right) \otimes (1 - z^{-1})x(z) \right] (1-z^{-1})G(z) \right\} + (1-z^{-1})e(z) \right\} \right\}^2 = \\
= E \left\{ \frac{\Delta T(n)}{T_s} [x(n) - x(n-1)] \otimes Z^{-1} \left[ \left( \frac{1-z^{-1}}{1+G(z)} \right) \right]^2 \right\} + E \left\{ Z^{-1} \left[ \left( \frac{1-z^{-1}}{1+G(z)} \right) \right]^2 \right\}
\]

where the fact that \( \Delta T(n) \) is a random signal with zero mean has been used to drop one of aris-
ing terms. Using the approximation in (40), an upper bound for the second-last term in (44) can be evaluated:

\[
E\left\{ \left( \frac{\Delta T(n)}{T_s} [x(n) - x(n-1)] \otimes Z^{-1}\left[\frac{(1-z^{-1})G(z)}{1 + G(z)}\right] \right)^2 \right\} \leq 
\leq E\left\{ \left( \frac{\Delta T(n)}{T_s} [x(n) - x(n-1)] \otimes Z^{-1}\left[(1-z^{-1})\right] \right)^2 \right\} = 
= E\left\{ \left( \frac{\Delta T(n)}{T_s} [x(n) - x(n-1)] - \frac{\Delta T(n-1)}{T_s} [x(n-1) - x(n-2)] \right)^2 \right\} = 
= 2E\left\{ \left( \frac{\Delta T(n)}{T_s} \right)^2 \right\} E\{[x(n) - x(n-1)]^2 \} \leq \sigma_{\Delta T}^2 \omega_{in}^2
\]  

(45)

where the same approximation as in (8) has been used.

The last part of this demonstration consists of proving that the last term in (45) satisfies the relation:

\[
\sigma_{\Delta T}^2 \omega_{in}^2 \leq E\left\{ \left( Z^{-1}\left[\frac{(1-z^{-1})e(z)}{1 + G(z)}\right] \right)^2 \right\}
\]  

(46)

The proof starts with the definition of \( N_{TF}(z) \):

\[
E\left\{ \left( Z^{-1}\left[\frac{(1-z^{-1})e(z)}{1 + G(z)}\right] \right)^2 \right\} = \frac{X_{FS}^2}{12(2^B - 1)} \frac{1}{2\pi} \int_0^\pi |(1-e^{-j\omega})|^2 |N_{TF}(e^{-j\omega})|^2 d\omega > 
> \frac{X_{FS}^2}{12(2^B - 1)} \frac{1}{2\pi} \int_{2\pi B w/f_s}^{\pi} |(1-e^{-j\omega})|^2 |N_{TF}(e^{-j\omega})|^2 d\omega
\]  

(47)

Using the fact that the Out of Band Gain of \( N_{TF}(e^{-j\omega}) \) is greater than one, a lower bound for (47) can be estimated,

\[
E\left\{ \left( Z^{-1}\left[\frac{(1-z^{-1})e(z)}{1 + G(z)}\right] \right)^2 \right\} > \frac{X_{FS}^2}{12(2^B - 1)} \frac{1}{2\pi} \int_{2\pi B w/f_s}^{\pi} |(1-e^{-j\omega})|^2 d\omega \geq \frac{X_{FS}^2}{6(2^B - 1)} \geq \frac{2A^2}{3(2^B - 1)^2}
\]  

(48)

using the fact that the integral in (48) is \( \approx 2\pi \) for \( M \geq 4 \) and that the maximum input amplitude
is $A \leq X_{FS}/2$. Now the following relation can be easily shown to hold even for extreme values of clock jitter and input signal,

$$\frac{2A^2}{3(2^B - 1)^2} \gg \sigma_{\Delta f}^2 \sigma_i^2 A^2$$  \hspace{1cm} (49)

As an example, for $B = 5$, $\sigma_{\Delta f} = 30$ ps and $f_i = 30$ MHz, the first term in (49) is more than 20 times the second term.

Using equations (45), (48) and (49) in (44), the following conclusion can be drawn:

$$E\{(\Delta q_n)^2\} \cong E\left\{ \left( Z^{-1} [(1 - z^{-1}) N_{TF}(z)e(z)] \right)^2 \right\} = \frac{X_{FS}^2}{12\pi (2^B - 1)} \int_0^{\pi} \left| (1 - e^{-j\omega}) N_{TF}(e^{-j\omega}) \right|^2 d\omega$$  \hspace{1cm} (50)

APPENDIX II: MATHEMATICAL EXPECTATION OF $[q(n)]^2$ IN STABLE SYSTEMS

This appendix proves that for a stable system, that is, a system with a $N_{TF}$ having all poles inside the unit circle, the value of $E\{q(n)^2\}$ is independent of time, provided that a reasonable number of samples is used to derive (19). In order to demonstrate this, let us consider the state-space representation of $N_{TF}(z)$ shown in Fig.4, which can be described by the finite difference equations in (13).

After diagonalization, the equation system in (10) can be expressed as:

$$\overline{v}(n+1) = \bar{F} \cdot \overline{v}(n) + \bar{p} \cdot e(n)$$
$$q(n) = \bar{g}^T \cdot \overline{v}(n) + e(n)$$  \hspace{1cm} (51)

where $\overline{v}(n)$ is the state vector of the diagonalized system and $\bar{F}$, $\bar{p}$ and $\bar{g}$ are respectively given by:

$$\bar{F} = \bar{T}^{-1} \cdot \bar{F}_0 \cdot \bar{T} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_L \end{bmatrix}$$
$$\bar{p} = \bar{T}^{-1} \cdot \bar{p}_0 = \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_L \end{bmatrix}$$
$$\bar{g}^T = \bar{g}_0^T \cdot \bar{T} = \begin{bmatrix} g_1 & g_2 & \cdots & g_L \end{bmatrix}$$  \hspace{1cm} (52)
where $\mathbf{T}$ is the matrix formed by the eigenvectors of $\mathbf{F}$, $\lambda_j$ are the eigenvalues and $L$ is the order of the modulator.

Once $\mathbf{F}_0$ has been diagonalized, the matrix product $\left[ g_0 \cdot \mathbf{F}_0^{n-1-k} \cdot p_0 \right]^2$ can be written as:

\[
\begin{align*}
\left[ g_0 \cdot \mathbf{F}_0^{n-1-k} \cdot p_0 \right]^2 &= \left[ g \cdot \mathbf{T}^{-1} \cdot \mathbf{F}_0^{n-1-k} \cdot \mathbf{p} \right]^2 \\
&= \left[ g \cdot (\mathbf{T}^{-1} \cdot \mathbf{F}_0^{n-1-k} \cdot \mathbf{p}) \right]^2 \\
&= \left[ g \cdot \mathbf{F}^{n-1-k} \cdot \mathbf{p} \right]^2 = \sum_{i=1}^{L} \sum_{j=1}^{L} g_i p_i g_j p_j \lambda_i^{n-1-k} \lambda_j^{n-1-k} 
\end{align*}
\]

(53)

Taking into account (53), the summation in (19) can now be expressed as:

\[
\begin{align*}
\sum_{k=0}^{n-1} \left[ g_0 \cdot \mathbf{F}_0^{n-1-k} \cdot p_0 \right]^2 &= \sum_{i=1}^{L} \sum_{j=1}^{L} g_i p_i g_j p_j \sum_{k=0}^{n-1} \lambda_i^{n-1-k} \lambda_j^{n-1-k} 
\end{align*}
\]

(54)

Let us define the partial sums as:

\[
S_{nij} = \sum_{k=0}^{n-1} \lambda_i^{n-1-k} \lambda_j^{n-1-k}
\]

(55)

Multiplying (55) by $\lambda_i^{-1} \lambda_j^{-1}$ and subtracting from the original, the following expression is obtained:

\[
S_{nij} - \lambda_i^{-1} \lambda_j^{-1} S_{nij} = \lambda_i^{n-1} \lambda_j^{n-1} - \lambda_i^{-1} \lambda_j^{-1} \Rightarrow S_{ijn} = \frac{\lambda_i^{n-1} - \lambda_j^{n-1} - \lambda_i^{-1} \lambda_j^{-1}}{1 - \lambda_i^{-1} \lambda_j^{-1}}
\]

(56)

If the system is stable, $|\lambda_i| < 1$, and hence $S_{ijn}$ has a finite limit given by:

\[
\lim_{n \to \infty} S_{ijn} = -\frac{\lambda_i^{-1} \lambda_j^{-1}}{1 - \lambda_i^{-1} \lambda_j^{-1}}
\]

(57)

In practical examples, it has been observed that for $n \geq 10$, $S_{ijn}$ is very close to the limit given above. Using (57), the last summation in (54) can be expressed as:
Finally, from (19), (54) and (58), the expected value of $q(n)^2$ can be expressed as:

$$E\{q(n)^2\} = E\{e(n)^2\} \left( 1 - \sum_{k=1}^{L} \sum_{j=1}^{L} g_k p_k g_j p_j \frac{\lambda_k^{-1} \lambda_j^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \right)$$

which is independent of time.

Following a similar procedure, it can be proved that $E\{q(n)q(n-1)\}$ is given by

$$E\{q(n)q(n-1)\} = E\{e(n)^2\} \left( -T \cdot \bar{p} - \sum_{k=1}^{L} \sum_{j=1}^{L} g_k p_k g_j p_j \frac{\lambda_k^{-1}}{1 - \lambda_k^{-1} \lambda_j^{-1}} \right)$$

### REFERENCES


