HOTELLING-DOWNS MODEL OF ELECTORAL COMPETITION AND THE OPTION TO QUIT

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ABSTRACT. We study a variant of the multi-candidate Hotelling-Downs model that recognizes that politicians, even after declaring candidacy, have the option of withdrawing before the election date. We find that this natural variation dramatically alters equilibrium predictions. Under a mild parametric restriction, an equilibrium always exists for an arbitrary finite number of candidates. Moreover, under some further restriction, every equilibrium must involve policy divergence.

Journal of Economic Literature Classification: C70, D70, D72
1. Introduction

Models of electoral competition are central to the growing field of political economics. The seminal Hotelling-Downs model (Hotelling, 1929; Downs, 1957) and its celebrated “median voter” result with two competing politicians have shaped virtually all subsequent research on electoral competition.1

The key idea of the Hotelling-Downs model is simple but powerful: in a two-candidate election, each candidate can increase her probability of winning by adopting a position closer to her opponent. The idea is also remarkably robust as long as there are only two candidates: see the survey by Osborne (1995) for an elegant discussion of its robustness to many variations of the underlying structure.

This idea, alas, fails to survive the natural extension to elections with more than two contestants. Indeed, the model itself loses its predictive power: Osborne (1993) shows that the model generically does not admit any (pure-strategy) Nash equilibrium when the number of politicians exceeds two, and concludes that “a straightforward extension of Hotelling’s model to the case of more than two candidates gives us little insight into the outcome of multi-candidate competition” (Osborne, 1995, p. 291).2

In this paper, we study multi-candidate electoral competition by merely augmenting the Hotelling-Downs model to recognize that politicians, even after declaring candidacy and entering the race, have the option of dropping out before the election date. Since this is surely a right that candidates for almost any election have,3 it seems natural to ask if incorporating this in formal models makes a significant difference. We find that it does: in fact, explicitly recognizing this right restores some of the predictive power to the model.

In our model, politicians first choose whether to enter an electoral contest as candidates, and if so, which position (on a one-dimensional policy space) to adopt, or whether to stay out of the fray altogether. Then, after observing the choices of others, the candidates have a further option to quit the race or to continue as contestants. Voters then vote sincerely for a candidate espousing their favorite position among those still in the race. The winner is decided by plurality rule. Politicians care only about the benefit (b) of winning office and the cost (c) of running a campaign till the election date. We assume that quitting before the election date saves the cost of continued campaign (some fraction of the total cost c).

In this model, quits can never occur along the path of a subgame perfect equilibrium: there is no uncertainty. The option to quit, however, alters the properties of equilibria. Our solution concept is subgame perfect equilibrium

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1Feddersen, Sened, and Wright (1990) remark that “[the] result has generated one of the most fertile research programs in the history of political science.”

2To be precise, Osborne (1993) considers a variant of the Hotelling-Downs model in which politicians bear a cost to enter the electoral race and have the option of staying out of the race.

3Indeed, it is routine in Presidential elections in the US for politicians to officially declare candidacy over a year before the scheduled election date, and then to drop out during the long primary season.
with the proviso that no randomization takes place along the path of an equilibrium. We give conditions for the existence of an equilibrium for an arbitrary finite number of candidates and any distribution of single-peaked preferences of the voters. We find that an equilibrium exists for a wide range of the parameter space; indeed, an equilibrium may fail to exist only if \( b \in (3c, 4c) \).

We also provide a partial characterization of the equilibrium outcomes and address the question of whether candidates for an election adopt the same policy or different policies. When there are exactly three politicians, there is always an equilibrium in which all adopt the median voters’ favorite position. In contrast, the candidates must choose at least two distinct policies in any pure-strategy subgame perfect equilibrium when there are four or more politicians (and the benefit-cost ratio of contesting an election is commensurately high).

The voters in our model vote sincerely, as in the original Hotelling-Downs formulation, and in Osborne (1993). Feddersen, Sened, and Wright (1990) analyze the model without the option to quit when the voters (assumed finite in number) vote strategically. They demonstrate that the behavioral hypothesis of strategic—rather than sincere—voting yields a sharp prediction: there is a Nash equilibrium for an arbitrary number of politicians, and the candidates adopt the median ideal position in all Nash equilibria in which voters use undominated strategies.

There are sound theoretical arguments and some empirical support for focusing on strategic voting in multi-candidate elections. Voting sincerely is not a dominant strategy, and even in large general elections, voters appear to display some strategic sophistication. But in models with strategic voting, every voter is pivotal in any equilibrium, and the equilibria are often supported through fine coordination of the voters’ strategies. In Feddersen, Sened, and Wright (1990), if a candidate deviates to a position other than the median, all voters indifferent between the deviant’s position and the median vote for a particular candidate at the median. As the authors note, this “implicit coordination (or cooperation) among voters” is a “disturbing feature of the equilibrium set” (p. 1014). In our model, the equilibrium strategies also rely on implicit coordination among the players, but the players are the politicians. In many elections, the number of voters is orders of magnitude larger than the number of candidates, and the interaction among voters is far more anonymous than the interaction among the candidates; to that extent, the implicit coordination is perhaps not as unpalatable.

An alternative model of electoral competition, of more recent vintage, that has proved very useful is the citizen-candidate model, proposed independently by Osborne and Slivinski (1996) and Besley and Coate (1997). Political candidates are citizens who choose to enter the electoral contest; like other citizens, they have preferences over policy positions; they cannot credibly commit to implement a policy different from their favorite if

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4In one of the earliest investigations of multi-candidate Hotelling-Downs model, Cox (1987) had shown that, when it exists, a Nash equilibrium must be “noncentrist”. In view of the nonexistence result in Osborne (1993), this observation loses its significance. Our result, when candidates have the option to quit, offers a confirmation of Cox’s intuition.
they are elected. In this class of models,\(^5\) an equilibrium exists very generally, and a two-candidate equilibrium, when it exists, must have candidates whose ideal positions are different. Sengupta and Sengupta (2005) contains a preliminary analysis of a variant of the citizen-candidate model in which the preferences of voters may be subject to some exogenous shock and the candidates have the option to quit the race after the realization of the shock but before the election date. There is no equilibrium in this variant if the candidates do not have the option to quit. The option to quit guarantees the existence of an equilibrium, and the uncertain prospect means that entry and subsequent exit can occur along the path of an equilibrium (which is impossible in the model in this paper).

In our view, different models of electoral competition—with citizen candidates or opportunistic candidates, with sincere voting or strategic voting—illuminate different aspects of the interplay of manifold factors that are part of the political process of policy-making. We therefore regard the parallel exploration of these models as complementary.

2. Model and Results

There is an uncountable set of citizens, each of whom has single-peaked preferences over a one-dimensional set of policy positions, taken to be the real line, \(\mathbb{R}\). The distribution function of the citizens’ most preferred, or ideal, positions on \(\mathbb{R}\) is given by \(F\). We assume that \(F\) is nonatomic, and that the support of \(F\) is an interval; the unique median of \(F\) is denoted \(m\).

There is a finite set \(N = \{1, \ldots, n\}, n \geq 2,\) of politicians who care only about winning office in an election. Winning office outright confers a benefit of \(b > 0\) to a politician while the cost of contesting the election is \(c > 0\). The cost reflects the expenses of mobilizing and administering a campaign. The payoff of a politician who stays out of the electoral race altogether is normalized to zero. The politicians are risk-neutral.

Electoral competition is modelled as unfolding in three stages. In the first stage, the politicians simultaneously decide whether to enter an electoral race and, if so, which policy position \(x \in \mathbb{R}\) to adopt, or to stay out of the race. Politicians who choose to enter the electoral race become candidates.

The second stage takes place once the candidates have chosen their policies but before votes are cast. The candidates observe the choices made in the first stage, and then simultaneously decide whether to quit the race or to continue to contest the election. Quitting the race saves the expenses of continuing the campaign. A candidate who quits saves \(\alpha c\), \(\alpha \in (0, 1)\), of the cost.\(^6\)

The third stage is non-strategic. Each citizen, having observed the positions of the candidates who remain to contest the election, casts a vote. We assume that voting is sincere: if \(k\) contestants adopt a position \(x \in \mathbb{R}\), then

\(^5\)In Osborne and Slivinski (1996), there is a continuum of voters who vote sincerely; in Besley and Coate (1997), voters are finite in number and they vote strategically.

\(^6\)Equivalently, we could let a politician bear an initial cost of \((1 - \alpha)c\) at the point of entry and a further cost of \(\alpha c\) should she continue to contest the election.
each attracts a fraction $1/k$ of the votes of the citizens whose ideal position is closer to $x$ than to any other proposed position. The winner is a candidate who receives the highest share of votes: ties for first place are broken with equal probability.

Formally, the stages described above defines a two-stage game in extensive form in which the politicians are the players. Let $O$ denote the action of staying out, $Q$ denote the action of quitting, and $C$ denote the action of continuing. The payoff of a player is zero if she chooses $O$; for $x \in \mathbb{R}$, a player’s payoff is $-(1-\alpha)c$ if she chooses $(x, Q)$, and $(b/k) - c$ if she chooses $(x, C)$ and ties for first place with $k - 1$ other players, $k \geq 1$ and $-c$ if she chooses $(x, C)$ and loses for sure.

We refer to the first stage of the game as the entry stage and refer to the second stage as the post-entry subgame. By continuation payoff we mean the payoff accruing in the post-entry subgame: $\alpha c$ from quitting, $b/k$ from continuing and tying with $k - 1$ other candidates.

By an equilibrium of the game we mean a subgame perfect equilibrium with the restriction that the strategy profile prescribes pure actions along an equilibrium path (although it may prescribe randomization off-the-equilibrium path).

**Remark 1.** Since the proportion of cost saved upon exit $\alpha \in (0,1)$, a candidate’s decision to continue can be optimal only if she wins with positive probability. It follows that if $k$ candidates remain to contest the election at an equilibrium, then each must win with probability $1/k$. Similarly, it follows that an equilibrium cannot call for a candidate to quit along the equilibrium path.

**Remark 2.** The model only admits trivial equilibria if the benefit-cost ratio $b/c$ is too low. No politician has an incentive to enter the competition if $b < c$; only one contests the election adopting the median ideal position if $b \in (c, 2c)$; there are trivial multiple equilibria if $b = c$ or $b = 2c$. We dispense with these by assuming henceforth that $b > 2c$.

The model, of course, generates the well-known median voter result if there are only two politicians: the unique equilibrium is for both politicians to enter, adopt the median position, and continue on to contest the election. Our objective is to study the model when there are more than two politicians.

Osborne (1993) has shown that, in the absence of the option to quit, the model generates a strong non-existence result if there are more than two politicians. There is no pure-strategy Nash equilibrium for almost any distribution of the ideal positions of the voters.

We give conditions under which an equilibrium exists for any distribution of the voters’ ideal positions, and also provide a partial characterization of equilibrium outcomes. The conditions for the existence of an equilibrium as

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7The players take no action in the third stage which merely determines the payoff of each politician as a function of the profile of actions chosen in the first two stages.

8In what follows, we use the terms politicians and players synonymously.

9When we refer to a pure-strategy equilibrium, we mean that the strategy profile prescribes a pure action at every information set.
well as the properties of equilibrium outcomes differ depending on whether there are exactly three politicians or more.\textsuperscript{10}

\section*{2.1. Results with Three Politicians}

Our first proposition concerns the case when there are exactly three politicians. It shows that an equilibrium always exists. Indeed, the Hotelling-Downs outcome with policy-convergence at the median can always be supported as an equilibrium. Moreover, in any equilibrium, at least one candidate must adopt the median position.

\textbf{Proposition 1.} Let the set of politicians be \{1, 2, 3\}.

(a) If \( b \in (2c, 3c) \), there is an equilibrium in which exactly two politicians enter as candidates, choose the median position \( m \), and continue on to contest the election; the remaining politician stays out.

(b) If \( b \geq 3c \), there is an equilibrium in which all three enter as candidates, choose the position \( m \), and contest the election.

(c) In any equilibrium, at least one of the candidates contesting the election must choose the position \( m \).

To prove the proposition, we begin by constructing a Nash equilibrium of the post-entry subgame when one candidate has adopted a position different from the median and confronts two candidates at the median.

\textbf{Lemma 1.} Consider a subgame that begins with all three politicians in the race, two candidates at the median position \( m \), and the third candidate at \( x \neq m \). Then there is a Nash equilibrium of the continuation subgame in which the continuation payoff of the player at \( x \neq m \) is no more than \( \max \{b/3, \alpha c\} \).

\textit{Proof.} Suppose, without loss of generality, that candidates 1 and 2 have adopted the median position \( m \) while candidate 3’s policy choice is \( x \neq m \). We specify a profile of actions for the ensuing subgame that constitutes a Nash equilibrium of the subgame. The prescription of the profile depends on player 3’s choice of position.

(i) If at the position \( x \), player 3’s share of votes is less than \( 1/3 \), players 1 and 2 continue on to contest the election while candidate 3 quits the race.

(ii) If at \( x \), player 3’s vote share is exactly \( 1/3 \), players 1 and 2 continue while player 3 continues if and only if \( b \geq 3\alpha c \).

(iii) If at \( x \), player 3’s vote share exceeds \( 1/3 \), player 1 continues, 2 quits with probability \( q_2 = (b - \alpha c)/b \), and 3 quits with probability \( q_3 = 2\alpha c/b \).

\textsuperscript{10}The results here apply only to the plurality rule where the payoff of a losing candidate is \( -c \) regardless of her vote share; they do not apply to situations in which a candidate’s payoff is positively related to her share of votes (for example, as in proportional voting). That the objective of a politician in this model is maximization of the probability of winning rather than the share of votes is also what distinguishes it from models of multi-firm spatial economic competition, as in Eaton and Lipsey (1975).
We now verify that the above profile constitutes a Nash equilibrium. The argument for case (i) is immediate; we provide the arguments for cases (ii) and (iii).

In case (ii), if $b \geq 3\alpha c$, it is optimal for each candidate to continue on since the continuation payoff of each is $b/3 \geq \alpha c$, the payoff from quitting. On the other hand, if $b < 3\alpha c$, player 3 is better off quitting given that players 1 and 2 continue on. Moreover, since $b > 2c$, it is optimal for players 1 and 2 to contest the election when 3 quits.

Under case (iii), player 1 wins outright when player 2 quits, so that player 1’s expected payoff from continuing is at least $q_2b = b - \alpha c > \alpha c$, her payoff from quitting. Player 2 wins with probability $1/2$ when player 3 quits, so that player 2’s expected payoff from continuing is $q_2b/2 = \alpha c$, her expected payoff from quitting. Finally, player 3 wins outright when players 1 and 2 are in the race, so that player 3’s expected payoff from continuing is $(1 - q_2)b = \alpha c$, her payoff from quitting.

This establishes that the profile constitutes a Nash equilibrium. Finally, we note that the payoff of player 3 from the above profile is no more than $\max\{b/3, \alpha c\}$.

Proof of Proposition 1. To prove part (a) of the proposition, consider a strategy profile that stipulates that exactly two of the players—without loss of generality, players 1 and 2—enter as candidates and choose the median position $m$, and that the third candidate stays out of the race. If players adhere to the prescription at the entry stage, the stipulation for the post-entry subgame is for the two candidates to continue on to contest the election. If player 3 entered as a candidate (contrary to the prescription of the strategy profile), the actions prescribed by the strategy profile for the subgame are as given in items (i), (ii), and (iii) of Lemma 1.

Candidate 3 does not have a profitable deviation at the entry stage: by Lemma 1, upon entry, her payoff could be at most $\max\{b/3, \alpha c\} - c < 0$. It is immediate that candidates 1 and 2 do not have a profitable deviation. In conjunction with Lemma 1, this establishes that the strategy profile constitutes an equilibrium.

To prove part (b) of the proposition, we modify the strategy profile so that, at the entry stage, it calls for all three players to enter as candidates, choose the median position, and then continue on to contest the election provided that the players followed the prescription at the entry stage. If all three entered but one player chose a position other than the median, the profile again calls for the actions given in items (i), (ii), and (iii) of Lemma 1. If some player stayed out contrary to the prescription, the profile calls for a candidate to quit if and only if she would lose the election. It is straightforward to verify that this is an equilibrium profile.

In the interest of brevity, we refrain from listing in detail the actions prescribed for every possible post-entry subgame. When the prescription following a deviation at the entry stage is obvious, it is omitted. For example, for part (a), consider a subgame in which player 3 did not enter as a candidate (as per the prescription), but player 1 deviated and chose a position $x \neq m$; then the profile of course calls for 1 to quit in the post-entry subgame.
To prove part (c), we first observe that if only two players enter the race, then both must choose the median $m$ in equilibrium. Moreover, all three players can enter only if $b \geq 3c$, and must enter if $b > 3c$. If no candidate chooses the position $m$ at an equilibrium, either (1) all of the chosen positions are on one side of $m$, or (2) two of the chosen positions are on one side of $m$ while the third—without loss of generality, say, player 3’s choice—is on the other side of $m$. In case (1), the candidate nearest to $m$ can win outright by switching to $m$. In case (2), candidate 3 can win outright by switching to $m$. Therefore, in any equilibrium, at least one candidate must adopt the median position. □

2.2. Results with More than Three Politicians

We now turn to the case when there are four or more politicians. The next proposition provides an existence result when the benefit-cost ratio is high enough that each politician (weakly) prefers a tie with three other politicians to staying out of the race.

In preparation for stating the proposition, we select a pair of policies, $x_{12}^*$ and $x_{34}^*$, on opposite sides of the median $m$, with the property that the entry of exactly four politicians as candidates, with two adopting the position $x_{12}^*$ and two adopting $x_{34}^*$, ensures a win for each with probability $1/4$. Formally, let $x_{12}^*$ and $x_{34}^*$ be such that

\[
x_{12}^* < m < x_{34}^*, \quad x_{12}^* + x_{34}^* = 2m, \quad \text{and} \quad F(x_{34}^*) - F(x_{12}^*) < F(x_{12}^*).
\]

Proposition 2. Let the set of politicians be $N = \{1, \ldots, n\}$, $n \geq 4$, and let $b \geq 4c$. Then, there is an equilibrium in which exactly four politicians enter as candidates, two choose the position $x_{12}^*$ and two the position $x_{34}^*$, and all four continue on to contest the election.

Remark 3. If $b \in (2c, 3c)$, it is easy to see that the conclusion of part (a) of Proposition 1 continues to hold: there is an equilibrium in which exactly two politicians enter as candidates, adopt the median position, and continue on to contest the election. Proposition 2 guarantees the existence of an equilibrium provided $b \geq 4c$. Thus, there is a gap: the case when $b \in (3c, 4c)$. An example in Appendix A demonstrates that an equilibrium may not exist when $b \in (3c, 4c)$. It follows from Remark 2, and Propositions 1 and 2 that it is the only subset of the parameter space for which an equilibrium may fail to exist.

Proof of Proposition 2. We construct a strategy profile and then verify that it is an equilibrium. The profile prescribes that a subset of exactly four players—without loss of generality, the set $\{1, 2, 3, 4\}$—enter as candidates, and that two players (for concreteness, 1 and 2) adopt the position $x_{12}^*$ and the other two players the position $x_{34}^*$. If players adhere to the prescription at the entry stage, the profile calls for all four candidates to continue on to contest the election. In the event that some player $j \notin \{1, 2, 3, 4\}$ entered as a candidate or some player $i \in \{1, 2, 3, 4\}$ chose a position different from the prescription, the stipulation for the post-entry subgame is for exactly one player at position $x_{12}^*$ and one player at position $x_{34}^*$ to continue to contest the election and for the others to quit.
Consider a player \( j \notin \{1, 2, 3, 4\} \) who enters as a candidate and chooses position \( x_j \). If \( x_j \leq x_{12}^* \), under the prescribed profile, a candidate that remains at \( x_{34}^* \) to contest the election will win for sure. Similarly, if \( x_j \geq x_{34}^* \), the candidate that remains at \( x_{12}^* \) to contest the election will win for sure. If \( x_j \in (x_{12}^*, x_{34}^*) \), the candidate at \( x_{12}^* \) will win for sure since her share of votes is at least \( F(x_{12}^*) > F(x_{34}^*) - F(x_{12}^*) \), the maximum possible vote share that player \( j \) can get. Therefore, player \( j \) will lose for sure if she continued, so that her optimal action in the post-entry subgame is to follow the prescription of the profile and quit. Note also that if exactly one player remains at \( x_{12}^* \) (resp. at \( x_{34}^* \)), then it is optimal for exactly one player to continue at \( x_{34}^* \) (resp. at \( x_{12}^* \)). Thus no player \( j \) who is currently out of the race has a profitable deviation.

An analogous argument establishes that if some candidate \( i \in \{1, 2, 3, 4\} \) were to chose a position different from \( x_{12}^* \) or \( x_{34}^* \), the prescriptions of the strategy profile in any subgame following a deviation are mutual best responses.

Further, the prescription for players 1 and 2 at \( x_{12}^* \) and 3 and 4 at \( x_{34}^* \) to continue if no player deviated at the entry stage results in a payoff of \( b/4 > \alpha c \), the payoff from quitting. This establishes that the profile induces a Nash equilibrium in the post-entry subgame.

Now consider the entry stage. At the proposed strategy profile, the payoff of any player \( i \in \{1, 2, 3, 4\} \) is \( (b/4) - c \geq 0 \) and that of player \( j \notin \{1, 2, 3, 4\} \) is zero. If player \( i \) were to stay out, her payoff would be zero; if player \( j \) were to enter, her payoff would be negative. It follows that the strategy profile is an equilibrium. □

Remark 4. We note that the strategy profile constructed above is in fact a pure-strategy subgame perfect equilibrium: it does not call for randomization even off-the-equilibrium path.

The next result provides a partial characterization of equilibrium outcomes. It provides conditions under which every equilibrium must involve policy divergence: that is, the winning candidates must adopt at least two distinct positions.\(^{12}\) Roughly, policy divergence is a necessary consequence of an equilibrium provided that most of the cost of fighting an election has to be borne at the entry stage (more precisely, if \( \alpha < 1/3 \)), or if the number of politicians is large and the benefit-cost ratio of winning office is commensurately high (more precisely, if \( n \geq 6 \) and \( b \geq 6c \)).

**Proposition 3.** Let the set of politicians be \( N = \{1, \ldots, n\} \). Then, in any equilibrium there must be candidates \( i \) and \( j \) who adopt positions \( x_i \neq x_j \) if (i) \( n \geq 4 \), \( b \geq 4c \), and \( \alpha < 1/3 \), or, (ii) \( n \geq 6 \), and \( b \geq 6c \).

The solution concept that we have used in this paper is a subgame perfect equilibrium that permits randomization off-the-equilibrium path. This was done to ensure the existence of an equilibrium when there are only three politicians. We can present a result cleaner and sharper than Proposition 3 if we restrict attention to pure-strategy subgame perfect equilibria. Indeed,

\(^{12}\)We have been unable to show that the distinct positions must be on opposite sides of the median ideal position.
under the conditions of Proposition 2 (i.e., \( n \geq 4 \) and \( b \geq 4c \)), every pure-strategy subgame perfect equilibrium must involve policy divergence.

**Proposition 4.** Let the set of politicians be \( N = \{1, \ldots, n\} \). Let \( n \geq 4 \) and \( b \geq 4c \). Then there is a pure-strategy subgame perfect equilibrium. Moreover, at any pure-strategy subgame perfect equilibrium, there must be candidates \( i \) and \( j \) who adopt positions \( x_i \neq x_j \).

The intuition behind these two propositions is simple: if all candidates were to choose the same position, it would have to be the median; that leaves enough ground for a politician to enter as a candidate either to the left (as at \( \tilde{x}_l \) below) or to the right and win if there are two or more candidates at the median.

Formally, let \( \tilde{x}_l < m \) be such that

\[
F\left(\frac{\tilde{x}_l + m}{2}\right) = \frac{1}{3}.
\]

Since the support of \( F \) is an interval in \( \mathbb{R} \), \( \tilde{x}_l \) is uniquely defined. By construction of \( \tilde{x}_l \), a candidate who adopts the position \( \tilde{x}_l \) and faces a set of candidates \( M \) at the median position (and no one else) will

- lose the election with probability 1 if and only if exactly one candidate in \( M \) stays on as a contestant;
- win with probability 1/3 if exactly two candidates in \( M \) remain as contestants;
- win with probability 1 if more than two candidates in \( M \) remain to contest the election.

The proofs of the propositions utilize a lower bound on the Nash equilibrium payoffs in the post-entry subgame for a candidate at \( \tilde{x}_l \) who faces at least two candidates at the median position and no one else.

**Lemma 2.** Consider a subgame that begins with a candidate \( i^* \) at \( \tilde{x}_l \), at least two candidates at the median position, and no one else. Then,

(a) in any pure-strategy Nash equilibrium of the post-entry subgame, the expected payoff of candidate \( i^* \) from continuing on must be at least \( b/3 \);

(b) in any Nash equilibrium of the post-entry subgame, the expected payoff of candidate \( i^* \) from continuing on must be at least \( (b/3) - ac \).

We provide below the straightforward proof for part (a) of the lemma concerning pure-strategy Nash equilibria, and relegate the proof of part (b) to the Appendix.

**Proof.** Let the set of candidates be \( M \cup \{i^*\} \), where each candidate \( j \in M \) has chosen the median position \( m \), \( \#M \geq 2 \), and \( i^* \) has chosen the position \( \tilde{x}_l \).

No more than two of the candidates in \( M \) can continue for sure at any Nash equilibrium of the post-entry subgame: otherwise, \( i^* \) could ensure an outright win by continuing so that a continuing contestant in \( M \) would be better off quitting.
Further, at a pure-strategy Nash equilibrium we now argue that at least two candidates in \( M \) must continue. If not, then with exactly one candidate remaining at \( m \), the optimal action for \( i^* \) is to quit the race but then a candidate in \( M \) who quit could have stayed in and ensured a payoff of \( b/2 > \alpha c \) by continuing. Therefore, exactly two candidates in \( M \) must continue at a pure-strategy Nash equilibrium. Candidate \( i^* \)'s optimal action in such a case then is to continue and get a payoff of \( b/3 > \alpha c \).

This proves part (a). The proof for part (b) of the Lemma is provided in Appendix B. \(\square\)

**Proof of Proposition 3.** First, we note that

\[
\frac{b}{3} - (1 + \alpha)c > \begin{cases} 
\frac{(b/6) - c}{c} & \geq 0 \text{ if } b \geq 6c, \\
\frac{(b/4) - c}{c} & \geq 0 \text{ if } \alpha < \frac{1}{3} \text{ and } b \geq 4c. 
\end{cases} 
\tag{1}
\]

Now, suppose that, contrary to the assertion of the proposition, there is an equilibrium in which all candidates choose the same position. Then, the common position must be the median \( m \): otherwise, one of the candidates could win outright by deviating to \( m \).

Since \( n \geq 4 \) and \( b > 3c \), at least three politicians must remain as contestants at such an equilibrium. We now argue that in fact *every* politician must be a candidate. Indeed, by part (b) of Lemma 2, a politician can enter, adopt the position \( \tilde{x}_i \), and ensure a continuation payoff of at least \( (b/3) - \alpha c \) (as she would face at least three candidates at the median). Then, her overall payoff would be \( (b/3) - (1 + \alpha)c > 0 \), by (1). This is thus a profitable deviation.

Since all politicians are candidates, the payoff of any such candidate at such an equilibrium is \( (b/n) - c \). Now consider a candidate \( i^* \), currently at \( m \) at such an equilibrium, deviating to the position \( \tilde{x}_i \). This would result in a post-entry subgame in which the conditions of Lemma 2 are satisfied, so that the payoff of \( i^* \) in the overall game is at least \( (b/3) - (1 + \alpha)c \). By (1), this will constitute a profitable deviation under the conditions of Proposition 3. \(\square\)

**Proof of Proposition 4.** The proof is virtually identical to that of Proposition 3. The only change needed is to replace the lower bound on the deviating player \( i^* \)'s Nash equilibrium payoff in the continuation subgame: it is now \( b/3 \). Thus, her overall payoff from the proposed deviation is \( (b/3) - c > \max \{(b/n) - c, 0\} \) for \( n \geq 4 \) and \( b \geq 4c \). Therefore, the deviation is profitable. \(\square\)

**Appendix A: A Non-Existence Example**

We provide an example to show that an equilibrium may fail to exist if \( b \in (3c, 4c) \).
Let the support of $F$ be $S = [a, b]$. Let $n \geq 4$ and $3c(1 + \alpha) < b < 4c$, with $\alpha < 1/3$. Let $F$ satisfy

\begin{align*}
(i) & \quad F\left(\frac{a + m}{2}\right) = 1 - F\left(\frac{m + b}{2}\right) = \frac{1}{3}, \quad \text{and} \\
(ii) & \quad F\left(\frac{a + x^*}{2}\right) \leq F\left(\frac{b + x^*}{2}\right) - F\left(\frac{a + x^*}{2}\right),
\end{align*}

for some $x^* \in (m, b)$.$^{13}$ We will argue that there is no equilibrium for this example.

Since $b < 4c$, no more than three candidates enter the race in any equilibrium. Applying part (c) of Proposition 1, we infer that at least one of the candidates must choose the median position $m$. Now, if fewer than three candidates were to enter the race, by Lemma 1, all must choose $m$. Since $n \geq 4$, there is a player who did not enter. Such a player could enter, choose $m$ and get a payoff of $(b/3) - c$. Since $b > 3c$, this would be a profitable deviation. Therefore, exactly three candidates must enter in any equilibrium.

Suppose that the three candidates choose distinct positions in equilibrium. By Proposition 1, part (c), one of them must choose $m$. Since $F$ satisfies $F((a + m)/2) = 1 - F((m + b)/2) = 1/3$, we conclude that the chosen positions are $a$, $m$ and $b$. Let the candidate at $m$ deviate and choose $x^*$ instead. By definition of $x^*$—see (ii) above—the deviating candidate will win with probability at least $1/2$ when the other two candidates continue in the race and with probability 1 if no more than one candidate stays in the race. Therefore in any equilibrium of the subgame that follows this deviation, the deviating candidate will stay in the race and get at least $b/2$. This is thus a profitable deviation.

Next, suppose that all three candidates choose the same position in an equilibrium: then it must be $m$. Consider a candidate who stayed out. Let her deviate by entering and choosing the position $a$. By Lemma 2, she can ensure a payoff of $b/3 - ac$ in any Nash equilibrium of the post-entry subgame following this deviation. The overall payoff of such a candidate then is at least $b/3 - (1 + \alpha)c > 0$. Thus, this is a profitable deviation.

Finally, suppose two candidates have chosen $m$ while the third candidate is located at one of the end points, say, $a$. Consider a player who stayed out. Suppose that she deviates by entering and choosing the other end point $b$. A somewhat tedious argument shows that, in any Nash equilibrium of the post-entry subgame following this deviation, the deviating candidate’s payoff is at least $(b/3) - c > 0$. Thus, this is a profitable deviation.

This completes the argument that the example admits no equilibrium.

\textbf{Appendix B: Proof of Lemma 2. (b)}

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$^{13}$These conditions on $F$ are not vacuous. Let $S = [0, 1]$ and $F$ be uniform on each of the intervals $[0, 1/4]$, $[1/4, 3/4]$ and $[3/4, 1]$ with $F(1/4) = 1/3 = 1 - F(3/4)$. Then it is easy to check that for any $x > 1/2$, $F(x/2) = F((1 + x)/2) - F(x/2)$. 

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Let the set of candidates be $M \cup \{i^*\}$, as in the proof of part (a) of Lemma 2.

We have already argued, in part (a), that no more than two of the candidates in $M$ can continue for sure at any Nash equilibrium of the post-entry subgame. Further, at least two candidates in $M$ must continue with positive probability: otherwise, some candidate who quit could have ensured a payoff of $b/2 > \alpha c$ by continuing.

Now, if exactly two candidates in $M$ continue for sure, candidate $i^*$’s optimal action is to continue and get a payoff of $b/3 > \alpha c$.

If fewer than two candidates in $M$ continues for sure, then, at least one candidate in $M$ must randomize: suppose $k$ candidates in $M$ randomize. Since the positions of these $k$ candidates are identical, at a Nash equilibrium of the post-entry subgame, their mixing probability must be the same. Let $q \in (0,1)$ denote the probability with which each of the $k$ candidates quits. Let $M^q$ denote this set of $k$ candidates.

Next, suppose exactly one candidate in $M$ continues for sure. If a candidate $j \in M^q$ were to continue, her expected payoff would be at least $q^{k-1}b/3$ since with probability $q^{k-1}$ she is the only candidate from $M^q$ to continue. Since $q > 0$, we must have

$$\frac{q^{k-1}b}{3} \leq \alpha c. \quad \text{(2)}$$

If candidate $i^*$, whose position is $\tilde{x}_l$, continues, she would face exactly two candidates at $m$ with probability $(1-q^k)$ and then win with probability $1/3$. Therefore, her expected payoff from continuing is at least

$$\frac{(1-q^k)b}{3} > \frac{b}{3} - \alpha c, \quad \text{by (2).} \quad \text{(3)}$$

Finally, suppose no candidate in $M$ continues for sure. Then, $M^q$ must contain at least two candidates. If a candidate in $M^q$ continues, she will win for sure when all of the other $k-1$ candidates in $M^q$ are out of the race and win with probability at least $1/3$ when exactly one other candidates in $M^q$ is also in the race. Since $q > 0$, we must have

$$\frac{q^{k-1} + (k-1)q^{k-2}(1-q)b}{3} \leq \alpha c. \quad \text{(4)}$$

Now, player $i^*$ loses only when exactly one of the $k$ candidates in $M^q$ is in and gets at least $b/3$ otherwise. The probability of the first event is $kq^{k-1}(1-q)$. Therefore, the expected payoff to $i$ if she continues is at least

$$\frac{(1-kq^{k-1}(1-q))b}{3} \geq \frac{1 - (q^{k-1} + (k-1)q^{k-2}(1-q))b}{3} \geq \frac{b}{3} - \alpha c, \quad \text{by (4).} \quad \text{(5)}$$

This completes the proof for part (b) of Lemma 2.
References


