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Time-Dependent Warping, Fluxes, and NCYM

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Abstract

We describe the supergravity solutions dual to D6-branes with both time-dependent and time-independent B-fields. These backgrounds generalize the Taub-NUT metric in two key ways: they have asymmetric warp factors and background fluxes. In the time-dependent case, the warping takes a novel form. Kaluza-Klein reduction in these backgrounds is unusual, and we explore some of the new features. In particular, we describe how a localized gauge-field emerges with an analogue of the open string metric and coupling. We also describe a gravitational analogue of the Seiberg-Witten map. This provides a framework in supergravity both for studying non-commutative gauge theories, and for constructing novel warped backgrounds.

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1 Introduction

The aim of this paper is to generalize the correspondence between D6-branes and multi-Taub-NUT metrics [1]. The case of particular interest to us is a D6-brane with NS-NS B_2-fields along certain directions of its world-volume. The type IIA geometry describing the back reaction of these branes has been studied in [2, 3], and for massive type IIA in [4]. For a particular choice of low-energy fields, the world-volume theory is a supersymmetric non-commutative Yang-Mill theory [5]. On the other hand, the M theory description of this background is a warped geometry with fluxes that preserves one-half of the supersymmetries. For certain choices of B_2-field, this background appeared in [6]. Our approach differs from the AdS/CFT style approach of [7, 8] because we expect to see a non-commutative gauge-field directly in our Kaluza-Klein spectrum. While it might be possible to see this singlet field in the gravity dual of lower-dimensional branes, the zero-mode analysis looks harder [9].

We will also consider branes with time-dependent fluxes of the kind described in [10], and studied in related work [11, 12, 13]. The world-volume theory is a non-commutative gauge theory, but with a time-dependent non-commutativity parameter [10]. In these cases, the solution is warped in an unusual way not seen in the string compactifications studied to date. We will see that the metric for the *internal* space is also warped with a time-dependent scale factor. A priori, it would have been hard to imagine that this kind of solution could give rise to localized degrees of freedom. Yet the existence of a brane dual suggests that this is indeed the case. The form of this solution also suggests the existence of a far larger class of compactifications where the internal space modulates in a time-dependent way. This direction will be investigated elsewhere [14].

In the following section, we begin by obtaining the explicit metrics and fluxes corresponding to the M theory duals of D6-branes with time-independent fluxes. In section three, we repeat this analysis for the case of D6-branes with time-dependent fluxes. This gives us solutions with explicit time-dependent warping.

In the final section, we investigate the new issues that arise in determining the Kaluza-Klein spectrum for asymmetrically warped metrics of this kind. In particular, we find that fluctuations of the metric and 3-form necessarily couple in these backgrounds. We show this in two ways: first, by a direct analysis of small fluctuations around the warped background. Second, by a duality chasing argument. One of our main goals in this investigation was to see how the non-commutative gauge group arises from supergravity. This coupling of fluctuations makes it clear that a class of gravity gauge transformations constitute part of the gauge theory symmetry group. We also explicitly find the harmonic 2-forms on these spaces which play a key role in giving rise to space-time gauge-fields. The couplings and metrics are deformed from the values we expect for closed strings toward the values we expect for open strings.

While we see a good deal of evidence for the emergence of non-commutative gauge symmetry in our Kaluza-Klein analysis, our account from a direct small fluctuation analysis is still incomplete. Our duality chasing argument suggests that the Seiberg-Witten map between commutative and non-commutative variables map [15] should follow naturally from T-duality! Indeed, we are able to obtain this map from gravity to quadratic order in the gauge-fields. Our analysis is really a natural extension of the argument by Cornalba [16] (see, also [17]) to gravity. Using similar reasoning, it should be possible to describe, in a uniform way, the coupling of the supergravity multiplet to non-commutative gauge-fields. There is also a great deal more to be understood about the time-dependent case. We should also point out that our results have interesting implications for the geometric approach to computing the NS 5-brane partition sum studied in [18]. Lastly, we note that asymmetrically warped backgrounds have been considered recently in string theory [19], and in non-stringy settings [20].

2 Black Branes and T-duality

2.1 The smeared 5-brane solution

Our starting point for most of the following discussion is the smeared black 5-brane solution of IIB supergravity. It can be obtained, for example, by T-dualizing a spherically symmetric (in the transverse directions) 6-brane solution along the brane. The 5-brane solution is determined by the following dilaton, Φ, R-R potentials, C_p, and metric:

$$\begin{aligned}
e^{2\Phi} &= g_s^2 H^{-1} \\
ds^2 &= H^{-\frac{1}{2}}\left(-dx_0^2 + \cdots + dx_5^2\right) + H^{\frac{1}{2}}\left(dx_6^2 + dr^2 + r^2 d\Omega_2^2\right) \\
C_6 &= g_s^{-1}\frac{c}{r}H^{-1} dx_0 \wedge \ldots \wedge dx_5 \\
C_2 &= g_s^{-1} c \cdot \cos\theta dx_6 \wedge d\psi
\end{aligned} \tag{1}$$

where

$$H = 1 + \frac{c}{r} \tag{2}$$

and c is a constant chosen to give the correct value for the 5-brane charge, i.e. so that

$$\frac{1}{(2\pi\ell_s)^2}\int_{\partial\mathcal{M}_\perp} dC_2 = \frac{1}{(2\pi\ell_s)^2}\int_{\partial\mathcal{M}_\perp} *dC_6 = Q_5 \tag{3}$$

where $\mathcal{M}_\perp$ is the space transverse to the brane (parametrized by x_6, r, θ, and ψ). For example, if x_6 is compact with radius R, then $\partial\mathcal{M}_\perp$ is $S^1_R \times S^2_\infty$, then we find

$$c = \frac{1}{2} g_s \frac{\ell_s^2}{R} Q_5. \tag{4}$$

In other cases this may be modified, as we shall see. We will always consider a single brane, so that $Q_5 = 1$.

We now proceed to generate some new solutions from this starting point. Our main tools are T-duality and lifting to M-theory. We outline our conventions in Appendix A.

2.2 The usual story

It is useful for us to begin with a brief discussion of the standard story which we plan to generalize. Typically, one starts with a spherically symmetric 6-brane, but for practice, we will instead T-dualize our smeared 5-brane along x_6. If x_6 is taken to have radius R, we obtain:

$$\begin{aligned} e^{2\Phi} &= \tilde{g}_s^2 H^{-\frac{3}{2}} \\ ds^2 &= H^{-\frac{1}{2}}\left(-dx_0^2 + \cdots + dx_5^2 + dx_6^2\right) + H^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_2^2\right) \\ C_7 &= \tilde{g}_s^{-1}\frac{c}{r}H^{-1} dx_0 \wedge \ldots \wedge dx_5 \wedge dx_6 \\ C_1 &= -\tilde{g}_s^{-1} c \cdot \cos\theta d\psi \end{aligned} \tag{5}$$

where $\tilde{g}_s = \ell_s g_s/R$ and H is given by eqs. (2) and (4). Gratifyingly, it is the 6-brane solution that we expect.

Now lifting to 11D (with $\tilde{g}_s = 1$ for simplicity) we obtain:

$$ds^2 = -dx_0^2 + \cdots + dx_6^2 + H\left(dr^2 + r^2 d\Omega^2\right) + H^{-1}\left(dy + c\cdot\cos\theta d\psi\right)^2. \tag{6}$$

This solution corresponds to a KK monopole solution, with space-time metric $\mathbb{R}^{1,6} \times \mathcal{M}$, where $\mathcal{M}$ is the Taub-NUT manifold.

2.3 The twisted compactification

We will now generalize this solution to the case where the D6-brane supports a rank 2 NS-NS B_2-field. Let us outline the steps: our starting point is a D5-brane oriented along $x_{0,1,2,3,4,5}$ and delocalised along x_6 (this is essentially T-dual to a D6-brane oriented along $x_{0,1,2,3,4,5,6}$ and localised at a point in $x_{7,8,9}$). The directions $x_{5,6}$ form a square torus. We then *twist* the directions in such a way that a second T-duality along the delocalised x_6 direction gives a D6-brane with a non-trivial B_2-field along $x_{5,6}$. We then lift this configuration to M theory where we obtain a warped analogue of the Taub-NUT metric of (6) with 4-form G_4-fluxes. As a matter of notation, note that $G_4 = dA_3$ where A_3 is the 11-dimensional supergravity potential. This will be the solution that should give rise to a non-commutative gauge-field. Let us give a detailed analysis of this procedure now.

Starting with the solution from section 2.1, we make a change of coordinates (found in [6]):

$$\begin{aligned} z_1 &= \cos\alpha x_5 - \sin\alpha x_6 \\ z_2 &= \frac{x_6}{R\cos\alpha}, \end{aligned} \tag{7}$$

or inverting,

$$\begin{aligned} x_5 &= \sec\alpha z_1 + R\sin\alpha z_2 \\ x_6 &= R\cos\alpha z_2. \end{aligned} \tag{8}$$

We take z_2 to be compact with unit radius. In these variables the 5-brane solution becomes

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}}\left(-dx_0^2 + \cdots + dx_4^2 + \sec^2\alpha dz_1^2 + 2R\tan\alpha dz_1 dz_2 + R^2\left(1 + \frac{c}{r}\cos^2\alpha\right)dz_2^2\right) \\ &\quad + H^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_2^2\right) \qquad (9) \\ C_6 &= g_s^{-1}\frac{c}{r}H^{-1}dx_0\wedge\ldots\wedge dx_4\wedge(\sec\alpha dz_1 + R\sin\alpha dz_2) \\ C_2 &= g_s^{-1}Rc\cdot\cos\alpha\cos\theta dz_2\wedge d\psi. \end{aligned}$$

After T-dualizing along z_2, we find

$$
\begin{aligned}
e^{2\Phi} &= \tilde{g}_s^2 H_1^{-\frac{1}{2}} H_2^{-1} \\
ds^2 &= H_1^{-\frac{1}{2}}\left(-dx_0^2+\cdots dx_4^2\right)+H_1^{\frac{1}{2}}H_2^{-1}\left(dz_1^2+dz_2^2\right)+H_1^{\frac{1}{2}}\left(dr^2+r^2d\Omega_2^2\right) \\
B_2 &= H_2^{-1}\tan\alpha dz_1\wedge dz_2 \\
C_7 &= \tilde{g}_s^{-1}\frac{R'}{2r}H_2^{-1}dx_0\wedge\ldots\wedge dx_4\wedge dz_1\wedge dz_2 \\
C_5 &= \tilde{g}_s^{-1}\frac{R'}{2r}H_1^{-1}\tan\alpha dx_0\wedge\ldots\wedge dx_4 \\
C_3 &= \frac{1}{2}\tilde{g}_s^{-1}R'H_2^{-1}\tan\alpha\cos\theta dz_1\wedge dz_2\wedge d\psi \\
C_1 &= -\frac{1}{2}\tilde{g}_s^{-1}R'\cos\theta d\psi
\end{aligned}
\tag{10}
$$

where we have defined

$$
\begin{aligned}
\tilde{g}_s &= \frac{\ell_s}{R}g_s \\
R' &= \tilde{g}_s\ell_s \\
H_1 &= 1+\frac{R'}{2r\cos\alpha} \\
H_2 &= 1+\frac{R'\cos\alpha}{2r}
\end{aligned}
\tag{11}
$$

and have scaled z_2 so that it has the natural T-dual radius of ℓ_s^2/R. The constant c in (9) has been fixed in (10) and (11) by computing $\int dC_1$.

If we now set $\tilde{g}_s = 1$ and lift to M theory we obtain the solution:

$$
\begin{aligned}
ds^2 &= H_1^{-\frac{1}{3}}H_2^{\frac{1}{3}}\left(-dx_0^2+\cdots+dx_4^2\right)+H_1^{\frac{2}{3}}H_2^{-\frac{2}{3}}\left(dz_1^2+dz_2^2\right)+H_1^{\frac{2}{3}}H_2^{\frac{1}{3}}\left(dr^2+r^2d\Omega_2^2\right) \\
&\quad +H_1^{-\frac{1}{3}}H_2^{-\frac{2}{3}}\left(dy+\frac{1}{2}R'\cos\theta d\psi\right)^2 \\
A_3 &= H_2^{-1}\tan\alpha dz_1\wedge dz_2\wedge\left(dy+\frac{1}{2}R'\cos\theta d\psi\right).
\end{aligned}
\tag{12}
$$

Note that the 3-form has a non-trivial field strength with 2 legs in the direction of B_2 and 2 legs along the internal space.

Let us make the map between parameters explicit. Viewing the type IIA configuration as our starting point, we began with a solution whose parameters are $\tilde{g}_s$, ℓ_s, and the angle α which determines the strength of the B-field via

$$
B_2(r=\infty)-B_2(r=0)=\tan\alpha dz_1\wedge dz_2. \tag{13}
$$

Our derivation of the IIA supergravity solution led to a B_2 which vanished at the origin, but shifting B_2 by any constant 2-form also satisfies the equations of motion. When we lift to 11 dimensions, we have new parameters R_{11}, ℓ_p, and the strengh of the 3-form, A_3. In terms of the IIA parameters, the relation is

$$\begin{aligned} R_{11} &= \tilde{g}_s \ell_s \\ \ell_p &= \tilde{g}_s^{\frac{1}{3}} \ell_s \\ A_{z_1 z_2 y}(\infty) - A_{z_1 z_2 y}(0) &= \tan\alpha. \end{aligned} \tag{14}$$

2.4 Generalizing to higher rank B_2-fields

It is a simple matter to repeat the analysis above with additional twisted compactifications, obtaining supergravity solutions corresponding to B_2-fields of rank 4 or 6.

In the rank 4 case, one starts with a black 4-brane smeared in two directions and performs two sets of coordinate redefinitions and T-dualities to obtain the following black 6-brane configuration with flux:

$$\begin{aligned} e^{2\Phi} &= \tilde{g}_s^2 H_0^{\frac{1}{2}} H_1^{-1} H_2^{-1} \\ ds^2 &= H_0^{-\frac{1}{2}} \left(-dx_0^2 + dx_1^2 + dx_2^2\right) + H_0^{\frac{1}{2}} H_1^{-1} \left(dz_3^2 + dz_4^2\right) + H_0^{\frac{1}{2}} H_2^{-1} \left(dz_5^2 + dz_6^2\right) \\ &\quad + H_0^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_2^2\right) \\ B_2 &= H_1^{-1} \tan\alpha_1 dz_3 \wedge dz_4 + H_2^{-1} \tan\alpha_2 dz_5 \wedge dz_6 \\ C_7 &= \tilde{g}_s^{-1} \frac{R'}{2r} H_0 H_1^{-1} H_2^{-1} dx_0 \wedge dx_1 \wedge dx_2 \wedge dz_3 \wedge dz_4 \wedge dz_5 \wedge dz_6 \\ C_5 &= \tilde{g}_s^{-1} \frac{R'}{2r} H_1^{-1} \tan\alpha_2 dx_0 \wedge dx_1 \wedge dx_2 \wedge dz_3 \wedge dz_4 \\ &\quad + \tilde{g}_s^{-1} \frac{R'}{2r} H_2^{-1} \tan\alpha_1 dx_0 \wedge dx_1 \wedge dx_2 \wedge dz_5 \wedge dz_6 \\ C_3 &= -\tilde{g}_s^{-1} \frac{R'}{2r} H_0^{-1} \tan\alpha_1 \tan\alpha_2 dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad + \frac{1}{2} \tilde{g}_s^{-1} R' H_1^{-1} \tan\alpha_1 \cos\theta dz_3 \wedge dz_4 \wedge d\psi + \frac{1}{2} \tilde{g}_s^{-1} R' H_2^{-1} \tan\alpha_2 \cos\theta dz_5 \wedge dz_6 \wedge d\psi \\ C_1 &= -\frac{1}{2} \tilde{g}_s^{-1} R' \cos\theta d\psi. \end{aligned} \tag{15}$$

We have defined

$$
\begin{aligned}
\tilde{g}_s &= g_s \frac{\ell_s^2}{R_4 R_6} \\
H_0 &= 1 + \frac{R'}{2r\cos\alpha_1\cos\alpha_2} \\
H_1 &= 1 + \frac{R'\cos\alpha_1}{2r\cos\alpha_2} \\
H_2 &= 1 + \frac{R'\cos\alpha_2}{2r\cos\alpha_1}.
\end{aligned}
\tag{16}
$$

The lift to M theory has the form (again setting $\tilde{g}_s = 1$)

$$
\begin{aligned}
ds^2 &= H_0^{-\frac{2}{3}} H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} \left(-dx_0^2 + dx_1^2 + dx_2^2\right) + H_0^{\frac{1}{3}} H_1^{-\frac{2}{3}} H_2^{\frac{1}{3}} \left(dz_3^2 + dz_4^2\right) \\
&+ H_0^{\frac{1}{3}} H_1^{\frac{1}{3}} H_2^{-\frac{2}{3}} \left(dz_5^2 + dz_6^2\right) + H_0^{\frac{1}{3}} H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} \left(dr^2 + r^2 d\Omega_2^2\right) \\
&+ H_0^{\frac{1}{3}} H_1^{-\frac{2}{3}} H_2^{-\frac{2}{3}} \left(dy + \frac{1}{2} R' \cos\theta d\psi\right)^2 \\
A_3 &= -\frac{R'}{2r} H_0^{-1} \tan\alpha_1 \tan\alpha_2 dx_0 \wedge dx_1 \wedge dx_2 \\
&+ \left(H_1^{-1} \tan\alpha_1 dz_3 \wedge dz_4 + H_2^{-1} \tan\alpha_2 dz_5 \wedge dz_6\right) \wedge \left(dy + \frac{1}{2} R' \cos\theta d\psi\right).
\end{aligned}
\tag{17}
$$

Similar considerations can be applied to the rank 6 case. We will write down only the form of the 11D solution:

$$
\begin{aligned}
ds^2 &= -H_0^{-1} H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} H_3^{\frac{1}{3}} dx_0^2 + H_1^{-\frac{2}{3}} H_2^{\frac{1}{3}} H_3^{\frac{1}{3}} \left(dz_1^2 + dz_2^2\right) + H_1^{\frac{1}{3}} H_2^{-\frac{2}{3}} H_3^{\frac{1}{3}} \left(dz_3^2 + dz_4^2\right) \\
&+ H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} H_3^{-\frac{2}{3}} \left(dz_5^2 + dz_6^2\right) + H_1^{\frac{1}{3}} H_2^{\frac{1}{3}} H_3^{\frac{1}{3}} \left(dr^2 + r^2 d\Omega^2\right) \\
&+ H_0 H_1^{-\frac{2}{3}} H_2^{-\frac{2}{3}} H_3^{-\frac{2}{3}} \left(dy + \frac{1}{2} R' \cos\theta d\psi + \frac{R}{2r} H_0^{-1} t_1 t_2 t_3 dx_0\right)^2
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
A_3 &= \frac{R}{2r} H_1^{-1} t_2 t_3 dx_0 \wedge dz_1 \wedge dz_2 + \frac{1}{2} R H_1^{-1} t_1 \cos\theta dz_1 w\ dz_2 \wedge d\psi \\
&+ \frac{R}{2r} H_2^{-1} t_1 t_3 dx_0 \wedge dz_3 \wedge dz_4 + \frac{1}{2} R H_2^{-1} t_2 \cos\theta dz_3 w\ dz_4 \wedge d\psi \\
&+ \frac{R}{2r} H_3^{-1} t_1 t_2 dx_0 \wedge dz_5 \wedge dz_6 + \frac{1}{2} R H_3^{-1} t_3 \cos\theta dz_5 w\ dz_6 \wedge d\psi
\end{aligned}
\tag{19}
$$

with

$$
\begin{aligned}
H_0 &= 1+\frac{R'}{2rc_1c_2c_3} \\
H_1 &= 1+\frac{R'c_1}{2rc_2c_3} \\
H_2 &= 1+\frac{R'c_2}{2rc_1c_3} \\
H_3 &= 1+\frac{R'c_3}{2rc_1c_2}.
\end{aligned}
\tag{20}
$$

(21)

and where $c_i \equiv \cos\alpha_i$ and $t_i \equiv \tan\alpha_i$. Note the appearance of a cross term in the metric between dy and dx_0. This is a consequence of the fact that a D6-brane with a rank 6 B_2-field induces a non-zero C_1 field (said differently, the D6-brane with a rank 6 B_2 carries D0-brane charge). This C_1 lifts to become the off-diagonal metric terms seen above.

3 Time-Dependent Cases

Another new solution can be obtained by performing the null-brane quotient [21,22] on the D5-brane, with the parabolic quotient acting along the brane (in directions x^+, x^-, and $x = x_2$), and the shift direction transverse ($z = x^6$). We recall that the null-brane quotient acts by

$$
\begin{aligned}
x^+ &\to x^+ \\
x^- &\to x^- + 2\pi x + 2\pi^2 x^+ \\
x &\to x + 2\pi x^+ \\
z &\to z + 2\pi R.
\end{aligned}
\tag{22}
$$

Now let us switch to the natural invariant coordinates:

$$
\begin{aligned}
\tilde{x}^- &= x^- - \frac{z}{R}x + \frac{z^2}{2R^2}x^+ \\
\tilde{x} &= x - \frac{z}{R}x^+ \\
\tilde{z} &= \frac{z}{R}.
\end{aligned}
\tag{23}
$$

In these coordinates, the quotient action is simply $\tilde{z} \to \tilde{z} + 2\pi$.

In terms of these invariant coordinates (for simplicity, we will drop the tildes from now on), the smeared 5-brane is given by

$$\begin{aligned}
e^{2\Phi} &= g_s H^{-1} \\
ds^2 &= H^{-\frac{1}{2}}\left(-2dx^+dx^- - 2x dx^+ dz + dx^2 + 2x^+ dx dz + (x^+)^2 dz^2 + dx_3^2 + dx_4^2 + dx_5^2\right) \\
&\quad + H^{\frac{1}{2}}\left(R^2 dz^2 + dr^2 + r^2 d\Omega_2^2\right) \\
C_6 &= -g_s^{-1}\frac{c}{r}H^{-1}dx^+ \wedge \left(dx^- \wedge dx + x^+ dx^- \wedge dz - x dx \wedge dz\right) \wedge dx_3 \wedge dx_4 \wedge dx_5 \\
C_2 &= g_s^{-1} Rc \cdot \cos\theta dz \wedge d\psi,
\end{aligned} \tag{24}$$

where,

$$H = 1 + \frac{c}{r}, \tag{25}$$

and c is the same as the case of an ordinary compact direction (see equation (4)).

Next, we T-dualize along z. We obtain a solution for a D6-brane with B_2 flux:

$$\begin{aligned}
e^{2\Phi} &= \tilde{g}_s^2 H^{-\frac{1}{2}} h^{-1} \\
ds^2 &= H^{-\frac{1}{2}}\left(-2dx^+dx^- + dx_3^2 + dx_4^2 + dx_5^2\right) + H^{\frac{1}{2}} h^{-1}\left(dx^2 + dz^2\right) \\
&\quad + H^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_2^2\right) + H^{-\frac{1}{2}} h^{-1} R^{-2}\left(-x^2 (dx^+)^2 + 2x^+ x dx^+ dx\right) \\
B_2 &= h^{-1}R^{-1}\left(-x dx^+ + x^+ dx\right) \wedge dz \\
C_7 &= -\tilde{g}_s^{-1}\frac{c}{r}h^{-1} dx^+ \wedge dx^- \wedge dx \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dz \\
C_5 &= \tilde{g}_s^{-1}R^{-1}\frac{c}{r}H^{-1}dx^+ \wedge \left(x^+ dx^- - x dx\right) \wedge dx_3 \wedge dx_4 \wedge dx_5 \\
C_3 &= \tilde{g}_s^{-1}R^{-1} c h^{-1}\cos\theta\left(-x dx^+ + x^+ dx\right) \wedge dz \wedge d\psi \\
C_1 &= -\tilde{g}_s^{-1} c \cdot \cos\theta d\psi.
\end{aligned} \tag{26}$$

We have defined

$$\tilde{g}_s = \frac{\ell_s}{R} g_s \tag{27}$$

which is the value of $\exp\left[\Phi(\infty)\right]$, and

$$h = 1 + \frac{c}{r} + \left(\frac{x^+}{R}\right)^2 \tag{28}$$

and where we have again rescaled z so that it has the natural T-dual radius ℓ_s^2/R.

Note that as $R \to \infty$, we have $h \to H$, $B_2 \to 0$, and the solution reduces to that of a standard spherically symmetric black 6-brane, as expected.

Finally, we would like to lift this configuration to M-theory. We set $\tilde{g}_s = 1$ to avoid cluttering the formulae:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{3}} h^{\frac{1}{3}} \left(-2dx^+ dx^- + dx_3^2 + dx_4^2 + dx_5^2\right) + H^{\frac{2}{3}} h^{-\frac{2}{3}} \left(dx^2 + dz^2\right) \\ &\quad + H^{-\frac{1}{3}} h^{-\frac{2}{3}} R^{-2} \left(-x^2 (dx^+)^2 + 2x^+ x dx^+ dx\right) + H^{\frac{2}{3}} h^{\frac{1}{3}} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2\right) \\ &\quad + H^{-\frac{1}{3}} h^{-\frac{2}{3}} \left(dy + c \cdot \cos\theta d\psi\right)^2 \qquad (29) \\ A_3 &= -R^{-1} h^{-1} \left(-x dx^+ + x^+ dx\right) \wedge dz \wedge \left(dy + c \cdot \cos\theta d\psi\right). \end{aligned}$$

As a simple check on the algebra, we note that the equation of motion for the M theory 3-form is obeyed; namely that $d * dA_3 = \frac{1}{2} dA_3 \wedge dA_3$ (both sides vanish in this case). This solution can be further generalized by adding static B_2-fields along various directions of the D6-brane.

4 Kaluza-Klein Reduction

4.1 Some preliminary comments

The metrics and fluxes obtained in our prior discussion define both M theory and type IIA compactifications. Our subsequent discussion assumes an M theory compactification, but similar comments apply to type IIA. It is worth first recalling how we obtain localized 7-dimensional modes for the case of vanilla Taub-NUT. We begin by considering small fluctuations around our supergravity solution,

$$\left(\begin{array}{c} \delta g \\ \delta A_3 \end{array} \right),$$

where δg parametrizes metric fluctuations, while δA_3 parametrizes 3-form fluctuations. In conventional situations without background fluxes, the two fluctuations decouple, and can be analyzed separately.

So we consider 3-form fluctuations of the form [23],

$$\delta A_3 = A_1 \wedge \omega + B_2 \wedge \xi, \qquad (30)$$

with A_1 a 7-dimensional gauge-field, and B_2 a 10-dimensional 2-form. The normalizable closed 2-form ω is related to the one-form ξ by the condition

$$\omega = d\xi.$$

Despite appearances, ω is not trivial in cohomology because ξ is not normalizable. Roughly, reducing on ξ gives us 10-dimensional propagating fields, while reducing on ω gives 7-dimensional fields. The gauge invariant field strength then takes the form,

$$d\delta A_3 = (F_2 + B_2) \wedge \omega,$$

which agrees with our expectations from string theory.

The $U(1)$ gauge symmetry

$$A_1 \to A_1 + d\lambda_0$$

visible at low-energies arises from the symmetry

$$A_3 \to A_3 + d\lambda_2$$

since

$$(A_1 + d\lambda_0) \wedge \omega = \delta A_3 + d(\lambda_0 \omega).$$

Note that the gauge symmetry of the full theory is much larger since the gauged 7-dimensional Poincaré group is still unbroken by this background. However, these two symmetry groups can be considered separately in the low-energy theory.

To proceed, let us actually construct the 2-form ω, which was found for Taub-NUT in [24, 25]. We will use the same approach to find forms on our generalized Taub-NUT metrics. The space of harmonic two-forms can be decomposed into two components, each containing either self-dual or anti-self-dual forms. Topologically, Taub-NUT is equivalent to $\mathbb{R}^4$ so any closed 2-form ω is exact, and can be written in the form $d\xi$. If ω is to be non-trivial then ξ cannot be normalizable. Our search therefore reduces to finding one-forms, ξ, satisfying $d\xi = \pm *_4 d\xi$. To generalize our discussion in a way that will be useful later, let us write our Taub-NUT space in the form

$$ds^2 = G_1(r)\left(dr^2 + r^2 d\Omega_2^2\right) + G_2(r)\left(dy + \beta \cdot \cos\theta d\psi\right)^2. \tag{31}$$

Here we assume that y has periodicity $g_s \ell_s$, which means that $\beta = \frac{1}{2} g_s^{2/3} \ell_s$ for the metric (31). For this metric, we define vierbeins

$$e^r = G_1^{\frac{1}{2}} dr, \quad e^\theta = r G_1^{\frac{1}{2}} d\theta, \quad e^\psi = r \sin\theta G_1^{\frac{1}{2}} d\psi, \quad e^y = G_2^{\frac{1}{2}}\left(dy + \beta \cdot \cos\theta d\psi\right) \cdot \tag{32}$$

We make the following ansatz for the form of ξ

$$\begin{aligned}
\xi &= g(r)\,(dy + \beta\cdot\cos\theta d\psi) \\
d\xi &= g'(r)dr\wedge(dy+\beta\cdot\cos\theta d\psi) - \beta g(r)\sin\theta d\theta\wedge d\psi \\
&= G_1^{-\frac{1}{2}}G_2^{-\frac{1}{2}}g'(r)e^r\wedge e^y - \frac{\beta}{r^2}G_1^{-1}g(r)e^\theta\wedge e^\psi \\
*d\xi &= G_1^{-\frac{1}{2}}G_2^{-\frac{1}{2}}g'(r)e^\theta\wedge e^\psi - \frac{\beta}{r^2}G_1^{-1}g(r)e^r\wedge e^y.
\end{aligned} \tag{33}$$

For $\omega = d\xi$ to be SD (ASD), we require that $g(r)$ satisfy

$$\begin{aligned}
g'(r) &= \mp\frac{\beta}{r^2}G_1^{-\frac{1}{2}}G_2^{\frac{1}{2}}g(r) \\
&\Longrightarrow\ g = \exp\left[\mp\beta\int^r G_1^{-\frac{1}{2}}G_2^{\frac{1}{2}}\frac{dr}{r^2}\right].
\end{aligned} \tag{34}$$

To check normalizability, we integrate

$$-\int\omega\wedge\omega = 2\beta\int g'(r)g(r)\sin\theta dr\,d\theta\,d\psi\,dy = 8\pi^2\beta g_s\ell_s\,[g(r)]^2\big|_0^\infty. \tag{35}$$

It will turn out that in all of the cases that we consider, the ASD solution is normalizable, and the SD solution is not. Also, we will find that generally $g(0)=0$, so the above formula reduces to

$$\int\omega\wedge*\omega = 8\pi^2\beta g_s\ell_s\,[g(\infty)]^2. \tag{36}$$

In order to fix the normalization constant, we use the following argument which appears in [26, 23]. The action for a membrane wrapping the directions r, y, and a transverse direction should give rise, on reduction along y, to the action of an open string ending on the D6-brane. The membrane action is

$$S = \tau_{M2}\int A_3 = \frac{g_s^{-\frac{2}{3}}}{(2\pi)^2\ell_p^3}\int g'(r)dr\wedge dy\int A_1 = \frac{g(\infty)}{2\pi g_s^{\frac{2}{3}}\ell_s^2}\int A_1 \tag{37}$$

while the open string world-sheet action has a piece

$$S = \int_{\partial\Sigma} A_1. \tag{38}$$

On comparing these two expressions, we find that $g(\infty) = 2\pi g_s^{2/3}\ell_s^2$, and so

$$g(r) = 2\pi\ell_p^2\exp\left[-\beta\int_r^\infty G_1^{-\frac{1}{2}}G_2^{\frac{1}{2}}\frac{dr}{r^2}\right]. \tag{39}$$

Returning to the case of standard Taub-NUT, we have $G_1 = H$ and $G_2 = H^{-1}$. The integral is particularly simple and gives,

$$g(r) = 2\pi \ell_p^2 H^{-1}. \tag{40}$$

Finally, let us see reduce part of the 11-dimensional SUGRA action using ω. Ignoring B_2 for the moment, the kinetic term for the 4-form gives

$$\begin{aligned} S &= -\frac{1}{2}\frac{1}{2\kappa_{11}^2}\int d\delta A_3 \wedge *_{11} d\delta A_3 \\ &= -\frac{1}{2(2\pi)^8 g_s^3 \ell_s^9}\, g_s^{-1} \int_{\mathbb{R}^{6,1}} dA_1 \wedge *_7 dA_1 \int_{\mathrm{TN}} \omega \wedge *_4 \omega \\ &= -\frac{1}{2(2\pi)^4 g_s \ell_s^3} \int_{\mathbb{R}^{6,1}} dA_1 \wedge *_7 dA_1. \end{aligned} \tag{41}$$

This is the correct 7D YM action with the correct coupling constant, $g_{YM}^2 \sim \ell_p^3$.

Let us imagine, for the moment, that we know the complete 7-dimensional effective action to all orders in ℓ_s. We could now contemplate moving in the space of SUGRA solutions by turning on a background $< B_2 > \neq 0$. By turning on this background, we reduce the full $Spin(6,1)$ Lorentz group to some subgroup. Nevertheless, the low-energy physics should be captured by the complete effective action which, from string theory, we expect takes the form

$$S_{eff} = \frac{1}{g_s (2\pi)^6 \ell_s^7} \int d^7x \sqrt{\det\left(\mathbf{1} + 2\pi \ell_s^2 (F_2 + B_2)\right)} + O(\partial F_2). \tag{42}$$

This is a completely commutative description of the low-energy physics which has, among other features, linear couplings to the background B_2. This is one way to describe the physics of our warped compactifications, but it requires knowledge of physics beyond supergravity. We now turn to a direct analysis of small fluctuations around the warped solutions.

4.2 The static warped case

Reduction on warped metrics introduces a number of novel issues to which we now turn. Let us begin by overviewing the key features of the supergravity solutions described earlier. The D6-brane world-volume supports a $6+1$-dimensional abelian gauge-field. We turn on an NS-NS B_2-field along certain directions of the world-volume. The B_2-field is characterized by its rank (2, 4, or 6). The presence of the B_2-field explicitly breaks the $Spin(6,1)$ Lorentz symmetry to a subgroup that depends on the rank. The corresponding M theory duals are the warped metrics of section 2 which generalize the Taub-NUT space. By a warped metric, we mean a metric that takes the form:

$$ds^2 = f(r) ds^2_{\text{space-time}} + ds^2_{\mathcal{M}}. \tag{43}$$

The coordinate r on which the warp factor f depends is along the compactification space $\mathcal{M}$. We use the term "compactification" here in an abuse of terminology since $\mathcal{M}$ for us is a non-compact manifold. Nevertheless, $\mathcal{M}$ supports normalizable modes which propagate in space-time.

The B_2-field in type IIA lifts to the 3-form, A_3, with field strength G_4 in M theory. The second important feature of these solutions is the presence of G_4 flux. It is typical in supergravity that warping is accompanied by fluxes. This complicates a Kaluza-Klein analysis since the metric and 3-form modes can mix in a non-trivial way. A discussion of how to find massless modes in situations like this appears in [27], which we will use as a guide. The first change from the usual case of (43) is that our warping is asymmetric. Let us parametrize space-time by coordinates $x_0, \ldots, x_4, z_1, z_2$, and let B_2, for simplicity, be non-vanishing in the z_1, z_2 directions. The metric takes the form,

$$ds^2 = f_1(r) ds^2_{\{x_0,x_3,x_4,x_5,x_6\}} + f_2(r) ds^2_{\{z_1,z_2\}} + ds^2_{\mathcal{M}}. \tag{44}$$

The accompanying M theory G_4 has 2 legs in the z_1, z_2 directions and 2 legs in $\mathcal{M}$. We want to describe the localized vector multiplets. More precisely, we want to describe the leading terms in the action for a fluctuation δA_3. The leading terms in the action are quadratic in the fluctuation with all the background parameters absorbed into the metric on the space of fluctuations,

$$S_{eff} = \int d\delta A_3 \wedge * d\delta A_3 + \ldots .$$

This is quite different from the commutative description of (42) in which the background $< B_2 >$ appears explicitly even for the leading terms. However, this is the usual procedure for determining the effective action and light degrees of freedom around a given SUGRA solution. This existence of (at least) two descriptions is very much along the lines described in [15]. This approach should, morally, give the non-commutative description. If this is true then at least the coupling constants and metric should be deformed toward the values we expect for open strings.

In the unwarped case, vectors arose by reducing A_3 on harmonic 2-forms of $\mathcal{M}$. We need to be more careful here. An A_3 fluctuation, δA_3, can be written in the form

$$\delta A_3 = \phi(x) C^{(3)} + A_1(x)\, C^{(2)} + \delta B_2\, C^{(1)} \tag{45}$$

where $C^{(m)}$ is an m-form on the internal space. The fields ϕ and A_1 have arbitrary dependence on (x, z). Since we want to consider vectors, let us set $\phi = 0$ and $\delta B_2 = 0$. Note that any vector A_1 is automatically part of a supermultiplet that includes 3 scalars. These additional scalars come from metric fluctuations. Now there is an immediate worry; namely, is A_1 a vector under $Spin(6,1)$ or under $Spin(4,1)$? Since we have broken the symmetry to $Spin(4,1)$ by an explicit $< B_2 >$, it seems more natural to consider an expansion like

$$\delta A_3 = A_\mu dx^\mu\, C_1^{(2)} + A_1 dz^1\, C_2^{(2)} + A_2 dz^2\, C_3^{(2)}, \tag{46}$$

where $\mu = 0, 1, 2, 3, 4$ and the $C_i^{(2)}$ are a priori independent. However, this decomposition does not seem natural if we want to see a gauge symmetry in the effective theory that mixes the z_i and x_μ directions. In this case, for example, both $A_\mu dx^\mu$ and $A_1 dz^1$ are needed to give a gauge covariant field strength, $F_{1\mu}\, dz^1 \wedge dz^\mu$. Another possibility is to insist on an expansion that involves just field strengths rather than potentials, but that seems unnatural. If we want an expansion in terms of the supergravity potential A_3 rather than the field strength G_4, it seems more natural to start by considering a fluctuation of the form

$$\delta A_3 = \left(A_\mu dx^\mu + A_1 dz^1 + A_2 dz^2 \right) \wedge \omega, \tag{47}$$

where we introduce one internal 2-form ω. We take this choice as our starting point, although we will see in section 4.3.1 that the more general ansatz of (46) is actually possible.

The fluctation is expanded in eigenmodes of the equation of motion

$$d\, \hat{*}\, d\, \delta A_3 = -G_4 \wedge d\delta A_3, \tag{48}$$

where $\hat{*}$ denotes the Hodge dual with respect to the warped metric. The right hand side of (48), which comes from the Chern-Simons interaction

$$\int G_4 \wedge G_4 \wedge A_3,$$

in M theory, is a $(4,4)$ form where (p, q) denotes a p form in space-time and a q form on $\mathcal{M}$. Using (47) which is a $(1,2)$ form, we see that the left hand side of (48) never gives a $(4,4)$ form so these terms decouple initially.

The left hand side can be expanded to give,

$$\begin{aligned} d\,\hat{*}\,d\delta A_3 &= d*F_2 \wedge (f_1^{a+1/2} f_2^{1-a} \, *\omega) + *F_2 \wedge d(f_1^{a+1/2} f_2^{1-a} \, *\omega) \\ &\quad + d*A_1 \wedge (f_1^{b+3/2} f_2^{1-b} * d\omega) + *A_1 \wedge d(f_1^{b+3/2} f_2^{1-b} * d\omega). \end{aligned} \tag{49}$$

The Hodge star products are now with respect to the unwarped space-time metric and the metric for $\mathcal{M}$. The numbers a, b depend on the number of legs that F_2 and A_1, respectively, have in the directions of $< B_2 >$. To make the last two terms vanish, we require that $d\omega = 0$.

The first term of (49) gives the equation of motion for A_1, and it is already clear that the metric and couplings will be asymmetric. We can see this explicitly. Let us assume, for the moment, that the only way gauge-field kinetic terms arise is from this first term. The analogue of (41) now gives a matrix of coupling constants,

$$\int \omega \wedge (f_1^{a+1/2} f_2^{1-a} \, *\omega). \tag{50}$$

The value of the coupling now depends on which component of F_2 we consider through the value of $a = 0, 1$, or 2. This is a feature forced on us by the asymmetric warp factors. For small B_2, we can evaluate (50) for a harmonic ω. The harmonic form on our internal space for the rank 2 B_2-field is determined by using the same ansatz as in (33). The function $g(r)$ is again given by (39) but with different warp factors. The result in this case (with the correct normalization) is

$$g(r) = 2\pi\ell_p^2\,(1+\cos\alpha)\left(1+\frac{1}{\cos\alpha}\right)\frac{2r}{\left(\sqrt{2r\cos\alpha + R'} + \sqrt{\frac{2r}{\cos\alpha} + R'}\right)^2}. \tag{51}$$

Using the explicit form for ω, we see that

$$S = \frac{1}{4g_{YM}^2}\sum_{a=0}^{2} f^{(a)}(B)\int_{\mathbb{R}^7} F^{(a)} \wedge *F^{(a)}. \tag{52}$$

The index a on F again refers to how many legs the field strength has lying along the B_2-field (e.g. $F^{(2)} \sim dz_1 \wedge dz_2$). The coupling g_{YM}^2 is the usual value for $B = 0$ so all of the B-dependence is absorbed into the functions $f^{(a)}$.

Explicitly we find, in the limit of small B that:

$$\begin{aligned} f^{(0)} &= 1 + \frac{B^2}{6} - \frac{B^4}{16} + \text{\O}(B^6) \\ f^{(1)} &= 1 - \frac{B^2}{6} + \frac{5B^4}{48} + \text{\O}(B^6) \\ f^{(2)} &= 1 - \frac{B^2}{2} + \frac{7B^4}{16} + \text{\O}(B^6). \end{aligned} \tag{53}$$

We now demand that our effective space-time action be covariant taking the form

$$S \sim \frac{1}{G_s} \int_{\mathbb{R}^7} \sqrt{G} G^{\alpha\beta} G^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \tag{54}$$

for some coupling G_s and some metric G which can both depend on B. Note that the indices $\alpha, \beta, \gamma, \delta$ in (54) run over all 7 space-time coordinates. If we assume that the B-dependence is of the form we expect from open string physics [15]

$$\begin{aligned} G_s &= g_s u(B) & (55) \\ G_{\alpha\beta} dx^\alpha dx^\beta &= dx^\mu dx^\mu + v(B)(dz_1^2 + dz_2^2) & (56) \end{aligned}$$

with $u(0) = v(0) = 1$, then $G^{zz} = v(B)^{-1}$, $\sqrt{G} = v(B)$, then we expect the functional dependence of the action (54) on B to be

$$S \sim \frac{1}{4g_{YM}^2} u(B)^{-1} \int \left[v(B) F_{xx} F_{xx} + F_{xz} F_{xz} + v(B)^{-1} F_{zz} F_{zz} \right]. \tag{57}$$

We can solve for these parameters in terms of the $f^{(a)}$ above

$$\begin{aligned} u(B) &= (f^{(1)}(B))^{-1} & (58) \\ v(B) &= \frac{f^{(0)}(B)}{f^{(1)}(B)} = \frac{f^{(1)}(B)}{f^{(2)}(B)}. & (59) \end{aligned}$$

The second of these equations imposes a consistency check on our solution. Indeed, if we expand our functions to arbitrary order in B

$$f^{(a)}(B) = 1 + \sum_{i=1}^{\infty} c_i^{(a)} B^{2i} \tag{60}$$

then to order B^2, the consistency check is simply that $c_1^{(0)} + c_1^{(2)} = 2c_1^{(1)}$, which is satisfied by our solution (53). However, at order B^4, the functions $f^{(a)}$ of (53) do not satisfy this constraint. This is not surprising for two reasons: first, higher derivative corrections to supergravity like terms of the schematic form $\int ||G_4||^8$ can, in principal, contribute at $O(B^4)$. Second, as we shall now see, there are many additional contributions to the gauge-field kinetic terms even at the level of supergravity.

To see this, let us return to our analysis of equation (49). More interesting, and problematic, than the first term is the second term. This term is a $(5,3)$ form, and so does not give a non-vanishing term in the action when wedged with δA_3. However, it does give couplings between the supergravity fields and the fields localized on the brane in a way described in [23, 28]. For example, a term on the brane of the form

$$\int \delta B_2 \wedge *F$$

arises this way. Usually, without warping, we would pick a gauge where ω is harmonic

$$d * \omega = 0 \tag{61}$$

to make this term vanish. This still works in the commutative directions where $a = 0$ if we choose the gauge,

$$d(* f_1^{1/2} f_2 \, \omega) = 0, \tag{62}$$

but not for the non-commutative terms with $a = 1, 2$.[3]

How are we to remedy this problem? Implicitly, we have decoupled metric fluctuations from our discussion so far, but now we are forced, by the asymmetric warping, to reintroduce δg fluctuations to satisfy the equations of motion. Said differently, there is a coupling of the form

$$\delta(\hat{*} G_4) \wedge d\delta A_3 \tag{63}$$

where $\delta(\hat{*} G_4)$ is a metric fluctuation. The background flux read from (12) has the form

$$G_4 = B_{12} \wedge \omega_b.$$

This leads to a non-vanishing coupling between δg and $d\delta A_3$.

The second term in (49) is a $(5,3)$ form. Let us take the case of $a = 1$. To cancel these terms by appropriate metric fluctuations, note that (63) contributes the following terms to the δA_3 equation of motion:

$$\begin{aligned} d\big(&\sqrt{g}\, \epsilon_{\mu_1 \cdots \mu_6 2}\, dx^{\mu_1} \cdots dx^{\mu_5}\, \delta g^{\mu_6 1} g^{22}\, B_{12} \, * \omega_b + \\ &\sqrt{g}\, \epsilon_{\mu_1 \cdots 1 \mu_7}\, dx^{\mu_1} \cdots dx^{\mu_5}\, g^{11} \delta g^{\mu_7 2}\, B_{12} \, * \omega_b\big). \end{aligned} \tag{64}$$

These terms give rise to both $(5,3)$ and $(6,2)$ forms. The first two $(5,3)$ terms of (64) can cancel the terms of (49) with $a = 1$ but not $a = 2$ if

$$\delta g^1_\mu \, B_{12} \sim F_{2\mu}, \qquad \delta g^2_\mu \, B_{12} \sim F_{1\mu}.$$

[3]There is some freedom here to choose a gauge so that the problematic terms with $a = 1$ or $a = 2$ vanish rather than $a = 0$. However, the problematic $a = 0$ terms cannot be cancelled by a metric fluctuation in the way described subsequently. Rather, if these gauge choices are sensible then the $a = 0$ term must cancel against a new induced δA_3 fluctuation.

The proportionality constant is a function of r. The $(6,2)$ terms have the form $d * F_2 \wedge \omega'$ for some internal space 2-form ω'. These terms modify the gauge-field kinetic terms.

In a similar way, to cancel the terms with $a = 2$, we also need to consider δg^{11} and δg^{22} fluctuations. To determine the correct combination of metric fluctuations, we should analyze the equation for metric fluctuations which comes from terms in the action with the schematic form,

$$\int \delta g \Delta_L \delta g + \delta(\hat{*} G_4) \wedge \delta(\hat{*} G_4) + \delta(\hat{*} G_4) \wedge d\delta A_3, \tag{65}$$

where Δ_L is the operator obtained by expanding $\sqrt{g}R$ to quadratic order in the metric fluctuations. With a suitable gauge choice, Δ_L is the Lichnerowicz Laplacian. The last two terms arise because of the non-vanishing background G_4.

Metric fluctuations are not the only fluctuations induced by our original choice (47). The $(2,2)$ form $d\delta A_3$ plugged into the right hand side of (48) gives a $(4,4)$ form. To cancel this term, we need to supplement the original δA_3 with an induced term of the form

$$\delta A_3' \sim *(B_{12} \wedge F_2)\, \zeta, \tag{66}$$

where ζ is a 0-form on the internal space. This $(3,0)$ form is chosen so that the $(4,4)$ pieces of

$$d\, \hat{*}\, d\, \delta A_3' = -G_4 \wedge d\delta A_3$$

cancel up to a possible $(5,3)$ term on the left hand side. In turn, this new induced fluctuation $\delta A_3'$ mixes, via the Chern-Simons term and the 4-form kinetic term, with δA_3 and δg. Rather than continue along this path, which is quite involved, let us turn to an alternate method based on duality.

4.3 Insights from duality chasing

Let us recall the method that we used to generate the warped supergravity solutions. We started with the 11-dimensional Taub-NUT solution. After reducing along the circle direction to get a IIA D6-brane, we T-dualized to get a smeared D5-brane of type IIB. We then performed a change of coordinates, and compactified a new direction. In the static case, we effectively T-dualized back to type IIA on a non-rectangular torus generating a B_2-field. In the time-dependent case, we switched to coordinates in which the null-brane quotient is a simple circle identification. Again, we T-dualized back to IIA. Generating higher rank B_2-fields simply required more T-dualities. In either case, the result is a IIA D6-brane with

B_2-fields along some world-volume directions. Finally, we lifted these configurations to 11 dimensions.

At the starting point of this duality chain, the correspondence between 11-dimensional supergravity and the low-energy theory on the brane world-volume is well understood. As discussed in section 4.1, the gauge fluctuations on the brane correspond to 3-form fluctuations of the form $\delta C_3 = A_1 \wedge \omega$, where A_1 becomes the gauge-field on the brane, and ω is the normalizable harmonic 2-form on Taub-NUT. The overall normalization of ω is determined by comparing the membrane and open string actions. On reduction, the correct Yang-Mills coupling and action emerge.

To take advantage of our understanding in the basic case, and to learn more about the warped backgrounds, it is natural to take the known localized fluctuations from Taub-NUT and push them through the chain of dualities. By construction, we should obtain fluctuations of the 11-dimensional fields which are localized on the brane, and which solve the equations of motion (so that they give rise to massless 7-dimensional fields). Let us perform this exercise first for the static case with a rank 2 B_2-field, and then for the time-dependent case.

4.3.1 Duality chasing the static background

We start with the ordinary Taub-NUT gauge fluctuations studied earlier,

$$\delta A_3 = \left(A_\mu dx^\mu + A_5 dx^5 + A_6 dx^6\right) \wedge \omega \tag{67}$$

where

$$\omega = 2\pi\ell_p^2\, d\left[H_1^{-1}\left(dy + \frac{R}{2\cos\alpha}\cos\theta d\psi\right)\right]. \tag{68}$$

The warp factor H_1 here, and H_2 appearing below, are defined in section 2.3. Note that R has been rescaled to $R/\cos\alpha$ to agree with the twisted solution. This is the same procedure that was followed in section 2.3, but this time we will carry the fluctuations with us under the successive dualities.

After chasing this fluctuation through the duality chain, we find that the A_μ fluctuations appear as 3-form fluctuations in the new background:

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2 A_\mu dx^\mu \wedge d\left[H_1^{-1}\left(dy + \frac{1}{2}R\cos\theta d\psi\right)\right] \\ &\quad -2\pi\ell_p^2 \sin\alpha H_1^{-1}\left(*_5 F_2\right).\end{aligned} \tag{69}$$

In the second line, the $*_5$ acts only in the μ, ν directions. Explicitly in components

$$\delta A_{\mu\nu\rho} = -2\pi\ell_p^2 \sin\alpha H_1^{-1} \epsilon_{\mu\nu\rho}{}^{\lambda\sigma} \partial_\lambda A_\sigma. \tag{70}$$

This induced 3-form fluctuation had to have been there to cancel the contribution to the equation of motion from the Chern-Simons coupling. This is precisely the induced $(3,0)$ form $\delta A_3'$ described in (66). However, here we have an explicit form for the fluctuation. We can indeed check that (69) satisfies

$$d \,\hat{*}\, d\, \delta A_3 = G_4 \wedge \delta A_3. \tag{71}$$

The A_5 and A_6 fluctuations give rise to both a 3-form fluctuation, and metric fluctuations, in a particular combination. The 3-form component is given by

$$\begin{aligned} \delta A_3 \;\; &= \;\; 2\pi\ell_p^2 \left(A_5 dz_1 + A_6 dz_2\right) \wedge \left[\frac{R}{2r^2} H_1^{-1} H_2^{-1} dr \wedge \left(dy + \frac{1}{2} R\cos\theta d\psi\right) \right. \\ &\left. \quad - \frac{R}{2\cos\alpha} H_1^{-1} \sin\theta d\theta \wedge d\psi \right] \end{aligned} \tag{72}$$

while the metric fluctuations are most simply written as a pair of vielbein fluctuations,

$$\begin{aligned} e^{z_1} \;\; &= \;\; H_1^{\frac{1}{3}} H_2^{-\frac{1}{3}} \left[dz_1 + 2\pi\ell_p^2 \frac{R}{2r^2} H_1^{-2} \tan\alpha A_6 dr \right] \\ e^{z_2} \;\; &= \;\; H_1^{\frac{1}{3}} H_2^{-\frac{1}{3}} \left[dz_2 - 2\pi\ell_p^2 \frac{R}{2r^2} H_1^{-2} \tan\alpha A_5 dr \right]. \end{aligned} \tag{73}$$

That a combination of metric and 3-form fluctuations are needed is in accord with our earlier direct analysis.

Suppose we consider a different set of fluctuations that differ by a gauge transformation

$$A_\mu{}' = A_\mu + \partial_\mu \lambda, \qquad A_i{}' = A_i + \partial_i \lambda.$$

Before chasing this fluctuation through the duality chain, we know that this corresponds to the same supergravity solution because it differs from our original configuration by a 3-form gauge transformation

$$A_3 \to A_3 + d(\lambda\omega).$$

After performing the dualities, we must therefore also have the same solution. However, now even the metric differs for A and A'. The resolution must be that the two answers differ by some combination of 3-form shift and diffeomorphism. In this way, what we would have thought of as a $U(1)$ gauge symmetry becomes mixed with diffeomorphisms, and in this way, the resulting theory can be reinterpreted as having a non-commutative gauge group.

With some foresight, let us define $A_1 = A_5 \cos\alpha$, $A_2 = A_6 \cos\alpha$. The latter of these two redefinitions is natural from the change of variables (8) between x^6 and z^2. The definition of A_1 can then be justified from the symmetry between z^1 and z^2. The two equations above may now be written

$$e^{z_i} = H_1^{\frac{1}{3}} H_2^{-\frac{1}{3}} \left[dz^i + 2\pi\ell_p^2 \tan\alpha\, d\left(H_1^{-1}\right) \epsilon^{ij} A_j\right]. \tag{74}$$

These results are exact, at least to the extent that supergravity can be trusted at each step in the duality chain. However, in our subsequent discussion in this section, we will work only to linear order in the gauge fluctuation A.

The form of the metric fluctuation suggests a natural change of coordinates

$$Z^i = z^i + 2\pi\ell_p^2 \tan\alpha H_1^{-1} \epsilon^{ij} A_j \tag{75}$$

where $i = 1, 2$. This diffeomorphism moves all the metric fluctuations into the world-volume; in other words, only components of the metric that have no Taub-NUT indices fluctuate. However, there is no unique choice of diffeomorphism. There are other diffeomorphisms that accomplish the same task since only the r-dependence is fixed by this constraint.

Specifically, let us consider a more general change of coordinates

$$Z^i = z^i + 2\pi\ell_p^2 \left(\tan\alpha H_1^{-1}\epsilon^{ij} - \theta^{ij}\right) A_j \tag{76}$$

where θ^{ij} is a constant anti-symmetric matrix. The suggestive label is no mere coincidence. As we shall see, this theta will have an interpretation as the non-commutativity parameter of the world-volume theory. Of course, we could consider more general diffeomorphisms, but it turns out that these particular ones are especially nice. Also, when this diffeomorphism, (76), is pulled back to the brane world-volume, i.e. computed at $r = 0$, then the first term drops out leaving simply

$$Z^i = z^i - 2\pi\ell_p^2 \theta^{ij} A_j. \tag{77}$$

Under (76), the metric fluctuations become

$$e^{z_i} = H_1^{\frac{1}{3}} H_2^{-\frac{1}{3}} \left[dZ^i - 2\pi\ell_p^2 \left(\tan\alpha H_1^{-1}\epsilon^{ij} - \theta^{ij}\right)\left(\partial_\mu A_j dx^\mu + \partial_k A_j dZ^k\right)\right]. \tag{78}$$

The background 3-form,

$$\langle A_3\rangle = H_2^{-1} \tan\alpha dz_1 \wedge dz_2 \wedge \left(dy + \frac{1}{2} R \cos\theta d\psi\right) \tag{79}$$

also changes. The combination of fluctuating 3-forms becomes, to linear order in A (also neglecting the induced 3-form from the second line of (69)),

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2\left(A_\mu dx^\mu + \frac{1}{\cos^2\alpha}A_i dZ^i\right)\wedge d\left[H_1^{-1}\left(dy+\frac{1}{2}R\cos\theta d\psi\right)\right]\\ &\quad -2\pi\ell_p^2\tan\alpha H_2^{-1}\left(\tan\alpha H_1^{-1}-\theta^{12}\right)\left(\partial_\mu A_i dx^\mu + \partial_j A_i dZ^j\right)\wedge dZ^i\\ &\quad \wedge\left(dy+\frac{1}{2}R\cos\theta d\psi\right). \end{aligned} \tag{80}$$

This expression can be cleaned up by making a 3-form gauge transformation. Specifically by adding an exact 3-form

$$d\left[2\pi\ell_p^2\tan\alpha H_2^{-1}\left(\tan\alpha H_1^{-1}-\theta^{12}\right)A_i dZ^i\wedge\left(dy+\frac{1}{2}R\cos\theta d\psi\right)\right], \tag{81}$$

we obtain the total fluctuating 3-form

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2 A_\mu dx^\mu\wedge d\left[H_1^{-1}\left(dy+\frac{1}{2}R\cos\theta d\psi\right)\right]\\ &\quad +2\pi\ell_p^2 A_i dZ^i\wedge d\left[\left(1+\tan\alpha\theta^{12}\right)H_2^{-1}\left(dy+\frac{1}{2}R\cos\theta d\psi\right)\right]. \end{aligned} \tag{82}$$

As mentioned earlier, we can choose any constant value for the parameter θ^{12}. Let us consider three particular choices of θ^{12} that simplify the above fluctuations.

The first choice we consider is $\theta^{12}=0$. For this choice, the diffeomorphism (76) vanishes at $r=0$. As we shall see, this means effectively that on the brane, we see only commutative gauge transformations. However, we cannot really escape non-commutativity in the full 11-dimensional theory in the sense that a commutative gauge transformation on A still maps to a diffeomorphism.

The next choice is $\theta^{12}=-1/\tan\alpha$, i.e. $\theta=B^{-1}$. In this case, the 3-form piece above vanishes and the A_i fluctuations move entirely into the metric. The metric fluctuations become explicitly

$$e^{z_i}=H_1^{\frac{1}{3}}H_2^{-\frac{1}{3}}\left[dZ^i-2\pi\ell_p^2\frac{1}{\sin\alpha\cos\alpha}H_1^{-1}H_2\epsilon^{ij}\left(\partial_\mu A_j dx^\mu+\partial_k A_j dZ^k\right)\right]. \tag{83}$$

At $r=0$ this reduces to

$$e^{z_i}=H_1^{\frac{1}{3}}H_2^{-\frac{1}{3}}\left[dZ^i-2\pi\ell_p^2\frac{1}{\tan\alpha}\epsilon^{ij}\left(\partial_\mu A_j dx^\mu+\partial_k A_j dZ^k\right)\right]. \tag{84}$$

One more choice worth mentioning is $\theta^{12}=-\sin\alpha\cos\alpha$. If we believe the correspondence between the θ appearing in the diffeomorphism and the θ of non-commutative

Yang-Mills, then this should correspond to pure NCYM. We find for this choice that the 3-form fluctuation becomes

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2 A_\mu dx^\mu \wedge d\left[H_1^{-1}\left(dy + \frac{1}{2}R\cos\theta d\psi\right)\right] \\ &\quad +2\pi\ell_p^2 A_i dZ^i \wedge d\left[\cos^2\alpha H_2^{-1}\left(dy + \frac{1}{2}R\cos\theta d\psi\right)\right].\end{aligned} \tag{85}$$

Near $r = 0$, $H_1^{-1} \simeq \cos^2\alpha H_2^{-1}$, so A_μ and A_i appear on equal footing. This is precisely what we would expect for pure NCYM.

Finally, let us consider what happens to the commutative gauge group of our starting point. Here, we take the point of view that we have fixed a diffeomorphism initially. The duality chain and the diffeomorphism then define a map from the original theory of supergravity on a Taub-NUT space to a new theory with flux. Under a gauge transformation $A \to A + d\lambda$, we are then instructed to perform a further diffeomorphism of the form (76) but with A_j replaced by $\partial_j\lambda$ (in addition to simply shifting A_μ and A_i by $\partial_\mu\lambda$ and $\partial_i\lambda$). This diffeomorphism acts non-trivially on all of the gauge fields, and on any other fields that we might consider, such as fields that correspond to scalars on the D6-brane. Explicitly, at $r = 0$, a field would transform as

$$\delta\Phi = 2\pi\ell_p^2\theta^{ij}\partial_j\lambda\partial_i\Phi. \tag{86}$$

So, in total, a gauge field like A_μ would transform as

$$\delta A_\mu = \partial_\mu\lambda + 2\pi\ell_p^2\theta^{ij}\partial_j\lambda\partial_i A_\mu. \tag{87}$$

These are, of course, the expected noncommutative gauge transformations to linear order in θ.

4.3.2 Relation to the Seiberg-Witten map

In this section we will set $2\pi\ell_p^2 = 1$. The diffeomorphism (76) for the case $\theta = B^{-1}$ has an interesting relation to the Seiberg-Witten map relating the non-commutative gauge field $\hat{A}$ to the ordinary gauge field A [15]. As shown in [16], in the presence of a background B_2-field, we can define (for $\theta = B^{-1}$) the following diffeomorphism on the world-volume of a single D-brane:

$$X^i = x^i + \theta^{ij}\hat{A}_j(x). \tag{88}$$

Under this diffeomorphism,

$$(F_{ij}(X)+B_{ij})\frac{\partial X^i}{\partial x^k}\frac{\partial X^j}{\partial x^l}=B_{kl} \tag{89}$$

so that the coordinates x are interpreted as the coordinates in which the commutative field strength F is constant. This diffeomorphism therefore moves the gauge fluctuations on the brane entirely into the metric. Moreover, the diffeomorphism (88) is not unique, but only defined up to diffeomorphisms that leave B invariant.

We have already seen that under the diffeomorphism (76) for $\theta = B^{-1}$, the gauge field fluctuations are moved entirely into the metric, at least to linear order in A. Based on its remarkable similarity to (88), it is natural to wonder whether we can move the gauge fluctuations into the metric to all orders in A. We will argue below that this is indeed the case provided we replace A in (76) by $\hat{A}$. Specifically, we will use the diffeomorphism

$$Z^i = z^i + \left(\tan\alpha H_1^{-1}\epsilon^{ij} - (B^{-1})^{ij}\right)\hat{A}_j(Z) = z^i + \frac{1}{\sin\alpha\cos\alpha}H_1^{-1}H_2\epsilon^{ij}\hat{A}_j(Z). \tag{90}$$

To facilitate comparison with (88), it is convenient to rewrite (90) as

$$z^i = Z^i - \frac{1}{\sin\alpha\cos\alpha}H_1^{-1}H_2\epsilon^{ij}\hat{A}_j(Z). \tag{91}$$

The coordinate z is thus the analogue of X in (88), while Z is the analogue of x. Instead of applying this diffeomorphism to the fields $\langle A_3\rangle$ and δA_3 respectively, it is more useful to apply it to the field strengths $\langle G_4\rangle = d\langle A_3\rangle$ and $\delta G_4 = d\delta A_3$. Using the explicit forms of the background field (79) and the fluctuation (72) obtained by the duality chasing, we compute

$$\begin{aligned}
\langle G_4\rangle &= \tan\alpha\; dz^1\wedge dz^2\wedge d[H_2^{-1}(dy+\frac{R}{2}\cos\theta\; d\psi)], && (92)\\
\delta G_4 &= \sec\alpha\;\left(F_{12}dz^1\wedge dz^2 + \partial_\mu A_i dx^\mu\wedge dz^i\right)\\
&\quad \wedge\left[\frac{R}{2r^2}H_1^{-1}H_2^{-1}dr\wedge(dy+\frac{R}{2}\cos\theta\; d\psi) - \frac{R}{2\cos\alpha}H_1^{-1}\sin\theta\; d\theta\wedge d\psi\right]\\
&\quad +\sec\alpha\tan^2\alpha\frac{R^2}{4r^2}H_1^{-2}H_2^{-1}\sin\theta A_i dz^i\wedge dr\wedge d\theta\wedge d\psi && (93)
\end{aligned}$$

where we have only shown the terms in δG_4 that are independent of A_μ. We now apply the diffeomorphism (91) to $\langle G_4\rangle + \delta G_4$. In order to compare the result with (89), which strictly speaking, is valid for maximal rank B_2-field, we will now ignore all dependence of the gauge fields on the coordinates x^μ which correspond to worldvolume coordinates transverse to the

B-field. Having thus dropped all terms involving $\partial_\mu A$, we find to lowest order, $\hat{A}_i = A_i$ and

$$\langle G_4 \rangle + \delta G_4 \mapsto \tan\alpha dZ^1 \wedge dZ^2 \wedge d\left[H_2^{-1}(dy + \frac{R}{2}\cos\theta\ d\psi)\right]. \qquad (94)$$

Thus, to lowest order, we can indeed interpret the Z^i as the coordinates in which the 4-form field strength G is constant (in Z), and (94) is the natural generalization of (89).

We shall now verify the second order correction, so we write $\hat{A}_i = A_i + a_i$. The second order piece, a_i, is a function of the space-time coordinates, and of r, and is explicitly given by:

$$\hat{A}_i = A_i + a_i = A_i + \frac{1}{2}\tilde{\theta}^{jk}\left(2A_k\partial_j A_i - A_k\partial_i A_j\right) + \mathcal{O}(A^3) \qquad (95)$$

where we have defined the natural r-dependent combination

$$\tilde{\theta}^{ij} = -\frac{1}{\sin\alpha\cos\alpha}H_1^{-1}H_2\epsilon^{ij}. \qquad (96)$$

Note that this is precisely the tensor contracted with $\hat{A}_j$ in (91). To this order, we now find

$$\langle G_4 \rangle + \delta G_4 \mapsto \tan\alpha dZ^1 \wedge dZ^2 \wedge d\left[H_2^{-1}(dy + \frac{R}{2}\cos\theta\ d\psi)\right] + P_i dZ^i \wedge dr \wedge d\theta \wedge d\psi, \quad (97)$$

where,

$$\begin{aligned} P_1 \;=\; & -\frac{R\sin\theta}{2\cos^2\alpha}\Big[\frac{R\sin^2\alpha}{2r^2\cos\alpha}H_1^{-2}H_2^{-1}a_1 + H_1^{-1}\partial_r a_1 \\ & -\frac{R\sin\alpha}{2r^2\cos^2\alpha}H_1^{-3}\left(2A_1\partial_2 A_1 - A_2\partial_1 A_1 - A_1\partial_1 A_2\right) + \mathcal{O}(A^3)\Big], \end{aligned} \qquad (98)$$

and similarly for P_2. In writing the above expression, we have used

$$A_1(z) = A_1(Z) - \frac{1}{\sin\alpha\cos\alpha}H_1^{-1}H_2\left(A_2\partial_1 A_1 - A_1\partial_2 A_1\right) + \mathcal{O}(A^3). \qquad (99)$$

We see from (97) that we can interpret the Z^i as the coordinates in which the 4-form field strength G is constant (in Z) only if the P_i vanish. Due to the presence of the derivatives in r, the constraint $P_i = 0$ provides a non-trivial consistency check of the diffeomorphism (91) with the choice of r-dependent noncommutativity parameter (96). Happily, the choice of a_i given by (95) indeed satisfies this constraint.

At $r = 0$, $\tilde{\theta}^{ij}$ reduces to the previous θ^{ij}, and (95) reproduces the Seiberg-Witten map between A and $\hat{A}$ to linear order in θ. In this sense we can view equation (95) as a lift of the usual Seiberg-Witten map [15] between commutative and non-commutative variables to the full supergravity theory. The diffeomorphism (91) is the corresponding lift of equation (88). It is very reasonable to expect that the above procedure can be iterated, and that this lift can be constructed to all finite orders in A (or equivalently to all finite orders in θ). We expect that all of the statements of the previous section should similarly extend to all orders.

4.3.3 Duality chasing the time-dependent background

Before jumping into a time-dependent duality chase, it is worth surveying the structure of the time-dependent solution (29), which takes the form:

$$\begin{aligned} ds^2 &= f_1(r,x^+)\left(-2dx^+dx^- + dx_3^2 + dx_4^2 + dx_5^2\right) + f_2(r,x^+)\left(dx^2+dz^2\right) \\ &\quad + f_3(r,x^+)\left(-x^2(dx^+)^2 + 2x^+ x dx^+ dx\right) + f_4(r,x^+)ds^2_{\mathcal{M}_1} + f_5(r,x^+)ds^2_{\mathcal{M}_2}. \end{aligned}$$

The accompanying M theory G_4 has both $(3,1)$ and $(2,2)$ pieces. There are multiple warp factors in the metric (100) but what is most remarkable is that the internal space warp factors depend on x^+. In this sense, space-time and the internal space are warped with respect to one another. A direct analysis of small fluctuations, along the lines of section 4.2, should demonstrate the existence of a localized gauge-field in an interesting way. However, the same issues that we met in the static case will also appear in this case. We will therefore follow the duality chasing tactic again.

Let us start again with the fluctuation $A_1 \wedge \omega$, where

$$\omega = 2\pi\ell_p^2\, d\left[H^{-1}\left(dy + \frac{1}{2}R\cos\theta d\psi\right)\right]. \tag{100}$$

On T-dualizing z, changing to invariant null-brane coordinates, T-dualizing back, and lifting to 11 dimensions, we find again a mixing of 3-form and metric fluctuations. Let us first define the following combinations of the original gauge fluctuations which naturally appear at the end of the duality chain:

$$\begin{aligned} \tilde{A}_+ &= A_+ + \tilde{z}A_x + \frac{1}{2}\tilde{z}^2 A_- \\ \tilde{A}_x &= A_x + \tilde{z}A_-. \end{aligned} \tag{101}$$

We drop the tildes from now on. Furthermore, we make the same definitions as before so that the coordinates agree with (29). Then the result is a 3-form fluctuation

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2\left[A_i dx^i + A_- dx^- + Hh^{-1}A_z dz + \left(A_+ + h^{-1}\left(\frac{x^+x}{R^2}A_x + \frac{x^2}{R^2}A_-\right)\right)dx^+\right.\\ &\quad \left.+\left(Hh^{-1}A_x - h^{-1}\frac{x^+x}{R^2}A_-\right)dx\right]\wedge\left[\frac{c}{r^2}H^{-2}dr\wedge(dy + c\cdot\cos\theta d\psi)\right] \\ &\quad -cH^{-1}\sin\theta\left(A_+dx^+ + A_-dx^- + A_xdx + A_zdz + A_idx^i\right)\wedge d\theta\wedge d\psi.\end{aligned} \tag{102}$$

The metric fluctuations can be most succinctly written by making some replacements in the metric (29)

$$\begin{aligned} dx^- &\to dx^- + 2\pi\ell_p^2\frac{c}{r^2}H^{-2}\frac{x}{R}A_z dr \\ dx &\to dx + 2\pi\ell_p^2\frac{c}{r^2}H^{-2}\frac{x^+}{R}A_z dr \\ dz &\to dz - 2\pi\ell_p^2\frac{c}{r^2}H^{-2}\left(\frac{x^+}{R}A_x + \frac{x}{R}A_-\right)dr. \end{aligned} \tag{103}$$

As before, a particular change of coordinates presents itself for use in moving the metric fluctuations purely into the world-volume directions. If we let μ run over the indices ($-$, x, z), then a natural diffeomorphism is

$$X^\mu = x^\mu - 2\pi\ell_p^2 H^{-1}\theta^{\mu\nu}A_\nu \tag{104}$$

where,

$$\begin{aligned} \theta^{z-} &= -\theta^{-z} = \frac{x}{R} \\ \theta^{zx} &= -\theta^{xz} = \frac{x^+}{R}, \end{aligned} \tag{105}$$

with all other components of θ zero. This agrees with the results reported in [10].

We close by noting that under this diffeomorphism, the fluctuating part of the 3-form becomes (to linear order in A)

$$\begin{aligned}\delta A_3 &= 2\pi\ell_p^2\left(A_+dx^+ + A_idx^i + A_\mu dx^\mu\right)\wedge d\left[H^{-1}(dy + c\cdot\cos\theta d\psi)\right] \\ &\quad -2\pi\ell_p^2 H^{-1}h^{-1}\left[\left(\frac{x^+}{R}\right)^2 d(A_zdz) + \left(\frac{x}{R^2}A_- + \frac{x^+}{R^2}A_x\right)dx^+\wedge dx\right. \\ &\quad \left. + \left(\frac{x}{R}dx^+ - \frac{x^+}{R}dx\right)\wedge\left(\frac{x^+}{R}dA_x + \frac{x}{R}dA_-\right)\right]\wedge(dy + c\cdot\cos\theta d\psi).\end{aligned} \tag{106}$$

Clearly, there is much more to be said. This analysis can be continued along the lines of section 4.3.1 and section 4.3.2, perhaps giving an analogue from gravity of the Seiberg-Witten map for the time-dependent case.

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A Conventions and Rules

In performing type II T-duality and lifts to M theory, we will (mostly) use the conventions found in [29]. Let ℓ_p denote the eleven-dimensional Planck scale, and ℓ_s denote the string scale.

A.1 Lifting IIA to M theory

A type IIA SUGRA configuration can be lifted to 11 dimensions. Let y parametrize the eleventh direction. The y coordinate is compact with periodicity 2π. Writing the 11-dimensional fields on the left and the IIA fields on the right, we have:

$$\begin{aligned} G_{\mu\nu} &= e^{-\frac{2}{3}\Phi} G_{\mu\nu} + e^{\frac{4}{3}\Phi} C_\mu C_\nu \\ G_{\mu y} &= -\ell_p e^{\frac{4}{3}\Phi} C_\mu \\ G_{yy} &= \ell_p^2 e^{\frac{4}{3}\Phi} \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho} \\ A_{\mu\nu y} &= \ell_p B_{\mu\nu}. \end{aligned} \tag{107}$$

A.2 T-Duality

We dualize in the direction x, which we scale to have periodicity 2π (and is hence dimensionless).

The NS-NS fields transform in the following manner:

$$
\begin{aligned}
e^{2\Phi'} &= \frac{\ell_s^2 e^{2\Phi}}{G_{xx}} \\
G'_{xx} &= \frac{\ell_s^4}{G_{xx}} \\
G'_{\mu x} &= \frac{\ell_s^2 B_{\mu x}}{G_{xx}} \\
G'_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu x}G_{\nu x} - B_{\mu x}B_{\nu x}}{G_{xx}} \\
B'_{\mu x} &= \frac{\ell_s^2 G_{\mu x}}{G_{xx}} \\
B'_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu x}G_{\nu x} - G_{\mu x}B_{\nu x}}{G_{xx}}.
\end{aligned} \tag{108}
$$

The transformation of the R-R potentials is given by

$$
\begin{aligned}
\ell_s^{-1} {C^{(n)}_{\mu\cdots\nu\alpha x}}' &= C^{(n-1)}_{\mu\cdots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|x}G_{|\alpha]x}}{G_{xx}} \\
\ell_s {C^{(n)}_{\mu\cdots\nu\alpha\beta}}' &= C^{(n+1)}_{\mu\cdots\nu\alpha\beta x} + nC^{(n-1)}_{[\mu\cdots\nu\alpha}B_{\beta]x} + n(n-1)\frac{C^{(n-1)}_{[\mu\cdots\nu|x}B_{|\alpha|x}G_{|\beta]x}}{G_{xx}}.
\end{aligned} \tag{109}
$$

References

[1] P. K. Townsend, "The eleven-dimensional supermembrane revisited," *Phys. Lett.* **B350** (1995) 184–187, hep-th/9501068.

[2] M. Alishahiha, Y. Oz, and M. M. Sheikh-Jabbari, "Supergravity and large N noncommutative field theories," *JHEP* **11** (1999) 007, hep-th/9909215.

[3] M. Alishahiha, H. Ita, and Y. Oz, "Graviton scattering on D6 branes with B fields," *JHEP* **06** (2000) 002, hep-th/0004011.

[4] H. Singh, "Note on (D6,D8) bound state, massive duality and non- commutativity," hep-th/0212103.

[5] A. Connes, M. R. Douglas, and A. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," *JHEP* **02** (1998) 003, hep-th/9711162.

[6] S. Chakravarty, K. Dasgupta, O. J. Ganor, and G. Rajesh, "Pinned branes and new non Lorentz invariant theories," *Nucl. Phys.* **B587** (2000) 228–248, hep-th/0002175.

[7] A. Hashimoto and N. Itzhaki, "Non-commutative Yang-Mills and the AdS/CFT correspondence," *Phys. Lett.* **B465** (1999) 142–147, hep-th/9907166.

[8] J. M. Maldacena and J. G. Russo, "Large N limit of non-commutative gauge theories," *JHEP* **09** (1999) 025, hep-th/9908134.

[9] T. Adawi, M. Cederwall, U. Gran, B. E. W. Nilsson, and B. Razaznejad, "Goldstone tensor modes," *JHEP* **02** (1999) 001, hep-th/9811145.

[10] A. Hashimoto and S. Sethi, "Holography and string dynamics in time-dependent backgrounds," hep-th/0208126.

[11] M. Alishahiha and S. Parvizi, "Branes in time-dependent backgrounds and AdS/CFT correspondence," *JHEP* **10** (2002) 047, hep-th/0208187.

[12] L. Dolan and C. R. Nappi, "Noncommutativity in a time-dependent background," hep-th/0210030.

[13] R.-G. Cai, J.-X. Lu, and N. Ohta, "NCOS and D-branes in time-dependent backgrounds," *Phys. Lett.* **B551** (2003) 178–186, hep-th/0210206.

[14] D. Robbins and S. Sethi, *under investigation.*

[15] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032, hep-th/9908142.

[16] L. Cornalba, "D-brane physics and noncommutative Yang-Mills theory," *Adv. Theor. Math. Phys.* **4** (2000) 271–281, hep-th/9909081.

[17] N. Ishibashi, "A relation between commutative and noncommutative descriptions of D-branes," hep-th/9909176.

[18] R. Dijkgraaf, E. Verlinde, and M. Vonk, "On the partition sum of the NS five-brane," hep-th/0205281.

[19] A. R. Frey, "String theoretic bounds on Lorentz-violating warped compactification," hep-th/0301189.

[20] C. Csaki, J. Erlich, and C. Grojean, "Gravitational Lorentz violations and adjustment of the cosmological constant in asymmetrically warped spacetimes," *Nucl. Phys.* **B604** (2001) 312–342, hep-th/0012143.

[21] J. Figueroa-O'Farrill and J. Simon, "Generalized supersymmetric fluxbranes," *JHEP* **12** (2001) 011, hep-th/0110170.

[22] J. Simon, "The geometry of null rotation identifications," *JHEP* **06** (2002) 001, hep-th/0203201.

[23] Y. Imamura, "Born-Infeld action and Chern-Simons term from Kaluza-Klein monopole in M-theory," *Phys. Lett.* **B414** (1997) 242–250, hep-th/9706144.

[24] J. P. Gauntlett and D. A. Lowe, "Dyons and S-Duality in N=4 Supersymmetric Gauge Theory," *Nucl. Phys.* **B472** (1996) 194–206, hep-th/9601085.

[25] K.-M. Lee, E. J. Weinberg, and P. Yi, "Electromagnetic Duality and $SU(3)$ Monopoles," *Phys. Lett.* **B376** (1996) 97–102, hep-th/9601097.

[26] A. Sen, "Kaluza-Klein dyons in string theory," *Phys. Rev. Lett.* **79** (1997) 1619–1621, hep-th/9705212.

[27] K. Dasgupta, G. Rajesh, and S. Sethi, "M theory, orientifolds and G-flux," *JHEP* **08** (1999) 023, hep-th/9908088.

[28] E. Bergshoeff, B. Janssen, and T. Ortin, "Kaluza-Klein monopoles and gauged sigma-models," *Phys. Lett.* **B410** (1997) 131–141, hep-th/9706117.

[29] C. V. Johnson, "D-brane primer," hep-th/0007170.