Fubini Theorems for Generalized Lebesgue-Bochner-Stieltjes Integral

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Let $R$ be the space of reals. If $Y_i, W (i=1, \ldots, k)$ are semi-normed spaces then by $L(Y_1, \ldots, Y_k; W)$ we shall denote the space of all operators $u$ which are $k$-linear and continuous from the product of the spaces $Y_i (i=1, \ldots, k)$ into the space $W$. The semi-norm of elements in the above spaces will be denoted by $| |$.

A family of sets $V$ of an abstract space $X$ will be called a pre-ring if for any two sets $A_1, A_2 \in V$ we have $A_1 \cap A_2 \in V$, and there exists disjoint sets $B_1, \ldots, B_k \in V$ such that $A_1 \setminus A_2 = B_1 \cup \cdots \cup B_k$.

A nonnegative function $v$ on the pre-ring $V$ will be called a volume if for every countable family of disjoint sets $A_t \in V (t \in T)$ such that $A_t \supseteq \bigcup_{t \in T} A_t \in V$ we have $v(A) = \sum_{t \in T} v(A_t)$.

A triple $(X, V, v)$ where $V$ is a pre-ring of sets of $X$ and $v$ is a volume on $V$, will be called a volume space. If the triples $(X_i, V_i, v_i) (i=1, \ldots, k)$ are volume spaces then the triple $(X, V, v)$ defined by $X = X_1 \times \cdots \times X_k$ and $V = V_1 \times \cdots \times V_k$ consisting of all sets of the form $A = A_1 \times \cdots \times A_k; A_t \in V_t$ with $v(A) = v(A_1) \cdots v(A_k)$ is a volume space.

Let $(X, V, v)$ be a fixed volume space. Denote by $M_q(v, Z) (1 < q < \infty)$ the space of all finite additive functions $\mu$ from the pre-ring $V$ into a Banach space $Z$ and such that $\mu(A) = 0$ if $v(A) = 0$ and

$$
\sup \{|\mu(A)| \mid v(A)^{1-q} \leq 1, \|\mu\|_q < \infty\}
$$

when $q \neq \infty$, where the supremum is taken over all finite families of disjoint sets $A_t \in V$ such that $v(A_t) > 0$. In the case when $q = \infty$, let $\sup \{|\mu(A)| \mid v(A)^{-1} \leq 1, A \in V\} = \|\mu\|_\infty < \infty$ where the supremum is taken over all sets $A \in V$ such that $v(A) > 0$.

Now if $1/p_i + 1/q_i = 1, p_i, q_i \geq 1, i=1, 2$ and $u \in L(Y_i, Y_i, Z; W)$, denote by $M(q_i, v_i, Z, u)$ the family of all functions $\mu(A_1, A_2)$ from $V_1 \times V_2$ into $Z$ which are additive in each variable $A_i$ separately and $\mu(A_1, A_2) = 0$ if $v_i(A_i) = 0$ or $v_i(A_2) = 0$; moreover assume that the following norm is finite $\|\mu\| = \sup \{|\mu(A_1, A_2)| : u(y_i, z_i, \mu(A_1, A_2)) (v_1(A_1))^{-1/p_1} (v_2(A_2))^{-1/p_2} |a_1| |a_2| \}$ where the supremum is taken over all finite systems such that $|y_i| \leq 1, |z_i| \leq 1, \sum |a_1|^{p_1} \leq 1, \sum |a_2|^{p_2} \leq 1$, where $A_1$ is a family of disjoint sets of the pre-ring $V_1$ such that $v_1(A_1) > 0$ and similarly $A_2$ is a finite family of disjoint sets of the pre-ring $V_2$ such that $v_2(A_2) > 0$. 

If \( q=q_i=q_z \) and \( u(y_i, y_z, z)=z(y_i, y_z) \) for \( y_i \in Y_i, z \in L(Y_i, Y_z; W) \) then we have \( M_3(v, Z) \subset M(q_3, q, v_3, v_z, Z, u) \).

**Theorem 1.** Let \((X, V, v)\) be the product volume space of the volume spaces \((X_i, V_i, v_i)\) \((i=1, \cdots, k)\). If \( \mu_i \in M_4(v_i, Z_i) \) where \( 1 \leq q \leq \infty \) and \( u \in L(Z_i, \cdots, Z_n; W) \) then \( \mu \in M_4(v, W) \) where

\[
\mu(A_1 \times \cdots \times A_k) = u(\mu(A_1), \cdots, \mu(A_k)) \quad \text{for} \quad A \in V
\]

Let \((X, V, v)\) be a volume space and \( Y \) be a fixed Banach space. Denote by \( S(V, Y) = S(Y) \) the set of all functions of the form \( h = y_1 x_{A_1} + \cdots + y_k x_{A_k} \) where \( y_i \in Y_i \) and \( A_i \in V \) are disjoint sets. Put \( \|h\|_V = |y_i| v(A_i) + \cdots + |y_k| v(A_k) \).

A sequence of functions \( s_n \) is called basic if there exist a sequence \( h_n \in S(Y) \) and a constant \( M > 0 \) such that \( s_n = h_1 + \cdots + h_n, \|h_n\| \leq M 4^{-n} \) for \( n = 1, 2, \cdots \).

A set \( A \subset X \) is called a null set if for every \( \varepsilon > 0 \) there exists a countable family of sets \( A_t \in V \) \((t \in T)\) such that \( A \subset \bigcup_t A_t \) and \( \sum_t v(A_t) < \varepsilon \).

A condition \( c(x) \) depending on a parameter \( x \in A \subset X \) is said to be satisfied almost everywhere on the set \( A \) if there exists a null set \( A \) such that condition is satisfied at every point of the set \( A \backslash A \).

Denote by \( L_b(v, Y) \) the space of all functions \( f \) such that there exists a basic sequence \( s_n \) convergent almost everywhere on the space \( X \) to the function \( f \). Put \( \|f\| = \lim \|s_n\| \). This definition is correct, that is, it doesn’t depend on the particular choice of the basic sequence. It follows from Theorem 1 [1], that the space \( (L_b(v, Y), \|\|) \) is a complete seminormed space. The set of simple functions \( S(V, Y) \) is dense in the space \( L_b(v, Y) \) according to Lemmas 1 and 4 [1].

Now let \( 1 \leq p < \infty \). Denote by \( a(y) = \|y\|^{p-1} y \) for \( y \in Y \). Since the function and its inverse \( a^{-1}(y) = \|y\|^{1/p -1} y \) for \( y \in Y \) are continuous on the space \( Y \) therefore it establishes a homeomorphism of the space onto itself.

Denote by \( L_p(v, Y) \) the space of all functions \( f \) from the set \( X \) into the space \( Y \) such that \( a \circ f \in L_b(v, Y) \). Put

\[
\|f\|_p = \left( \int |a \circ f|^p \, dv \right)^{1/p} = \left( \int |f(x)|^p \, dv \right)^{1/p}.
\]

The space \( (L_p(v, Y), \|\|_p) \) is a complete seminormed space and the set \( S(V, Y) \) is dense in it according to Theorem 1 [4].

Now let \((X, V, v)\) be the product space of the volume spaces \((X_i, V_i, v_i)\) \((i=1, 2)\). Take any simple functions \( s_i \in S(V_i, Y_i) \) and assume that \( s_i = \sum_i y_{n_i} x_{A_{n_i}} \). Let \( \mu \in M_3(v, Z) \) and let \( u \) be a multilinear continuous operator from the product of the Banach spaces \( Y_i, Y_z, Z \) into a Banach space \( W \). Define
\[ \int u(s_1, s_2, d\mu) = \sum_{n_1, n_2} u(y_{n_1}, y_{n_2}; \mu(A_{n_1} \times A_{n_2})). \]

It is easy to see that the definition is correct. Put \( U = L(Y_1, Y_2, Z; W) \).

The integral operator just defined is linear in each variable \( u, s_1, s_2, \mu \) separately and is defined on a dense set of the product of the spaces \( U, L_p(v_1, Y_1), L_p(v_2, Y_2), M_q(v, A) \), where \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \).

Now from the inequality
\[ \left| \int u(s_1, s_2, d\mu) \right| \leq |u| \| s_1 \|_p \| s_2 \|_p \| \mu \|_q \]

and from the completeness of the space \( W \) we get that there exists a unique extension of the operator to a multilinear continuous operator defined on \( U \times L_p(v_1, Y_1) \times L_p(v_2, Y_2) \times M_q(v, Z) \).

In a similar way one could define the integral operator \( \int u_0(f, d\mu) \) for \( f \in L_p(v, Y) \), \( \mu \in M_q(v, X) \), \( u_0 \in L(Y, Z; W) \). When it is important to indicate the variable of integration which shall use the symbol
\[ \int u_0(f(x), \mu(dx)). \]

## § Fubini's Theorem for the integral \( \int u(f_1, f_2, d\mu) \)

Take any multilinear continuous operator \( u \in L(Y_1, Y_2, Z; W) = U \).

Define an operator \( u_0(y, z) = u(\cdot, y, z) \) for \( y \in Y_1, z \in Z \). We see that \( u_0 \in L(Y_1, Z; W) = U_1 \) for \( Z_0 = L(Y_1, W) \). Define also the operator \( u_0(y, z_0) = z_0(y_1) \) for \( y_1 \in Y_1, z_0 \in Z_0 \). We have \( u_0 \in L(Y_1, Z; W) \) and
\[ |u| = |u_1|, \quad |u_0| = 1. \]

Let \( (X, V, v) \) be the product volume space of the volume spaces \( (X_1, V_1, v_1) \) \((i=1, 2)\). Assume that \( 1 \leq p < \infty \) and \( 1/p + 1/q = 1 \). We have the following theorem.

**Theorem 1.**

1. If \( \mu \in M_q(v, Z) \) then for all \( A_1 \in V_1 \) the vector function \( \mu_{A_1} \) defined by the formula
   \[ \mu_{A_1}(A_2) = \mu(A_1 \times A_2) \]
   \( A_2 \in V_2 \)
   belongs to the space \( M_q(v_2, Z) \).

2. The operator \( \mu_2 = r(f_2, \mu) \) defined by means of the integral
   \[ \mu_2(A_1) = \int u(f_2, d\mu_{A_1}) \]
   for all \( A_1 \in V_1 \)
   is bilinear from the product \( L_p(v_1, Y_2) \times M_q(v, Z) \) into the space \( M_q(v, Z_0) \)
   and
   \[ || \mu_2 || \leq |u| || f_2 ||_p || \mu ||_q \]
   for all \( f_2 \in L_p(v_1, Y_2), \mu \in M_q(v, Z) \).

3. Moreover the following equality holds
   \[ \int u(f_1, f_2, d\mu) = \int u_0(f_1, d\mu(f_2, \mu)) \]
   for all \( f_2 \in L_p(v_1, Y_2) \) \((i=1, 2), \mu \in M_q(v, Z) \).

(The above theorem can be easily generalized to the case when \( f_1 \in L_p(v_1, Y_1), f_2 \in L_p(v_2, Y_2), \) and \( \mu \in M(q_1, q_2, v_1, v_2, Z, w) = M \).
If we take the trilinear operator \( u(y_1, y_2, z) = z(y_1, y_2) \) for \( y_i \in Y_i \), \( z \in Z \) and define \( Z = L(Y_1, Y_2; W) \), then the space \( M \) is isomorphic and isometric to the space of all bilinear continuous operators \( h \) from the product \( L_p(v_1, Y_1) \times L_p(v_2, Y_2) \) into the space \( W \).

Consider the following example. Let \( Y_1, Z, W \) be equal to the space \( C \) of complex numbers. Let \( u(y_1, y_2, z) = y_1 y_2 z \). Then we have \( u_1(y_2, z) = y_2 z \) and \( u_0(y_1, z_0) = y_1 z_0 \). If \( f_i \in L_p(v_i, C) \), \( \mu \in M_\mu(v, C) \) then we get from the theorem

\[
\int f_i(x_i) f_2(x_2) d\mu(dx_1 \times dx_2) = \int f_i(x_i) d\mu_i(dx_i)
\]

where \( \mu_i(A) = \int f_i(x_i) d\mu_i(A \times dx_2) \) for all \( A \in V_i \).

\[ \text{R_2} \] Fubini's theorem for generalized Lebesgue-Bochner-Stieltjes integral.

Denote by \( (X, V, v) \) the product volume space of the volume spaces \( (X_i, V_i, v_i) \). Let \( 1 \leq p < \infty \) and \( 1/p + 1/q = 1 \).

Let \( Y, Z, W \) be Banach spaces. Assume that \( u \in U = L(Y, Z_i, Z_2; W) \) and define a new operator \( u(y, z_i, z_2) = u(y, \cdot, z_i) \) for \( y \in Y \), and \( z_2 \in Z_2 \). We see that \( u_i \in L(Y, Z_i, Y_1) \), where \( Y_i = L(Z_i; W) \). Define \( u_0(y, z_2) = y(z_2) \) for \( y \in Y_i \) and \( z_2 \in Z_2 \). Notice that \( u_0 \in L(Y, Z_i, W) \) and \( \| u \| = \| u_1 \| \) and \( \| u_0 \| = \| 1 \| \).

Put \( N = \{ f \in L_p(v_1, Y_1); \| f \|_p = 0 \} \). The set \( N \) is linear and according Theorem 1 [1], coincides with the set of all functions \( f \) from the set \( X_i \) into the space \( Y_i \) such that \( f(x) = 0 \) \( v_1 \)-a.e.

Consider the quotient space \( L_p(v_1, Y_1)/N \) and define the norm of a class \( [f] = f + N \) by \( \| [f] \|_p = \| f \|_p \). This definition is correct. Notice that in order to determine a class \( [f] \) it is enough to give the values of the function \( f(x_i) \) \( v_i \)-almost everywhere.

Since the integral operator \( \int u_0(f, d\mu) \) is linear in the variable \( f \), and we have the estimation

\[
\int u_0(f, d\mu) \leq |u_0| \| f \|_p \| \mu \|_q,
\]

therefore the following definition

\[
\int u_0([f], d\mu) = \int u_0(f, d\mu)
\]

is correct where \( [f] \in L_p(v_1, Y_1)/N \). The operator defined in this way \( \int u_0(g, d\mu) \) is bilinear and we have

\[
\int u_0(g, d\mu) \leq |u_0| \| g \|_p \| \mu \|_q
\]

where \( g \in L_p(v_1, Y_1)/N \) and \( \mu \in M_\mu(v_1, Z_i) \).

**Theorem 3.**

1. If \( f \in L_p(v, Y) \), there exists a \( v \)-null set \( C \) such that \( f(x_i, \cdot) \in L_p(v_i, Y) \) if \( x_i \in C \).

2. The operator \( \tilde{f_i} = r(f, \mu_2) \) defined by the formula
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\[ \tilde{f}(x_i) = \int u_0(f(x_i, \cdot), d\mu) \]

is bilinear from the product \( L^p(v, Y) \times M_q(v, Z) \) into the space \( L^p(v, Y)/N \) and

\[ ||\tilde{f}||_p \leq ||u||_p ||f||_p ||\mu||_q \]

for all \( f \in L^p(v, Y) \) and \( \mu \in M_q(v, Z) \).

(3) Moreover \( u(f, d\mu_1, d\mu_2) = \int u_0(f, \mu, d\mu) \) for all \( f \in L^p(v, Y) \), \( \mu \in M_q(v, Z) \) \((i=1, 2)\).

Consider the following example. Let \( Y = \mathbb{W} \) be a complex Banach space and let \( Z = C \) be the space of complex numbers. Define \( u(y, z_i) = z_i y \) for all \( z_i \in C \), \( y \in Y \). We see that we may identify \( Y = \mathbb{W} \). Thus we have \( u_0(y, z_i) = y z_i \) and also \( u_0(y, z_i) = z_i y \).

Now if \( f \in L^p(v, Y) \) and \( \mu \in M_q(v, C) \) then \( f(x_i, \cdot) \in L^p(v, Y) \)

for \( v_i \)-almost all \( x_i \in X_i \). For the function \( h(x_i) = \int f(x_i, \cdot) d\mu \) we have \( h \in L^p(v, Y) \) and

\[ \int h d\mu = \int \left( \int f(x_i, x_i) d\mu \right) (dx) = \int \int (\mu_1 \times \mu_2) (dx) \]

For the case \( p = 1 \) we get the classical Fubini theorem for Bochner summable functions (compare Dunford and Schwartz: Linear Operators, p. 193).

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References


References


