Equation of state for systems with Goldstone bosons

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Abstract

We discuss some recent determinations of the equation of state for the XY and the Heisenberg universality class.

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In the last few years there has been a significant progress in the determination of the critical properties of $O(N)$ models; see, e.g., Ref. [1] for a comprehensive review. First of all, high-precision estimates of critical exponents and of several high-temperature universal ratios have been obtained by using *improved* Hamiltonians. Improved models are such that the leading nonanalytic correction is absent in the expansion of any thermodynamic quantity near the critical point. The idea is quite old [2–4]. However, the early attempts that used high-temperature techniques were not able to determine improved models with high accuracy, so that final results did not significantly improve the estimates of standard analyses. Recently [5–12], it has been realized that Monte Carlo simulations using finite-size scaling techniques are very effective for this purpose, obtaining accurate determinations of several improved models in the Ising, XY, and O(3) universality class. Once an improved models is accurately determined, one can use standard high-temperature techniques in order to obtain very precise determinations of the critical exponents. For instance, for the experimentally relevant cases, we obtained [13,14,10,12]:

$$\gamma = 1.2373(2), \qquad \nu = 0.63012(16), \qquad N = 1,$$

$$\gamma = 1.3177(5), \qquad \nu = 0.67155(27), \qquad N = 2,$$
$$\gamma = 1.3960(9), \qquad \nu = 0.7112(5), \qquad N = 3.$$

Beside the critical exponents, experiments may determine several other universal properties. We consider here the equation of state that relates the magnetic field $\vec{H}$, the magnetization $\vec{M}$, and the reduced temperature $t \equiv (T - T_c)/T_c$. In a neighborhood of the critical point $t = 0$, $\vec{H} = 0$, it can be written in the scaling form

$$\vec{H} = (B_c)^{-\delta}\,\vec{M} M^{\delta-1} f(x), \qquad x \equiv t(M/B)^{-1/\beta}, \tag{1}$$

where B_c and B are the amplitudes of the magnetization on the critical isotherm and on the coexistence curve,

$$M = B_c H^{1/\delta} \qquad t = 0. \tag{2}$$
$$M = B(-t)^{\beta} \qquad H = 0,\ t < 0. \tag{3}$$

With these choices, the coexistence line corresponds to $x = -1$, and $f(-1) = 0$, $f(0) = 1$. Alternatively, one can write

$$\vec{H} = k_1 \frac{\vec{M}}{M} |t|^{\beta\delta} F_{\pm}(|z|), \qquad z \equiv k_2 M t^{-\beta}, \tag{4}$$

where $F_+(z)$ applies for $t > 0$ and $F_-(|z|)$ for $t < 0$. The constants k_1 and k_2 are fixed by requiring

$$F_+(z) = z + \frac{1}{6} z^3 + \sum_{n=3} \frac{r_{2n}}{(2n-1)!} z^{2n-1} \tag{5}$$

for $z \rightarrow 0$ in the high-temperature phase. The behavior of the functions $f(x)$ and $F_-(|z|)$ at the coexistence curve depends crucially on N. For $N = 1$ they vanish linearly. On the other hand, for $N \geq 2$, the presence of the Goldstone modes implies in three dimensions [15–19]:

$$f(x) \approx c_f (1 + x)^2. \tag{6}$$

The nature of the corrections to this behavior is not clear [17–20]. In particular, logarithmic terms are expected [20].

In order to obtain approximations of the equation of state, we parametrize the thermodynamic variables in terms of two parameters θ and R:

$$M = m_0 R^{\beta} m(\theta), \qquad t = R(1 - \theta^2), \qquad H = h_0 R^{\beta\delta} h(\theta). \tag{7}$$

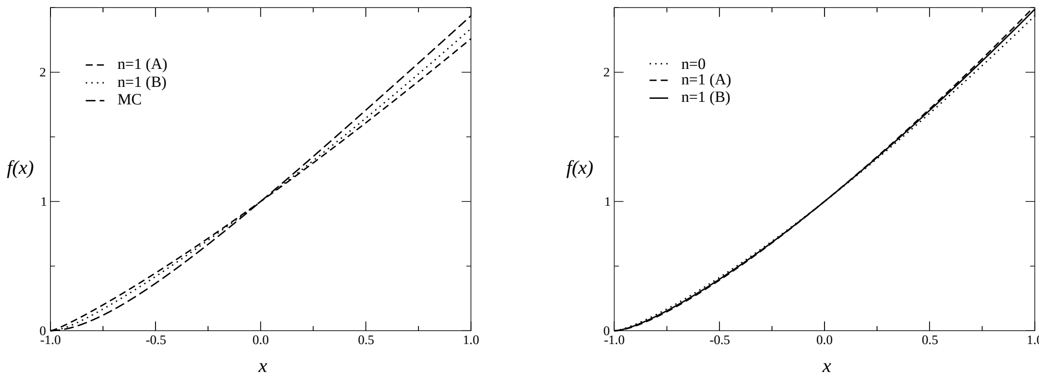

Fig. 1. Graph of the function $f(x)$ for $N = 2$ (left) and $N = 3$ (right). For $N = 2$ we also report the Monte Carlo results of Ref. [27].

Here, m_0 and h_0 are nonuniversal constants, $m(\theta)$ and $h(\theta)$ are odd functions of θ, normalized so that $m(\theta) = \theta + O(\theta^3)$ and $h(\theta) = \theta + O(\theta^3)$. The variable R is nonnegative and measures the distance from the critical point in the (t, H) plane, while the variable θ parametrizes the displacement along the lines of constant R. In particular, $\theta = 0$ corresponds to the high-temperature line $t > 0$, $H = 0$, $\theta = 1$ to the critical isotherm $t = 0$, and $\theta = \theta_0$, where θ_0 is the smallest positive zero of $h(\theta)$—it must satisfy of course $\theta_0 > 1$—to the coexistence line. Such a mapping has been extensively used in the Ising case and provides accurate approximations if one uses low-order polynomials for $m(\theta)$ and $h(\theta)$ [21–24,13,25]. In systems with Goldstone bosons we must additionally ensure the condition (6). For this purpose, it is enough to require [26] $h(\theta) \sim (\theta_0 - \theta)^2$ for $\theta \to \theta_0$.

In Refs. [26,10,12] we obtained the equation of state in the scaling limit by using two different approximation schemes for the functions $m(\theta)$ and $h(\theta)$:

$$
\begin{aligned}
\text{scheme}\quad (\mathrm{A}): \qquad & m(\theta) = \theta \left(1 + \sum_{i=1}^{n} c_i \theta^{2i} \right), \\
& h(\theta) = \theta \left(1 - \theta^2/\theta_0^2 \right)^2, \qquad (8) \\
\text{scheme}\quad (\mathrm{B}): \qquad & m(\theta) = \theta, \\
& h(\theta) = \theta \left(1 - \theta^2/\theta_0^2 \right)^2 \left(1 + \sum_{i=1}^{n} c_i \theta^{2i} \right). \qquad (9)
\end{aligned}
$$

The constants c_i and θ_0 were fixed by requiring $F_+(z)$ to have the expansion (5), with the coefficients determined by high-temperature expansion techniques. Since we were able to compute accurately only r_6 and r_8, we used the two schemes for $n = 0$ and $n = 1$. The results, especially those for $N = 3$, see Fig. 1, are quite independent of the scheme used, indicating the good convergence of the method.

By using the equation of state, one can determine several amplitude ratios. We mention here the experimentally relevant

$$U_0 = \frac{A^+}{A^-}, \qquad R_\chi = \frac{C^+ B^{\delta-1}}{B_c^\delta}, \tag{10}$$

where C^+ and $A^\pm$ are related to the critical behavior of the susceptibility χ and of the specific heat C for $H = 0$:

$$\begin{aligned} &\chi = C^+ t^{-\gamma}, && t > 0, \\ &C = A^\pm (\pm t)^{-\alpha} + B && \pm\, t > 0. \end{aligned}$$

Using the approximate equation of state we obtain [14,10,12]: $U_0 = 1.062(4)$, $R_\chi = 1.35(7)$ for $N = 2$ and $U_0 = 1.57(4)$, $R_\chi = 1.33(8)$ for $N = 3$.

For $N = 3$ the approximate equation of state can be compared with experiments. We observe good qualitative and quantitative agreement. For $N = 2$ we can use our results to predict critical properties of the λ-transition in ^{4}He. In this case, the equation of state does not have a direct physical meaning, but we can still compare the predictions for the singular specific-heat ratio U_0. A precise determination of the exponent α and of U_0 was done recently by means of a calorimetric experiment in microgravity [28] (see also reference 4 in Ref. [10]) obtaining $\alpha = -0.01056(38)$ and $U_0 \approx 1.0442$. The result for U_0 is lower than the theoretical one. This is strictly related to the disagreement in the value of α (see also Ref. [29]). Indeed, using hyperscaling we find $\alpha = -0.0146(8)$, that significantly differs from the experimental estimate. The origin of this discrepancy is unclear and further theoretical and experimental investigations are needed. A new generation of experiments in microgravity environment that is currently in preparation should clarify the issue on the experimental side [30].

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