

A resolution of the transition to turbulence paradox

Understanding the transition to turbulence has been impeded by a modeling oversight

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Despite being around for over a century, the transition to turbulence problem remains central in fluid dynamics. This phenomenon was apparently known to Leonardo da Vinci [15], who in 1507 introduced the term “la turbulenza”, and nowadays it has an impact on practically every field ranging from astrophysics and atmosphere dynamics to nuclear reactors and oil pipelines. Beginning with the systematic experimental studies in a pipe by Osborne Reynolds [1] in the 1880s, it is known that the flow becomes turbulent at finite flow rate, usually measured by Reynolds number $Re = LU/\nu$ (see the definitions in figure 1). Similar observations have been made in other flows, in particular Couette flow – the flow between two plates moving parallel to each other, cf. figure 1(a), where the transition is observed at $Re \simeq 350$ (see, for instance, [2]). Reconciliation of these experimental observations with theory [3, 14] failed because the eigenvalue analysis of the linearized Navier-Stokes equations (NSEs), which govern fluid motion, yields eigenvalues λ in the left half-plane at all values of Re , which implies that all small initial disturbances should decay exponentially like $e^{\lambda t}$, as time increases and thus one should have stability. The basic mathematical setup in classical works [3] treats the stability problem in an infinite channel $x \in (-\infty, +\infty)$, as in figure 1(a). In this work we demonstrate that this infinite channel assumption is a sticking point that has prevented one from understanding the primary instability in the transition to turbulence. Our analysis on semi-infinite channel-domain $x \in [0, +\infty)$, cf. figure 1(b), which is more relevant to the way experiments are usually done, predicts *instability* and thus explains many important features of these phenomena in a simple and basic way.

First, let us recall the NSEs, which can be written as an evolution equation for the velocity field \mathbf{u} ,

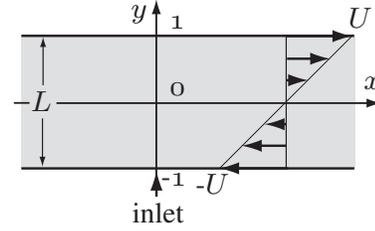
$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u} + \mathbf{N}(\mathbf{u}), \quad (1)$$

with the linear and nonlinear terms given by

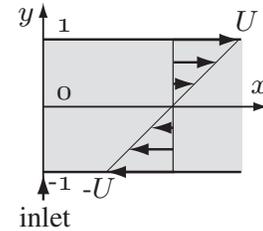
$$\mathbf{A}\mathbf{u} = \mathbb{P} [-\mathbf{U} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{U} + Re^{-1} \Delta \mathbf{u}], \quad (2a)$$

$$\mathbf{N}(\mathbf{u}) = -\mathbb{P} [\nabla \cdot (\mathbf{u} \otimes \mathbf{u})], \quad (2b)$$

where \mathbb{P} is the projection on the space of divergence free vector fields and \mathbf{U} is the base flow. In the class of weak solutions, Romanov [14] established linear and nonlinear stability, based on the absence of linearly unstable eigenmodes, of the Couette base flow $\mathbf{U} = (y, 0)$ on an infinite domain $x \in (-\infty, +\infty)$. This, of course, leads to a conceptual difficulty since instability is observed in experiments.



(a) Infinite channel setup, $x \in (-\infty, +\infty)$.



(b) Semi-infinite channel setup, $x \in [0, +\infty)$.

Figure 1: Couette flow of a fluid with kinematic viscosity ν in a channel of width L ; the base flow is $\mathbf{U} = (y, 0)$.

The failure of the hydrodynamic stability theory based on (1) to predict the transition to turbulence motivated various alternative explanations, including the idea of a very small basin of attraction of the stable base flow and the transient growth idea [4, 5]. It is also understood that the transition to turbulence belongs to a general class of counter-intuitive dissipation-induced instabilities based on the recent theory [10, 11, 12]. While all these approaches are still in development, it is worth mentioning the line of logic of the transient growth concept. Namely, based on the ansatz that the nonlinear terms (2b) of the NSEs (1) are energy conserving and since the linear terms (2a) can produce energy only transiently in time, then the transient growth is the only explanation of the fact that we observe non-zero deviations from the laminar base state \mathbf{U} in the aforementioned flows. The transient growth itself is related to the sensitive and non-normal nature

of (2a) [4], i.e. $AA^* \neq A^*A$, where A^* is the operator adjoint to A . While this picture has been appended with various dynamical systems scenarios [6], such as chaotic saddles, and it also appears to be useful as a transient effect [7], there is still no theory which would be able to predict a transition robustly. Here we propose a more direct resolution of this long-standing problem by demonstrating the existence of linearly unstable eigenmodes.

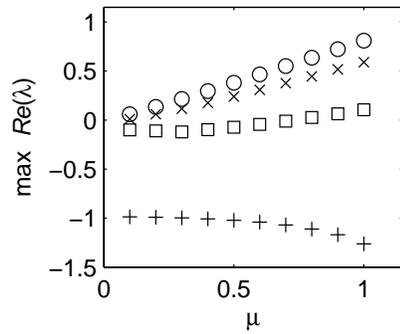
Before giving a resolution of the mismatch of theory and experiment, we would like to remind the reader that the fact of instability implies an existence of at least one eigenmode $f_\lambda(x)$ such that the corresponding eigenvalue λ has a positive real part. Perhaps because of the translational invariance of the base state $\mathbf{U} = (y, 0)$, cf. figure 1(a), or for convenience, the stability of Couette flow has always been studied on an infinite domain, $-\infty < x < +\infty$. However, if one recalls the way the experiments on the transition are usually done, i.e. one introduces disturbances at some fixed inlet location, say $x = 0$, and observes how they evolve downstream, then it becomes clear that the semi-infinite domain, $x \in [0, +\infty)$, as in figure 1(b) is more relevant as a mathematical idealization. Then, it is convenient to study the linear eigenvalue problem – the classical Orr-Sommerfeld (OS) equation – by assuming that the disturbance eigenfunction is of the form $\sim a_{\lambda\mu}(y)e^{\lambda t}e^{-\mu x}$, where $\lambda \in \mathbb{C}$ is the eigenvalue and $\mu \geq 0$ is an analog of the wavenumber:

$$\left[\frac{d^2}{dy^2} + \mu^2 - Re(\lambda - \mu y) \right] \left(\frac{d^2}{dy^2} + \mu^2 \right) a(y) = 0, \quad (3a)$$

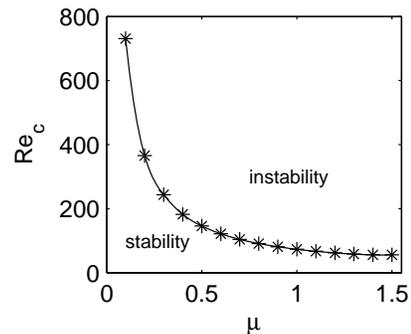
$$y = -1, 1 : a = a_y = 0, \quad (3b)$$

where we dropped the indices λ and μ . The basis functions $e^{-\mu x}$, $\mu \geq 0$, are clearly not members of the space of bounded functions on the whole real line, $x \in (-\infty, +\infty)$, which are used in the classical analysis of this problem, but they do belong to the space of functions bounded on the semi-infinite domain, $x \in [0, +\infty)$. Therefore, these eigenfunctions were not captured in the traditional approach. Alternatively, equation (3) could be treated with cosine and sine Fourier-transforms, which would lead to the same results but with a considerably more complicated version of (3). We choose to work with exponential functions $e^{-\mu x}$, since our goal here is to demonstrate the existence of unstable eigenmodes in a direct way; other reasons will be clear from the subsequent discussion. Also, this choice of functions leads to the OS equation analogous to the classical one [3]: the only difference of (3) from the classical case of the OS equation [3] studied on an infinite

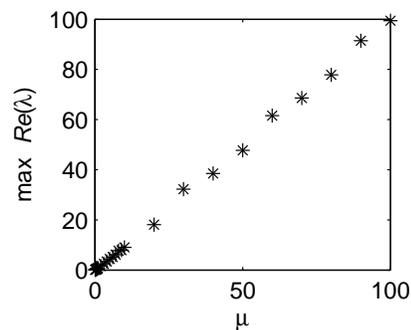
domain is μ versus $-ik$, where k is the wavenumber.



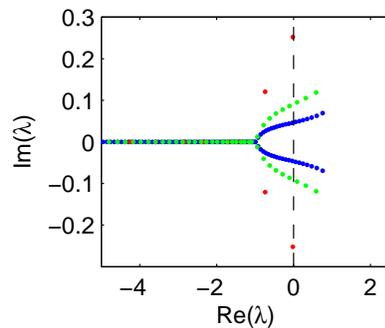
(a) $\max Re(\lambda)$; \circ : $Re = 10^4$, \times : $Re = 10^3$, \square : $Re = 10^2$, $+$: $Re = 10^1$.



(b) Critical Reynolds number $Re_c(\mu)$.



(c) Asymptotics $\max Re(\lambda)$ for $Re = 10^4$.



(d) Spectrum: distribution of eigenvalues λ_n , $n \in \mathbb{N}$ of (3) in the complex plane (eigenvalues continue to the negative part of the real axis for $\mu = 1$ and $Re = 70$ (red dots), $Re = 1000$ (green dots), and $Re = 5000$ (blue dots)).

Figure 2: Stability picture of the Couette flow.

The eigenvalue problem (3) is solved numerically by expanding $a(y)$ in Chebyshev polynomials [8]. As one learns from figure 2, the stability picture on a semi-infinite domain is in sharp contrast to what one has on an infinite domain [3], but conforms well with the usual intuitive understanding of instability phenomena: for some values of μ there are eigenvalues with positive real part and thus Couette flow is absolutely unstable. In fact, since in general all values of μ may be present in a real flow, then figure 2(a) suggests that the transition in the Couette flow is not a critical phenomenon. Indeed, figure 2(c) indicates that the instability in the Couette flow is in fact a short-wave instability, since the value of $\max \operatorname{Re}(\lambda)$ is increasing with μ , and larger μ means that the disturbance is localized around the inlet. It is also notable that the structure of the eigenfunctions corresponding to the leading eigenvalues increases in complexity with increasing μ , as illustrated in figure 3, which may explain the tangled flow picture observed experimentally in the Couette flow: see [16] and references therein. However, if in a particular experiment, the admissible magnitudes of μ are restricted to a range of small values, then one can observe critical phenomena, as in figure 2(b). These are clearly of Hopf bifurcation type, common in various fluid dynamics problems [13], as follows from the distribution of eigenvalues in the complex plane in figure 2(d), which illustrates that the leading (rightmost) eigenvalues cross the imaginary axis as Reynolds number increases. In this case one can expect that the leading eigensolutions are the usual Tollmien-Schlichting waves [10, 3] appearing via the Hopf bifurcation. We have to stress, however, that in general, if all values of μ are present, then the transition to turbulence is not a critical phenomenon and thus not a Hopf bifurcation, similar to the Rayleigh-Taylor instability, i.e. instability of a heavy fluid accelerating into a light one.

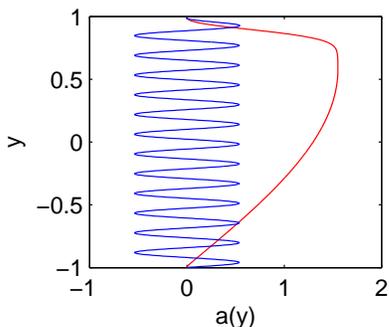


Figure 3: The eigenfunctions corresponding to the rightmost eigenvalue for the Couette flow at $Re = 5000$: red curve corresponds to $\mu = 1$ and $\operatorname{Re} \lambda_{\max} \simeq 38.53$, blue curve corresponds

to $\mu = 40$ and $\operatorname{Re} \lambda_{\max} \simeq 0.76$.

There is much more to this stability picture and a lot remains to be understood about the properties of equation (3), as well as a full function-analytic nonlinear analysis of (1) with careful treatment of the inlet boundary conditions is needed similar to [12]. However, the key feature – the existence of unstable eigenmodes – originating from the semi-infinite domain setup is well illustrated above. Analogous computations performed for the plane Poiseuille flow also revealed a similar instability picture, which suggests that the right mathematical setup used above is a universal explanation of the transition to turbulence in the aforementioned “troublesome” flows.

This counter-intuitive difference in the stability results between the semi-infinite and infinite domains can be appreciated with the assistance of the sketch in figure 4.

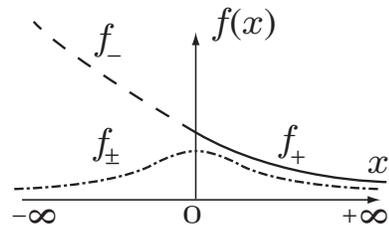


Figure 4: Eigenmodes in the stability analysis: semi- versus infinite domain.

Namely, if, for example, one restricts (eigen-) functions $f(x)$ to be bounded for all x as dictated by the fact that the physical solution should be bounded, then eigenfunctions defined on $x \in (-\infty, +\infty)$ are more restrictive compared to eigenfunctions defined on $x \in [0, +\infty)$. Indeed, if one can construct a function f_+ bounded on $x \in [0, +\infty)$ which also satisfies the OS equation (3a), then continuation of this function onto $x \in (-\infty, 0]$ may lead to an unbounded function f_- , as dictated by the structure of the linear operator (2a) and as illustrated in figure 4. Our exponential eigenfunctions $e^{-\mu x}$ are a good example of functions bounded on the right half-line and unbounded on the left half-line, while still satisfying the OS equation (3). These observations can be illustrated with the following elementary eigenvalue problem:

$$\lambda \frac{d^2 \phi(x)}{dx^2} + \frac{d^3 \phi(x)}{dx^3} = 0, \quad (4)$$

which clearly contrasts the problems on infinite and semi-infinite domains:

- $x \in (-\infty, +\infty)$ and $\phi(x) \in L_2$: applying Fourier transform we get $\lambda = -ik$, $k \in \mathbb{R}$, i.e. marginal stability.
- $x \in [0, +\infty)$ and $|\phi(x)| < \infty$: instability is present since

there are eigenfunctions $\phi \sim e^{-\mu x}$, $\mu \geq 0$, with eigenvalues $\lambda = \mu$.

Note that the above arguments also explain the sensitivity of the experimentally observed critical Reynolds number Re_c to the properties of disturbances at the domain inlet, $x = 0$: while their amplitudes do not play a role in view of the linearity of the problem, gradient-like properties of the disturbances do! Namely, by varying gradient-like properties of disturbances at $x = 0$ effectively changes the boundary conditions at $x = 0$ and thus the size of the eigenspace. Since restricting the domain to $x \in [0, +\infty)$ enlarges the function space, one can expect that the spectrum enlarges as well and may lead to instabilities. This simple fact, as we saw above, explains the mechanism behind the transition to turbulence!

Finally, note that there is nothing unusual about increasing growth rate for increasing μ , observed in figure 2(c), which is a common feature in many fundamental hydrodynamic instabilities in the short-wave limit, such as the Rayleigh-Taylor instability. Of course, in the nonlinear setting one does not observe infinite growth rates, because they are suppressed by the nonlinear terms, which play a stabilizing role opposing the formation of singularities even in the case when they are energy conserving [9]; in our case the nonlinear terms can also be dissipative. The latter fact is in apparent contrast with the above mentioned ansatz of the transient growth story, i.e. that the nonlinear terms are energy-conserving. This ansatz is valid if the disturbance field in the Couette flow problem is considered on an infinite domain with the boundary conditions at $x = \pm\infty$ corresponding to decay to zero, but not on a semi-infinite domain. Indeed, multiplying (1) with \mathbf{u}^T and integrating over the semi-infinite strip-like flow domain Ω we arrive at the Reynold-Orr equation for the kinetic energy $E(t) = \|\mathbf{u}\|^2/2$:

$$\begin{aligned}
 -\frac{dE}{dt} &= \int_{\partial\Omega} n_i u_i p \, ds + \nu \int_{\partial\Omega} n_j u_i u_{i,j} \, ds + \nu \|\nabla \mathbf{u}\|^2 \\
 &+ \langle D_{ij}, u_i u_j \rangle + \boxed{\frac{1}{2} \int_{\partial\Omega} n_j u_j u_i u_i \, ds} + \frac{1}{2} \int_{\Omega} n_j U_j u_i u_i \, dx,
 \end{aligned} \tag{5}$$

where \mathbf{n} is the outward normal of $\partial\Omega$. If the domain Ω is unbounded and open, as in the Couette or pipe flows, then the effect of the nonlinear terms (cubic term in (5)) does not disappear, since disturbances are non-zero at the inlet and, if they lead to an instability, do not necessarily decay at infinity.

References

- [1] O. Reynolds. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels. *Proc. R. Soc. London A*, 35:84–99, 1883.
- [2] A. Lundbladh and A. V. Johansson. Direct simulation of turbulent spots in plane Couette flow. *J. Fluid Mech.*, 229:499–516, 1991.
- [3] P.G. Drazin and W.H. Reid. *Hydrodynamic stability*. Cambridge University Press, 1984.
- [4] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll. Hydrodynamic stability without eigenvalues. *Science*, 261:578–584, 1993.
- [5] D. S. Henningson and S. C. Reddy. On the role of linear mechanisms in transition to turbulence. *Phys. Fluids*, 6:1396–1398, 1994.
- [6] B. Eckhardt, T. M. Schneider, B. Hof, and J. Westerweel. Turbulence transition in pipe flow. *Annu. Rev. Fluid Mech.*, 39, 2007.
- [7] B. Bamieh and M. Dahleh. Energy amplification in channel flows with stochastic excitation. *Phys. Fluids*, 13:3258–3269, 2001.
- [8] D. Gottlieb and S. A. Orszag. *Numerical Analysis of Spectral Methods: Theory and Applications*. SIAM, Philadelphia, Pennsylvania, 1977.
- [9] T. Y. Hou and Z. Lei. On the Stabilizing Effect of Convection in 3D Incompressible Flow. *Commun. Pure Appl. Math.*, 2008.
- [10] C. C. Lin. *The Theory of Hydrodynamic Stability*. Cambridge University Press, New York, 1955.
- [11] R. Krechetnikov and J. E. Marsden. Dissipation-induced instabilities in finite dimensions. *Rev. Mod. Phys.*, 79:519–553, 2007.
- [12] R. Krechetnikov and J. E. Marsden. Dissipation-induced instability phenomena in infinite-dimensional systems. 2008. to appear in *Arch. Rat. Mech. Anal.*
- [13] J. E. Marsden and M. McCracken. *The Hopf Bifurcation and Its Applications. Applied Mathematical Sciences*, 19. Springer-Verlag, New York, NY, 1976.

- [14] V. A. Romanov. Stability of plane-parallel Couette flow. *Functional Anal. & its Applications*, 7:137–146, 1973.
- [15] J. P. Richter. Plate 20 and Note 389. In *The Notebooks of Leonardo Da Vinci*. Dover Publications, New York, 1970.
- [16] H. I. Andersson, M. Lygren, and K. Kristoffersen. Roll cells in turbulent plane Couette flow: Reality or artifact? In *Lecture Notes in Physics*, 515:117–122, 1998.