Single-Stage Transmit Beamforming Design for MIMO Radar

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Abstract

MIMO radar beamforming algorithms usually consist of a signal covariance matrix synthesis stage, followed by signal synthesis to fit the obtained covariance matrix. In this paper, we propose a radar beamforming algorithm (called Beam-Shape) that performs a single-stage radar transmit signal design; i.e. no prior covariance matrix synthesis is required. Beam-Shape’s theoretical as well as computational characteristics, include: (i) the possibility of considering signal structures such as low-rank, discrete-phase or low-PAR, and (ii) the significantly reduced computational burden for beampattern matching scenarios with large grid size. The effectiveness of the proposed algorithm is illustrated through numerical examples.

Keywords: Beamforming, multi-input multi-output (MIMO) radar, peak-to-average-power ratio (PAR), signal design

1. Introduction

A key problem in the radar literature is the transmit signal design for matching a desired beampattern. In contrast to conventional phased-array radar, multiple-input multiple-output (MIMO) radar uses its antennas to transmit independent waveforms, and thus provides extra degrees of freedom (DOF) [1][2]. As a result, MIMO radars can achieve beampatterns which might be impossible for phased-arrays [3][4]. The MIMO radar transmit beampattern design...
approaches in the literature require two stages in general (see, e.g. [3]-[12]). The first stage consists of the design of the transmit covariance matrix $R$. The design of $R$ can be typically performed using convex optimization tools. Next, the transmit signals (under practical constraints) are designed in order to fit the obtained covariance matrix.

In this paper, we present a novel approach (which we call $Beam-Shape$) for “shaping” the transmit beam of MIMO radar via a single-stage transmit signal design. We consider the transmit beamspace processing (TBP) scheme [15] for system modeling (see Section 2 for details). Due to different practical (or computational) demands, two optimization problems are considered for both TBP weight matrix design as well as a direct design of the transmit signal. In comparison to the two-stage framework of beamforming approaches in the literature:

- $Beam-Shape$ is able to directly consider in its formulation the matrix rank or signal constraints (such as low peak-to-average-power ratio (PAR), or discrete-phase); an advantage which generally is not shared with the covariance matrix design. As a result, the matching optimization problem will produce optimized solutions considering all the constraints of the original problem at once, and may thus avoid the optimality losses imposed by a further signal synthesis stage. See Section 4 for some numerical illustrations.

- In beamforming scenarios with large grid size, $Beam-Shape$ appears to have a significantly smaller computational burden compared to the two-stage framework. See the related discussions in Sections 3 and 4.

**Notation:** We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero vectors/matrices. The symbol $\odot$ stands for the Hadamard (element-wise) product of matrices. $\|\mathbf{x}\|_n$ or the $l_n$-norm of the vector $\mathbf{x}$ is defined as $(\sum_k |\mathbf{x}(k)|^n)^{1/n}$ where $\{\mathbf{x}(k)\}$ are the entries of $\mathbf{x}$. The Frobenius
norm of a matrix $X$ (denoted by $\|X\|_F$) with entries $\{X(k,l)\}$ is equal to $\left(\sum_{k,l} |X(k,l)|^2\right)^{\frac{1}{2}}$. We use $\mathcal{R}(X)$ to denote the matrix obtained by collecting the real parts of the entries of $X$. Finally, $Q_p(X)$ yields the closest $p$-ary phase matrix with entries from the set $\{2k\pi/p : k = 0, 1, \cdots, p-1\}$, in an element-wise sense, to an argument phase matrix $X$.

2. Problem Formulation

Consider a MIMO radar system with $M$ antennas and let $\{\theta_l\}_{l=1}^L$ denote a fine grid of the angular sector of interest. Under the assumption that the transmitted probing signals are narrow-band and the propagation is non-dispersive, the steering vector of the transmit array (at location $\theta_l$) can be written as

$$a(\theta_l) = (e^{j2\pi f_0 \tau_1(\theta_l)}, e^{j2\pi f_0 \tau_2(\theta_l)}, \ldots, e^{j2\pi f_0 \tau_M(\theta_l)})^T,$$

where $f_0$ denotes the carrier frequency of the radar, and $\tau_m(\theta_l)$ is the time needed by the transmitted signal of the $m^{th}$ antenna to arrive at the target location $\theta_l$.

In lieu of transmitting $M$ partially correlated waveforms, the TBP technique employs $K$ orthogonal waveforms that are linearly mixed at the transmit array via a weighting matrix $W \in \mathbb{C}^{M \times K}$. The number of orthogonal waveforms $K$ can be determined by counting the number of significant eigenvalues of the matrix [15]:

$$A = \sum_{l=1}^L a(\theta_l)a^H(\theta_l).$$

The parameter $K$ can be chosen such that the sum of the $K$ dominant eigenvalues of $A$ exceeds a given percentage of the total sum of eigenvalues [15]. Note that usually $K \ll M$ (especially when $M$ is large) [15][18]. Let $\Phi$ be the matrix containing $K$ orthonormal TBP waveforms, viz.

$$\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_K)^T \in \mathbb{C}^{K \times N}, \quad K \leq M$$
where $\varphi_k \in \mathbb{C}^{N \times 1}$ denotes the $k^{th}$ waveform (or sequence). The transmit signal matrix can then be written as $S = W\Phi \in \mathbb{C}^{M \times N}$, and the transmit beampattern becomes

$$P(\theta_l) = \|S^H a(\theta_l)\|^2_2$$

$$= a^H(\theta_l)W\Phi\Phi^H W^H a(\theta_l)$$

$$= a^H(\theta_l)WW^H a(\theta_l)$$

$$= \|W^H a(\theta_l)\|^2_2.$$  

(4)

Eq. (4) sheds light on two different perspectives for radar beampattern design. Observe that matching a desired beampattern may be accomplished by considering $W$ as the design variable. Doing so, one can control the rank ($K$) of the covariance matrix $R = SS^H = WW^H$ by fixing the dimensions of $W \in \mathbb{C}^{M \times K}$. This idea becomes of particular interest for the phased-array radar formulation with $K = 1$. Note that considering the optimization problem with respect to $W$ for small $K$ may significantly reduce the computational costs.

On the other hand, imposing practical signal constraints (such as discrete-phase or low PAR) while considering $W$ as the design variable appears to be difficult. In such cases, one can resort to a direct beampattern matching by choosing $S$ as the design variable.

In light of the above discussion, we consider beampattern matching problem formulations for designing either $W$ or $S$ as follows. Let $P_d(\theta_l)$ denote the desired beampattern. According to the last equality in (4), $P_d(\theta_l)$ can be synthesized exactly if and only if there exist a unit-norm vector $p(\theta_l)$ such that

$$W^H a(\theta_l) = \sqrt{P_d(\theta_l)} p(\theta_l).$$  

(5)

Therefore, by considering $\{p(\theta_l)\}_l$ as auxiliary design variables, the beampattern matching via weight matrix design can be dealt with conveniently via the
optimization problem:

$$\min_{W, \alpha, \{p(\theta_l)\}} \sum_{l=1}^{L} \left\| W^H a(\theta_l) - \alpha \sqrt{P_d(\theta_l)} p(\theta_l) \right\|_2^2$$  \hspace{1cm} (6)$$

s.t. 

$$\mathbf{W} \odot \mathbf{W}^* \mathbf{1} = \frac{E}{M} \mathbf{1},$$  \hspace{1cm} (7)$$

$$\|p(\theta_l)\|_2 = 1, \ \forall \ l,$$  \hspace{1cm} (8)$$

where (7) is the transmission energy constraint at each transmitter with $E$ being the total energy, and $\alpha$ is a scalar accounting for the energy difference between the desired beampattern and the transmitted beam. Similarly, the beampattern matching problem with $S$ as the design variable can be formulated as

$$\min_{S, \alpha, \{p(\theta_l)\}} \sum_{l=1}^{L} \left\| S^H a(\theta_l) - \alpha \sqrt{P_d(\theta_l)} p(\theta_l) \right\|_2^2$$  \hspace{1cm} (9)$$

s.t. 

$$\mathbf{S} \odot \mathbf{S}^* \mathbf{1} = \frac{E}{M} \mathbf{1},$$  \hspace{1cm} (10)$$

$$\|p(\theta_l)\|_2 = 1, \ \forall \ l,$$  \hspace{1cm} (11)$$

$$S \in \Psi,$$  \hspace{1cm} (12)$$

where $\Psi$ is the desired set of transmit signals. The above beampattern matching formulations pave the way for an algorithm (which we call Beam-Shape) that can perform a direct matching of the beampattern with respect to the weight matrix $W$ or the signal $S$, without requiring an intermediate synthesis of the covariance matrix.

3. Beam-Shape

We begin by considering the beampattern matching formulation in (6). For fixed $W$ and $\alpha$, the minimizer $p(\theta_l)$ of (6) is given by

$$p(\theta_l) = \frac{W^H a(\theta_l)}{\|W^H a(\theta_l)\|_2}.$$  \hspace{1cm} (13)$$

Let $P \triangleq \sum_{l=1}^{L} P_d(\theta_l)$. For fixed $W$ and $\{p(\theta_l)\}$ the minimizer $\alpha$ of (6) can be obtained as

$$\alpha = \Re \left\{ \left( \sum_{l=1}^{L} \sqrt{P_d(\theta_l)} p^H(\theta_l) W^H a(\theta_l) \right) / P \right\}.$$  \hspace{1cm} (14)$$
Using (13), the expression for $\alpha$ can be further simplified as

$$\alpha = \left( \sum_{l=1}^{L} \sqrt{P_d(\theta_l)} \left\| W^H a(\theta_l) \right\|_2^2 \right) / P. \quad (15)$$

Now assume that $\{p(\theta_l)\}$ and $\alpha$ are fixed. Note that

$$Q(W) = \sum_{l=1}^{L} \left\| W^H a(\theta_l) - \alpha \sqrt{P_d(\theta_l)} p(\theta_l) \right\|_2^2$$

$$= \text{tr}(WW^H A) - 2\Re\{\text{tr}(WB)\} + P\alpha^2 \quad (16)$$

where $A$ is as defined in (2), and

$$B = \sum_{l=1}^{L} \alpha \sqrt{P_d(\theta_l)} p(\theta_l) a^H(\theta_l). \quad (17)$$

By dropping the constant part in $Q(W)$, we have

$$\tilde{Q}(W) = \text{tr}(WW^H A) - 2\Re\{\text{tr}(WB)\} \quad (18)$$

$$= \text{tr} \left( \begin{pmatrix} W & \ \\ I \end{pmatrix}^H \begin{pmatrix} A & -B^H \\ -B & 0 \end{pmatrix} \begin{pmatrix} W \\ I \end{pmatrix} \right).$$

Therefore, the minimization of (6) with respect to $W$ is equivalent to

$$\min_W \text{tr} \left( \begin{pmatrix} \tilde{W}^H & \tilde{C} \end{pmatrix} \tilde{W} \right) \quad (19)$$

s.t. \quad $$(W \odot W^*) 1 = \frac{E}{M} 1, \quad (20)$$

$$\tilde{W} = \left( W^T I \right)^T. \quad (21)$$

As a result of the energy constraint in (20), $\tilde{W}$ has a fixed Frobenius norm, and hence a diagonal loading of $C$ does not change the solution to (19). Therefore, (19) can be written in the following equivalent form:

$$\max_W \text{tr} \left( \begin{pmatrix} \tilde{W}^H & \tilde{C} \end{pmatrix} \tilde{W} \right) \quad (22)$$

s.t. \quad $$(W \odot W^*) 1 = \frac{E}{M} 1, \quad (23)$$

$$\tilde{W} = \left( W^T I \right)^T. \quad (24)$$
where $\tilde{C} = \lambda I - C$, with $\lambda$ being larger than the maximum eigenvalue of $C$. In particular, an increase in the objective function of (22) leads to a decrease of the objective function in (6). Although (22) is non-convex, a monotonically increasing sequence of the objective function in (22) may be obtained (see the Appendix for a proof) via a generalization of the power method-like iterations proposed in [19] and [20], namely:

$$W^{(t+1)} = \sqrt{\frac{E}{M}} \eta \left( \left( \begin{array}{cc} I_{M \times M} & \mathbf{0} \end{array} \right)^T \tilde{C} W^{(t)} \right)$$

where the iterations may be initialized with the latest approximation of $W$ (used as $W^{(0)}$), $t$ denotes the internal iteration number, and $\eta(\cdot)$ is a row-scaling operator that makes the rows of the matrix argument have unit-norm.

Next we study the optimization problem in (9). Thanks to the similarity of the problem formulation to (6), the derivations of the minimizers $\{p(\theta_l)\}$ and $\alpha$ of (9) remain the same as for (6). Moreover, the minimization of (9) with respect to the constrained $S$ can be formulated as the following optimization problem:

$$\max_S \quad \text{tr} \left( \tilde{S}^H \tilde{C} \tilde{S} \right)$$

s.t. \quad $$(S \odot S^*)(1) = \frac{E}{M} \mathbf{1},$$

$$\tilde{S} = \left( S^T I \right)^T, \quad S \in \Psi$$

with $\tilde{C}$ being the same as in (22). An increasing sequence of the objective function in (26) can be obtained via power method-like iterations that exploit the following nearest-matrix problem (see the Appendix for a sketched proof):

$$\min_{S^{(t+1)}} \left\| S^{(t+1)} - \left( \begin{array}{cc} I_{M \times M} & \mathbf{0} \end{array} \right)^T \tilde{C} S^{(t)} \right\|_F$$

s.t. \quad $$(S^{(t+1)} \odot S^{*(t+1)}) \mathbf{1} = \frac{E}{M} \mathbf{1}, \quad S^{(t+1)} \in \Psi.$$
unimodular, or $p$-ary matrices is straightforward, viz.

$$S^{(t+1)} = \begin{cases} \sqrt{E_M} \eta \left( \mathbb{R} \left\{ \hat{S}^{(t)} \right\} \right) , & \Psi = \text{real-values matrices}, \\ e^{j \arg(\hat{S}^{(t)})} , & \Psi = \text{unimodular matrices}, \\ e^{j Q_p(\arg(\hat{S}^{(t)}))} , & \Psi = \text{$p$-ary matrices}, \end{cases}$$

(31)

where

$$\hat{S}^{(t)} = \left( \begin{array}{c} I_{M \times M} \\ 0 \end{array} \right)^T \tilde{C} \tilde{S}^{(t)}.$$

(32)

Furthermore, the case of PAR-constrained $S$ can be handled efficiently via a recursive algorithm devised in [21].

Finally, the Beam-Shape algorithm for beampattern matching via designing the weight matrix $W$ or the transmit signal $S$ is summarized in Table 1.

Remark: A brief comparison of the computational complexity of the Beam-Shape algorithm and the two-stage beamforming approaches in the literature is as follows. The design of the covariance matrix $R \in \mathbb{C}^{M \times M}$ for the two-stage framework can be done using a semi-definite program (SDP) representation with $O(L)$ constraints. The corresponding SDP may be solved with $O(\max\{M, L\}^4 M^{1/2} \log(1/\epsilon))$ complexity, where $\epsilon > 0$ denotes the solution accuracy [22]. Using the formulation in [4], the design of $W$ or $S$ (for fitting the given covariance matrix) leads to an iterative approach with an iteration complexity of $O(M^2 K + KM^2 + K^3)$, or $O(M^2 N + NM^2 + N^3)$, respectively. On the other hand, Beam-Shape is an iterative method with an iteration complexity of $O(M(L + KH)(M + K))$ for designing $W$, and $O(M(L + NH)(M + N))$ for designing $S$; where $H$ denotes the number of required internal iterations of the power method-like methods discussed in (25) or (29). The above results suggest that Beam-Shape may be more computationally efficient when the grid size $(L)$ grows large. The next section provides numerical examples for further computational efficiency comparison between the two approaches.
Table 1: The Beam-Shape algorithm for MIMO radar beamforming

**Step 0:** Calculate the matrix $A$ using (2). Choose random $\alpha$ and $\{p(\theta_l)\}$ and initialize the matrix $B$ using (17).

**Step 1:** Use the power method-like iterations in (25) (until convergence) to obtain $W$, or (29) to obtain $S$.

**Step 2:** Update $\{p(\theta_l)\}$, $\alpha$, and $B$ using (13), (15), and (17), respectively.

**Step 3:** Repeat steps 1 and 2 until a stop criterion is satisfied, e.g. $\|W^{(v+1)} - W^{(v)}\|_F < \epsilon$ for some given $\epsilon > 0$, where $v$ denotes the total iteration number.

4. Numerical Examples with Discussions

In this section, we provide several numerical examples to show the potential of Beam-Shape in applications. Consider a MIMO radar with a uniform linear array (ULA) comprising $M = 32$ antennas with half-wavelength spacing between adjacent antennas. The total transmit power is set to $E = MN$. The angular pattern covers $[-90^\circ, 90^\circ]$ with a mesh grid size of $1^\circ$ and the desired beampattern is given by

$$P_d(\theta) = \begin{cases} 1, & \theta \in [\hat{\theta}_k - \Delta, \hat{\theta}_k + \Delta] \\ 0, & \text{otherwise} \end{cases}$$

(33)

where $\hat{\theta}_k$ denotes the direction of a target of interest and $2\Delta$ is the chosen beamwidth for each target. In the following examples, we assume 3 targets located at $\hat{\theta}_1 = -45^\circ$, $\hat{\theta}_2 = 0^\circ$ and $\hat{\theta}_3 = 45^\circ$ with a beamwidth of $24^\circ$ ($\Delta = 12^\circ$). The results are compared with those obtained via the covariance matrix synthesis-based (CMS) approach proposed in [3] and [4]. For the sake of a fair comparison, we define the mean square error (MSE) of a beampattern matching as

$$\text{MSE} = \sum_{l=1}^{L} \left| a^H(\theta_l) Ra(\theta_l) - P_d(\theta_l) \right|^2$$

(34)

which is the typical optimality criterion for the covariance matrix synthesis in the literature (including the CMS in [3] and [4]).

We begin with the design of the weight matrix $W$ using the formulation in (6). In particular, we consider $K = M$ corresponding to a general MIMO radar,
and $K = 1$ which corresponds to a phased-array. The results are shown in Fig. 1. For $K = M$, The MSE values obtained by Beam-Shape and CMS are 1.79 and 1.24, respectively. Note that a smaller MSE value was expected for CMS in this case, as CMS obtains $R$ (or equivalently $W$) by globally minimizing the MSE in (34). On the other hand, in the phased-array example (Fig. 1(b)), Beam-Shape yields an MSE value of 3.72, whereas the MSE value obtained by CMS is 7.21. Such a behavior was also expected due to the embedded rank constraint when designing $W$ by Beam-Shape, while CMS appears to face a considerable loss during the synthesis of the rank-constrained $W$.

Next we design the transmit signal $S$ using the formulation in (9). In this example, $S$ is constrained to be unimodular (i.e. $|S(k, l)| = 1$), which corresponds to a unit PAR. Fig. 2 compares the performances of Beam-Shape and CMS for two different lengths of the transmit sequences, namely $N = 8$ (Fig. 2(a)) and $N = 128$ (Fig. 2(b)). In the case of $N = 8$, Beam-Shape obtains an MSE value of 1.80 while the MSE value obtained by CMS is 2.73. For $N = 128$, the MSE values obtained by Beam-Shape and CMS are 1.74 and 1.28, respectively. Given the fact that $M = 32$, the case of $N = 128$ provides a large number of DOFs for CMS when fitting $SS^H$ to the obtained $R$ in the covariance matrix.
Finally, it can be interesting to examine the performance of Beam-Shape in scenarios with large grid size $L$. To this end, we compare the computation times of Beam-Shape and CMS for different $L$, using the same problem setup for designing $S$ (as the above example) but for $N = M = 32$. According to Fig. 3, the overall CPU time of CMS is growing rapidly as $L$ increases, which implies that CMS can hardly be used for beamforming design with large grid sizes (e.g. $L \gtrsim 10^3$). In contrast, Beam-Shape runs well for large $L$, even for $L \sim 10^6$ on a standard PC. The results leading to Fig. 3 were obtained by averaging the computation times for 100 experiments (with different random initializations) using a PC with Intel Core i5 CPU 750 @2.67GHz, and 8GB memory.

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**Appendix A. Power Method-Like Iterations Monotonically Increase the Objective Functions in (22) and (26)**

In the following, we study the power method-like iterations for designing $W$ in (22). The extension of the results to the design of $S$ in (26) is straightforward. For fixed $W^{(t)}$, observe that the update matrix $W^{(t+1)}$ is the minimizer of the
Figure 3: Comparison of computation times for Beam-Shape and CMS with different grid sizes $L$.

Criterion

$$\left\| \tilde{W}^{(t+1)} - \tilde{C}W^{(t)} \right\|_2^2 = \text{const} - 2\Re \left\{ \text{tr} \left( \tilde{W}^{(t+1)} H \tilde{C}W^{(t)} \right) \right\}$$  \hspace{1cm} (A.1)

or, equivalently, the maximizer of the criterion

$$\Re \left\{ \text{tr} \left( \tilde{W}^{(t+1)} H \tilde{C}W^{(t)} \right) \right\}$$  \hspace{1cm} (A.2)

in the search space satisfying the given fixed-norm constraint on the rows of $W$ (for $S$, one should also consider the constraint set $\Psi$). Therefore, for the optimizer $\tilde{W}^{(t+1)}$ of (22) we must have

$$\Re \left\{ \text{tr} \left( \tilde{W}^{(t+1)} H \tilde{C}W^{(t)} \right) \right\} \geq \text{tr} \left( \tilde{W}^{(t)} H \tilde{C}W^{(t)} \right).$$  \hspace{1cm} (A.3)

Moreover, as $\tilde{C}$ is positive-definite:

$$\text{tr} \left( \left( \tilde{W}^{(t+1)} - \tilde{W}^{(t)} \right) H \tilde{C} \left( \tilde{W}^{(t+1)} - \tilde{W}^{(t)} \right) \right) \geq 0$$  \hspace{1cm} (A.4)

which along with (A.3) implies

$$\text{tr} \left( \tilde{W}^{(t+1)} H \tilde{C}W^{(t+1)} \right) \geq \text{tr} \left( \tilde{W}^{(t)} H \tilde{C}W^{(t)} \right),$$  \hspace{1cm} (A.5)

and hence, a monotonic increase of the objective function in (22).

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