Deriving fuzzy weights of criteria from inconsistent fuzzy comparison matrices by using goal programming method

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Abstract. In this paper, deriving the fuzzy weights of criteria from the fuzzy pair-wise comparison matrix with triangular fuzzy elements is investigated. In the proposed method we first use the ranking function to convert the triangular fuzzy data into crisp one. Then by using the goal programming method we derive the fuzzy weights of criteria. We propose two new methods, and it is shown that these methods are able to derive the weights of criteria in the case of crisp elements. We compare the results of the presented method with some of the existing methods. The approach is illustrated by some numerical examples.

Keywords: Triangular fuzzy number, Fuzzy pair-wise comparison matrix, Goal programming; Fuzzy weight, Ranking function

1. Introduction

Multicriteria decision making (MCDM) models are characterized by the need to evaluate a finite set of alternatives with respect to multiple criteria. The main purpose in most MCDM problems is to measure the overall preference values of the alternatives on some permissible scale. Alternatives are generally first evaluated explicitly with respect to each of the criteria to obtain some sort of criterion specific priority scores which are then aggregated into overall preference values.

Determining criteria weights is a central problem in MCDM. Weights are used to express the relative importance of criteria in MCDM. When the decision maker is unable to rank the alternatives holistically and directly with respect to a criterion, pairwise comparisons are often used as intermediate decision support. Because of ease of understanding and application, pairwise comparisons play an important role in assessing the priority weights of decision criteria. Geoffrion’s gradient search method [16], Haines’ surrogate worth tradeoff method [17], Zions-Wallenius’ method [35], Saaty’s analytic hierarchy process [25], Cogger and Yu’s eigenvector method [14], Takeda, Cogger and Yu’s GEM [28], and the logarithmic least square method are just some methods which are primarily based on pairwise comparisons.

The classical pair-wise comparison matrix requires the decision maker (DM) to express his/her preferences in the form of a precise ratio matrix encoding a valued preference relation. However it can often be difficult for the DM to express exact estimates of the ratios of importance and therefore express his/her estimates as fuzzy numbers. The theory of fuzzy numbers is based on the theory of fuzzy sets which Zadeh introduced in 1965. First, Bellman and Zadeh [5] incorporate the concept of fuzzy numbers into decision analysis. The methodology presented in this paper is useful in assisting decision makers to determine criteria fuzzy weights from criteria, and it is helpful in alternative selection when these fuzzy weights are used with one of the techniques of MCDM. To deriving the weights of criteria from this fuzzy pair-wise comparison matrix is an important problem. Many methods for estimating the preference values from the pairwise comparison matrix have been proposed and their effectiveness comparatively evaluated. Some of the proposed estimating methods presume interval-scaled preference values (Barzilai, [4] and Salo, [26]). Islam et al. [18] and
Wang [29] developed a lexicographic goal programming to generate weights from inconsistent pair-wise interval comparison matrices. Buckley [6] proposed a method that extends hierarchical analysis to the case where the participants are allowed to employ fuzzy ratios in place of exact ratios. Chang [7] introduced a new approach for handling fuzzy AHP, with the use of triangular fuzzy numbers for pairwise comparison scale of fuzzy AHP, and the use of the extent analysis method for the synthetic extent value of the pairwise comparison. Cheng [11] proposed a method for ranking fuzzy numbers by distance method. That method is based on calculating the centroid point, where the distance means from original point to the centroid point. The proposed method used ranking function distance index as the order quantities in a vague environment, therefore that method can rank more than two fuzzy numbers simultaneously, and the fuzzy numbers need not be normal. Furthermore, that paper also proposed the coefficient of variation (CV index) to improve Lee and Li’s [20] method. Mikhailov [21] proposed a new approach for deriving priorities from fuzzy pair-wise comparison judgements. The proposed method is based on α - cuts decomposition of the fuzzy judgements into a series of interval comparisons. Jankowski et al. [19] proposed a method for assessing contractor selection criteria weights with fuzzy AHP method application in group decision environment. Their proposed method maximizes the group satisfaction with the final group solutions, the results are closer to the opinions expressed by the particular decision makers and, at the same time, closer to the geometric mean of the opinions, the mean considered as the synthesized group judgment. Wang and Chin [30] proposed an eigenvector method (EM) to generate interval or fuzzy weight estimate from an interval or fuzzy comparison matrix. Also in Xu and Chen [33], the concepts of adding and multiplicative consistent interval fuzzy preference relations defined, and some simple and practical linear programming models for deriving the priority weights of various interval fuzzy preference relations established. Wang and Chin [31], proposed a sound yet simple priority method for fuzzy AHP which utilized a linear goal programming model to derive normalized fuzzy weights for fuzzy pairwise comparison matrices. Ramik and Korviny [22] propose a method for obtaining inconsistency rate of pair-arise comparison matrix with fuzzy elements based on geometric mean. Taha and Rostam [27], proposed a decision support system for machine tool selection in flexible manufacturing cell using fuzzy analytic hierarchy process (fuzzy AHP) and artificial neural network. A program is developed in that model to find the Priority weights of the Evaluation Criteria and Alternative’s Ranking called PECAR for fuzzy AHP model. Ayag and Özdemir [2], proposed a fuzzy ANP-based approach to evaluate a set of conceptual design alternatives developed in a NPD environment in order to reach to the best one satisfying both the needs and expectations of customers, and the engineering specifications of company.

In this paper we apply the goal programming method to derive fuzzy weights of criteria. Goal programming was originally proposed by Charnes and Cooper [9], and is an important technique for DMs to consider simultaneously several objectives in finding a set of acceptable solutions see also [8]. Romero [23, 24] proposed an extended lexicographic goal programming and a general structure of achievement function for a goal programming model.

Also in order to compare the criteria and convert each fuzzy number to crisp one, we use the ranking function proposed by Asady and Zendehnam [1]. Asady and Zendehnam have suggested an interest approach to crisp point approximation of fuzzy numbers. Their method leads to the crisp point, which is the best related to a certain measure of distance between the fuzzy number and a crisp point of support function. Their proposed method constructed a very realistic ranking for fuzzy numbers. It is shown in [1] that their proposed method removes the shortcomings of previous methods such as Choobineh and Li [12], Yager [34], Chen [10], Baldwin and Guild [3], Chu and Tsao [13], Cheng [11] and Wang et al. [32].

By using the ranking function scores not only we able to compare the criteria but also we can remember that scores as a crisp weights of criteria.

The structure of the rest of this paper is following: The following section provides some required preliminaries. The third section of the paper gives a goal programming approach for deriving weights of criteria. Two examples are presented in section 4. The paper ends with conclusion.

2. Preliminaries

In this section we review some required concepts, about fuzzy numbers that are given from Asady and Zendehnam [1].

2.1. Some basic definitions

Fuzzy numbers are one way to describe the vagueness and lack of precision of data. The theory of fuzzy
A fuzzy number is a fuzzy set like $A : \mathbb{R} \rightarrow I = [0, 1]$ which satisfies:

- $A$ is continuous,
- $A(x) = 0$ outside some interval $[a, d]$,
- There are real numbers $b, c$ such that $a \leq b \leq c \leq d$ and
  1. $A(x)$ is increasing on $[a, b]$,
  2. $A(x)$ is decreasing on $[c, d]$,
  3. $A(x) = 1$, $b \leq x \leq c$.

We denote the set of all fuzzy numbers by $F(\mathbb{R})$.

Parametric form of fuzzy numbers is defined in Asady and Zendehnam \cite{1} as follows:

**Definition 2.2.** A fuzzy number $\tilde{A}$ in parametric form is a pair $(\tilde{A}(r), \tilde{A}(r))$ of functions $A(r), \tilde{A}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\tilde{A}(r)$ is bounded increasing continuous function,
2. $\tilde{A}(r)$ is bounded decreasing continuous function,
3. $0 \leq \tilde{A}(r) \leq 1$.

A crisp number $\lambda$ is simply represented by $\tilde{A}(r) = \tilde{A}(r) = \lambda$, $0 \leq r \leq 1$.

**Definition 2.3.** A trapezoidal fuzzy number is denoted as $\tilde{A} = (a, b, c, d)$, (see Fig. 1).

The membership function of a trapezoidal fuzzy number is express as:

\[
A(x) = \begin{cases} 
\frac{x-a}{b-a} & a \leq x \leq b, \\
1 & b \leq x \leq c, \\
\frac{d-x}{d-c} & c \leq x \leq d, \\
0 & \text{otherwise},
\end{cases}
\]

And the parametric form of a trapezoidal fuzzy number $\tilde{A}$ is as $\tilde{A} = (A(r), \tilde{A}(r))$ where:

\[
A(r) = a + (b - a)r, \quad \tilde{A}(r) = d - (d - c)r
\]

**Special case:**

If $b = c$, we have a triangular fuzzy number. The parametric form of a triangular fuzzy number denoted by $\tilde{A} = (a, b, c)$ is as follows (see Fig. 2):

\[
A(r) = a + (b - a)r, \quad \tilde{A}(r) = c - (c - b)r
\]

**Definition 2.4.** A fuzzy number $\tilde{A} = (a, b, c, d)$ is set to be non-negative fuzzy number, if and only if $a \geq 0$.

2.2. Basic operations on fuzzy numbers

Let $\tilde{A} = (A(r), \tilde{A}(r))$ and $\tilde{B} = (B(r), \tilde{B}(r))$ be two fuzzy numbers in parametric form, the arithmetic operations between them are defined by the extension principle and can be represented as follows:

**Definition 2.5.** Fuzzy addition of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as: $\tilde{A} \oplus \tilde{B} = (A(r) + B(r), \tilde{A}(r) + \tilde{B}(r))$ where

\[
A(r) + B(r) = A(r) + B(r), \quad \text{and} \quad \tilde{A}(r) + \tilde{B}(r) = \tilde{A}(r) + \tilde{B}(r)
\]

**Definition 2.6.** Fuzzy multiplication of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as: $\tilde{A} \odot \tilde{B} = (A(r)B(r), \tilde{A}(r)\tilde{B}(r))$ where

\[
A(r)B(r) = \min(A(r)B(r), A(r)\tilde{B}(r), \tilde{A}(r)B(r), \tilde{A}(r)\tilde{B}(r)), \quad \text{and} \quad \tilde{A}(r)\tilde{B}(r) = \max(A(r)B(r), A(r)\tilde{B}(r), \tilde{A}(r)B(r), \tilde{A}(r)\tilde{B}(r))
\]
Property 2.1. If \( \hat{A} = (A(r), \overline{A}(r)) \) and \( \hat{W} = (W(r), \overline{W}(r)) \) be two non-negative fuzzy numbers, then \( \hat{A} \otimes \hat{W} = (A\overline{W}(r), \overline{A}W(r)) \) where
\[
\frac{A\overline{W}(r)}{\overline{A}W(r)} = \frac{A(r)W(r)}{\overline{A}(r)\overline{W}(r)},
\]
(2.6)

Example 2.1. If \( \hat{A} = (A_L, A_M, A_U) \) and \( \hat{W} = (w_L, w_M, w_U) \) be two non-negative triangular fuzzy numbers, then \( \hat{A} \otimes \hat{W} = (A\overline{W}(r), \overline{A}W(r)) \) where
\[
A\overline{W}(r) = A_L w_U + [A_M(w_M - w_L)] + (A_U - A_M)w_L + (A_U - A_M)w_M w_M - w_L w_U + (A_U - A_M)w_M w_L - w_M w_U + \frac{w_M}{w_U} - 12,\]
(2.7)

2.3. Comparison between two fuzzy numbers

In this subsection, in order to compare two fuzzy numbers, we use the concept of ranking function.

A ranking function is a function \( g : F(\mathbb{R}) \rightarrow \mathbb{R} \), which maps each fuzzy number into the real line, where a natural order exists. Asady and Zendehnam in [1] proposed a defuzzification using minimizer of the distance between two the fuzzy number. They introduced distance minimization of a fuzzy number \( \hat{A} \) which was defined as follows:
\[
M(\hat{A}) = \frac{1}{2} \int_0^1 (\overline{A}(r) - \overline{A}(r))dr
\]
(2.8)

This ranking function have the following properties:

Property 2.2. If \( \hat{A} \) and \( \hat{B} \) be two fuzzy numbers then:
- \( M(\hat{A}) > M(\hat{B}) \) iff \( \hat{A} > \hat{B} \)
- \( M(\hat{A}) < M(\hat{B}) \) iff \( \hat{A} < \hat{B} \)
- \( M(\hat{A}) = M(\hat{B}) \) iff \( \hat{A} \sim \hat{B} \)

Property 2.3. If \( \hat{A} \) and \( \hat{B} \) be two fuzzy numbers then:
\[
M(\hat{A} \otimes \hat{B}) = M(\hat{A}) \otimes M(\hat{B})
\]
(2.9)

Property 2.4. If \( \hat{A} = (a, b, c, d) \) be a trapezoidal fuzzy number, then we have:
\[
M(\hat{A}) = \frac{1}{4}(a + b + c + d)
\]
(2.10)

Property 2.5. If \( \hat{A} = (a, b, c) \) be a triangular fuzzy number, then we have:
\[
M(\hat{A}) = \frac{1}{3}(a + 2b + c)
\]
(2.11)

Example 2.2. If \( \hat{A} = (A_L, A_M, A_U) \) and \( \hat{W} = (w_L, w_M, w_U) \) be two non-negative triangular fuzzy numbers, then \( M(\hat{A} \otimes \hat{W}) = \frac{1}{12} \int_0^1 (A\overline{W}(r) + \overline{A}W(r))dr \) and by formula (2.7) is obtained as
\[
M(\hat{A} \otimes \hat{W}) = \frac{1}{12} \left[ \left( \frac{1}{3}A_L + \frac{1}{6}A_M \right) w_U + \left( \frac{1}{3}A_L + \frac{1}{6}A_M \right) w_M + \left( \frac{1}{3}A_M + \frac{1}{6}A_U \right) w_L \right]
\]
(2.12)

2.4. Fuzzy pairwise comparison matrix

Suppose the decision maker provides fuzzy judgments instead of precise judgments for a pair-wise comparison. Without loss of generality we assume that we deal with pair-wise comparison matrix with triangular fuzzy numbers being the elements of the matrix. We consider a pair-wise comparison matrix where all its elements are triangular fuzzy numbers as follows
\[
\hat{A} = \begin{pmatrix}
(\hat{a}_{11}, \hat{a}_{12}, \hat{a}_{13}) & \cdots & (\hat{a}_{1n}, \hat{a}_{1n+1}, \hat{a}_{1n+2}) \\
\vdots & \ddots & \vdots \\
(\hat{a}_{m1}, \hat{a}_{m1+1}, \hat{a}_{m1+2}) & \cdots & (\hat{a}_{mn}, \hat{a}_{mn+1}, \hat{a}_{mn+2})
\end{pmatrix}
\]
We say that \( \hat{A} \) is reciprocal, if the following condition is satisfied:
for all \( i, j = 1, \ldots, n, \hat{a}_{ij} = (a_{ij}, a_{ij}^*, a_{ij}^{**}) \) implies that \( \hat{a}_{ij} = \left( \frac{1}{a_{ij}}, \frac{1}{a_{ij}^*}, \frac{1}{a_{ij}^{**}} \right) \).

2.5. Goal programming

Consider the following problem:
\[
\max \ (f_1(x), \ldots, f_k(x))
\]
\[
s.t. \quad x \in X
\]
(2.13)

where \( f_1, \ldots, f_k \) are objective functions and \( X \) is non-empty feasible region. Model (2.13) is called multiple objective programming. The goal programming (GP), which is introduced by Charnes and Cooper [9], is a technique for solving decision making problems with multiple objectives by achieving a set of compromising solutions. Goal programming is now an important area of multiple criteria optimization. The idea of goal programming is to establish a goal level of
achievement for each criterion. In goal programming method requires the decision maker to set goals for each objective that he/she wishes to obtain. A preferred solution is then defined as the one which minimizes the deviations from the set goals. Depending on the norm used, the solution arrived at can be interpreted either as one in which the consensus between all the measures is maximized (penalizing the more conflicting measures in favor of those that are more representative of the majority trend) or as one where preference is given to the most conflicting measures (thereby penalizing the measures that share the most information with the rest).

In the first case, the absolute difference between the multicriteria performance and the single-criterion performances is minimized (norm $L_1$); in the second case, it is the greatest difference between the multicriteria performance and the single-criterion performances (norm $L_\infty$) that is minimized (see [15]). The model in norm $L_1$ is the oldest and still most widely used form of achievement function for GP is called simple linear goal programming model (Simple LGP) and represented as model (2.14):

$$\min \sum_{i=1}^{k} (d_i^+ + d_i^-)$$

s.t.

$$f_j(x) + d_i^- - d_i^+ = b_i, \quad i = 1, \ldots, k,$$ (2.14)
$$x \in X,$$
$$d_i^+, d_i^- = 0, \quad i = 1, \ldots, k,$$
$$d_i^+, d_i^- \geq 0, \quad i = 1, \ldots, k.$$

The norm $L_\infty$ is implemented by the GP model called MINMAX (model 2.15), in which $D$ represents the maximum deviation between the multicriteria performance and the single-criterion performances. Then MINMAX GP can be formulated as the following achievement function:

$$\min \quad D$$

s.t.

$$d_i^+ + d_i^- \leq D, \quad i = 1, \ldots, k,$$
$$f_j(x) + d_i^- - d_i^+ = b_i, \quad i = 1, \ldots, k,$$ (2.15)
$$x \in X,$$
$$d_i^-, d_i^+ = 0, \quad i = 1, \ldots, k,$$
$$d_i^-, d_i^+ \geq 0, \quad i = 1, \ldots, k.$$

The DMs for their goals set some acceptable aspiration levels, $b_i$ ($i = 1, 2, \ldots, k$), for these goals, and try to achieve a set of goals as closely as possible. The purpose of GP is to minimize the deviations between the achievement of goals, $f(x)$, and these acceptable aspiration levels, $b_i$ ($i = 1, 2, \ldots, k$). Also, $d_i^+$ and $d_i^-$ are, respectively, over- and under-achievement of the $i$th goal.

### 3. Deriving the fuzzy weights of criteria

In the conventional case, if a pair wise comparison matrix $A$ be reciprocal and consistent then the weights of each criterion is simply calculated as:

$$w_i = \frac{a_{ii}}{\sum_{k=1}^{n} a_{ij}}, \quad i = 1, \ldots, n.$$ (3.1)

where $j \in \{1, \ldots, n\}$ is the index of an arbitrary column. That is $a_{ii} = \frac{a_{ij}}{a_{ji}}$ or $a_{ij} = a_{ji}$.

In the case of fuzzy matrix, we must obtain the fuzzy importance weight $\tilde{w}_i = (w^L_i, w^M_i, w^U_i)$, $i = 1, \ldots, n$, such that $\tilde{w}_i \tilde{w}_j = \tilde{w}_j = 0$.

From relation (2.9) this is equivalent to find $\tilde{w}_i = (w^L_i, w^M_i, w^U_i)$, $i = 1, \ldots, n$, such that $M(\tilde{a}_{ij} \tilde{w}_j - \tilde{w}_i) = 0$ and also by relation (2.11) this is equivalent to find $\tilde{w}_i = (w^L_i, w^M_i, w^U_i)$, $i = 1, \ldots, n$, such that:

$$\frac{1}{2} \left( \left( \frac{1}{2} w^L_i + \frac{1}{6} w^M_i \right) w^L_j + \frac{1}{6} w^M_i + \frac{2}{3} w^U_i \right) w^L_j + \frac{1}{6} w^M_i + \frac{1}{3} w^U_i \right) w^M_j + \frac{1}{3} w^U_i \right) w^U_j = 0,$$

$$i, j = 1, \ldots, n.$$ (3.2)

Therefore in the case of uncertainty, for deriving the fuzzy weights of criteria from inconsistent fuzzy comparison matrix for $i, j = 1, \ldots, n$ we introduce deviation variables $p_{ij}$ and $q_{ij}$ which leads to:

$$\frac{1}{2} \left( \left( \frac{1}{2} w^L_i + \frac{1}{6} w^M_i \right) w^L_j + \frac{1}{6} w^M_i + \frac{2}{3} w^U_i \right) w^L_j + \frac{1}{6} w^M_i + \frac{1}{3} w^U_i \right) w^M_j + \frac{1}{3} w^U_i \right) w^U_j = 0,$$

$$+ p_{ij} - q_{ij} = 0.$$ (3.3)

where $p_{ij}$ and $q_{ij}$ are both nonnegative real numbers, but can’t be positive at the same time, i.e. $p_{ij} = 0$. Now
we apply the goal programming method. It is desirable that the deviation variables $p_{ij}$ and $q_{ij}$ are kept to be small as possible, which leads to the following two goal programming models:

(Simple LGP)

$$d^* = \min \sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} + q_{ij})$$

s.t.

$$\frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_i^p + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_i^L \right\} + \frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_i^U + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_i^L \right\} + \frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_j^p + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_j^L \right\}$$

$$w_i^M - \frac{w_i^L}{w_i^U} \geq 1, \quad j = 1, \ldots, n,$$

$$w_j^M - \frac{w_j^L}{w_j^U} \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^U - w_j^L \geq 0, \quad j = 1, \ldots, n,$$

$$p_{ij}, q_{ij} \geq 0, \quad i, j = 1, \ldots, n. \quad (3.4)$$

and

(MINMAX GP)

$$h^* = \min D$$

s.t.

$$\frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_i^p + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_i^L \right\} + \frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_i^U + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_i^L \right\} + \frac{1}{2} \left\{ \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_j^p + \left( 1 - \frac{aM_{ij}}{aM_{ij}} \right) w_j^L \right\}$$

$$w_i^M - \frac{w_i^L}{w_i^U} \geq 1, \quad j = 1, \ldots, n,$$

$$w_j^M - \frac{w_j^L}{w_j^U} \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^U - w_j^L \geq 0, \quad j = 1, \ldots, n,$$

$$p_{ij} + q_{ij} \leq D, \quad i, j = 1, \ldots, n,$$

$$w_i^M - w_i^L \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^M - w_j^L \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^U, p_{ij}, q_{ij} \geq 0, \quad i, j = 1, \ldots, n. \quad (3.5)$$

By solving models (3.4 and 3.5) the optimal fuzzy weight vector $\tilde{w}_i = (w_i^p, w_i^M, w_i^L)$, $i = 1, \ldots, n$, which show the fuzzy importance of each criterion will be obtained. We can use these weights in the process of solving a multiple criteria decision making problem. Also, these weights show that which criterion is more important than others.

**Remark 3.1.** This method is also useable, even if all data of comparison matrix be in exact form. In such case we obtain the crisp weights for criteria.

**Theorem 3.1.** Model (3.4) is always feasible.

**Proof.** Consider the vector $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$, where $\tilde{w}_i = (w_i^p, w_i^M, w_i^L)$ is such that:

$$\frac{1}{2} \left\{ \sum_{j=1}^{n} w_j^p + \sum_{j=1}^{n} w_j^M + \sum_{j=1}^{n} w_j^L \right\} = 1,$$

$$w_i^M - w_i^L \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^M - w_j^L \geq 0, \quad j = 1, \ldots, n,$$

$$w_j^U \geq 0, \quad j = 1, \ldots, n.$$ 

Then we define $p_{ij}$ and $q_{ij}$ as follows:

$$
\begin{align*}
p_{ij} & = \max \left\{ \frac{1}{2} \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_j^p + \frac{1}{2} w_j^L + \frac{1}{2} aM_{ij} \right\} \\
q_{ij} & = \max \left\{ \frac{1}{2} \left( \frac{aM_{ij}}{aM_{ij}} + \frac{aL_{ij}}{aL_{ij}} \right) w_j^U + \frac{1}{2} w_j^L + \frac{1}{2} aM_{ij} \right\} \\
& \quad \vdots
\end{align*}
$$

It is clear that $(\tilde{w}, p_{ij}, q_{ij})$ is a feasible solution for model (3.4). □

**Corollary 3.1.** Model (3.5) is always feasible.

**Remark 3.2.** In order to ranking of these criteria, we assign the rank 1 to the criterion with the maximal value of $M(\tilde{w}_j)$, etc., in a decreasing order of $M(\tilde{w}_j)$.

**Special case:** The case of matrix with crisp elements. In the case of matrix with crisp data, in order to deriving the weights of criteria from the inconsistent pair-wise
comparison matrix, the goal programming model (3.4) can be converted to the following model:

\[
d^* = \min \sum_{i,j} (p_{ij} + q_{ij})
\]

s.t.

\[
a_{ij} w_j - w_i + p_{ij} - q_{ij} = 0, \quad i, j = 1, \ldots, n,
\]

\[
\sum_{i} w_j = 1,
\]

\[
w_j, p_{ij}, q_{ij} \geq 0, \quad i, j = 1, \ldots, n.
\]

(3.6)

where \( p_{ij} \) and \( q_{ij} \) are deviation variables. By solving model (3.6) the optimal weight vector \( w_i \), \( i = 1, \ldots, n \), which show the importance of each criterion will be obtained.

**Theorem 3.2.** In the case of exact data, the pairwise comparison matrix \( A \) is consistent if and only if, in model (3.6), \( d^* = 0 \).

**Proof.** Let us first prove that, if \( d^* = 0 \) then matrix \( A \) is consistent.

Since \( d^* = 0 \) we have \( p_{ij} = q_{ij} = 0 \). Therefore \( a_{ij} w_j - w_i = 0 \) and hence \( a_{ij} = \frac{w_i}{w_j} \). This gives \( a_{jk} w_k = a_{ik} \), and we conclude that matrix \( A \) is consistent.

Conversely, suppose that matrix \( A \) is consistent. That is

\[
a_{ij} a_{jk} = a_{ik}, \quad i, j, k = 1, \ldots, n.
\]

Now, if we define

\[
\begin{align*}
\pi_j &= \frac{a_{i,j}}{\sum_{k=1}^n a_{i,k}}, \\
p_{ij} &= \pi_j, \quad q_{ij} = 0,
\end{align*}
\]

then it is easy to check that \((\pi, p_{ij}, q_{ij})\) is feasible for model (3.6). Since model (3.6) has minimization form, we conclude that \( d^* = 0 \). \( \square \)

**Theorem 3.3.** Model (3.6) is always feasible.

**Proof.** By Theorem 1, proof is evident. \( \square \)

**Remark 3.3.** The same is hold for model 3.5.

4. Illustrating examples

In this section we present some illustrating examples showing that the proposed approaches are convenient tools not only for calculating the fuzzy weights of criteria of a pair-wise comparison matrices with fuzzy elements, but also for calculating the weights of criteria of crisp pair-wise comparison matrices. We apply model (3.4) for these examples. Similarly, we can apply model (3.5) for these data.

4.1. Example 1: Matrix with crisp elements

Consider the following \( 3 \times 3 \) reciprocal matrix with crisp elements:

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 4 \\
3 & 4 & 1
\end{pmatrix}
\]

We can easily check that the pairwise comparison matrix \( A \) is reciprocal but it is inconsistent. Now, for deriving the weights of criteria we apply a goal programming model (3.4) to matrix \( A \). Therefore, we must solve the following goal programming model

\[
d^* = \min p_{12} + q_{12} + p_{13} + q_{13} + p_{21} + q_{21} + p_{23} + q_{23} + p_{31} + q_{31} + p_{32} + q_{32}
\]

s.t.

\[
\begin{align*}
0.50w_2 - w_1 + p_{12} - q_{12} &= 0, \\
0.25w_1 - w_1 + p_{13} - q_{13} &= 0, \\
2.00w_1 - w_2 + p_{21} - q_{21} &= 0, \\
0.25w_3 - w_2 + p_{23} - q_{23} &= 0, \\
4.00w_1 - w_3 + p_{31} - q_{31} &= 0, \\
4.00w_2 - w_3 + p_{32} - q_{32} &= 0, \\
w_1 + w_2 + w_3 &= 1, \\
w_i, p_{ij}, q_{ij} &\geq 0, \quad 1 \leq i, j \leq 3.
\end{align*}
\]

(4.1)

By solving model (4.1), we obtain the optimal vector \( W = (w_1, \ldots, w_3) \). We assign the rank 1 to the criteria with the maximal value of \( w_j \), etc., in a decreasing order of \( w_j \). The result is shown in Table 1. The optimal objective of model (4.1) is \( d^* = 0.249 \), which shows that the pairwise comparison matrix \( A \) is inconsistent by Theorem 2.

Clearly, criterion 3 is the most important criterion between these criteria.
4.2. Example 2: Matrix with fuzzy elements

Consider the following $3 \times 3$ reciprocal matrix with triangular fuzzy elements:

$$\lambda = \begin{pmatrix}
(1, 1, 1) & (2, 3, 4) & (4, 5, 6) \\
(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}) & (1, 1, 1) & (3, 4, 5) \\
(\frac{1}{1}, \frac{1}{1}, \frac{1}{1}) & (\frac{1}{4}, \frac{1}{5}, \frac{1}{6}) & (1, 1, 1)
\end{pmatrix}$$

For deriving the fuzzy weights of criteria, similar to model (3.4), we construct the corresponding goal programming model. The results of the model is shown in Table 2.

Table 2. The fuzzy weights for example 2 and ranking

<table>
<thead>
<tr>
<th>DMU</th>
<th>The obtained weights $M(w_1)$</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$w_1 = (0.55301, 0.66667, 0.78301)$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$w_2 = (0.69355, 0.21155, 0.32155)$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$w_3 = (0.12293, 0.12293, 0.24293)$</td>
<td>3</td>
</tr>
</tbody>
</table>

By relation (2.11) we convert the fuzzy data of Table 2, to exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 2. Therefore, we can remember it as a crisp weight.

5. Comparing with the existing methods

In this section, we provide five numerical examples to illustrate the potential applications of the proposed method. And also we apply them for compare the proposed methods with some of the existing methods. These methods propose some approaches to derive weights from the fuzzy pairwise comparison matrices. Among the existing methods, we consider the following methods:

- Mikhailov [21] proposed a new approach for deriving priorities from fuzzy pairwise comparison judgements.
- Wang and Chin [30], proposed an eigenvector method (EM) to generate interval or fuzzy weight estimate from an interval or fuzzy comparison matrix.
- Wang and Chin [31], proposed a sound yet simple priority method for fuzzy AHP which utilized a linear goal programming model to derive normalized fuzzy weights for fuzzy pairwise comparison matrices.
- Taha and Rostam [27], proposed a decision support system for machine tool selection in flexible manufacturing cell using fuzzy analytic hierarchy process (fuzzy AHP) and artificial neural network.
- A program is developed in that model to find the Priority weights of the Evaluation Criteria and Alternative’s Ranking called PECAR for fuzzy AHP model.
- Ayag and Özdemir [2], proposed a fuzzy ANP-based approach to evaluate a set of conceptual design alternatives developed in a NPD environment in order to reach to the best one satisfying both the needs and expectations of customers, and the engineering specifications of company.

5.1. Example 1

Consider the following fuzzy comparison matrix which is derived from Mikhailov [21].

$$\tilde{A} = \begin{pmatrix}
(1, 1, 1) & (\frac{2}{2}, \frac{1}{2}, \frac{2}{2}) & (\frac{3}{2}, \frac{1}{2}, \frac{3}{2}) \\
(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & (1, 1, 1) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\
(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & (1, 1, 1)
\end{pmatrix}$$

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Mikhailov [21] are shown in Table 3.

By using the ranking function we convert the fuzzy data of Table 3, to the exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 3. In this example we see that the both of the proposed methods produce the fuzzy weights, but the Mikhailov [21] method produce the crisp weights. However, if we defuzzificate the obtained fuzzy weights of proposed methods by ranking function, we can see that the results of three methods are very close.

5.2. Example 2

Consider the following fuzzy comparison matrix which is derived from Wang and Chin [30].

$$B = \begin{pmatrix}
(1, 1, 1) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\
(2, 3, 4) & (1, 1, 1) & (1, 1, 1) \\
(2, 3, 4) & (1, 1, 1) & (1, 1, 1)
\end{pmatrix}$$
5.3. Example 3

Consider the following fuzzy comparison matrix which is derived from Wang and Chin [31].

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 0.5 & 1 \\
0.5 & 1 & 1 & 0.5 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Wang and Chin [30] method are shown in Table 4. By using the ranking function we convert the obtained fuzzy weights of Table 4 to the exact ones. Therefore we can compare these criteria. The results are shown in two last columns of Table 4. In this example we see that the both of the proposed methods and Wang and Chin [30] method produce the fuzzy weights and when we defuzzifize them by ranking function we can see that the results are very close.

5.3. Example 3

Consider the following fuzzy comparison matrix which is derived from Wang and Chin [31].

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 2 \\
1 & 0.5 & 1 \\
\end{bmatrix}
\]

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Wang and Chin [31] are shown in Table 5. By using the ranking function we convert the fuzzy data of Table 5, to exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 5. In this example we see that the both of the proposed method and Wang and Chin [31] method produce the fuzzy weights and when we defuzzifize them by ranking function we can see that the results are very close.

### Table 3

<table>
<thead>
<tr>
<th>Method</th>
<th>The obtained fuzzy weights</th>
<th>( M(W) )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed LGP</td>
<td>( (0.000, 0.012, 0.400) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Proposed minMax</td>
<td>( (0.582, 0.504, 0.617) )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Proposed simple LGP</td>
<td>( (0.500, 0.302, 0.370) )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Mihailov [21]</td>
<td>( 0.105 )</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>Method</th>
<th>The obtained fuzzy weights</th>
<th>( M(W) )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed LGP</td>
<td>( (0.136, 0.144, 0.217) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Proposed minMax</td>
<td>( (0.404, 0.427, 0.345) )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Proposed simple LGP</td>
<td>( (0.145, 0.157, 0.179) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Wang and Chin [30] method</td>
<td>( (0.000, 0.152, 0.250) )</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

### Table 5

<table>
<thead>
<tr>
<th>Method</th>
<th>The obtained fuzzy weights</th>
<th>( M(W) )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed LGP</td>
<td>( (0.000, 0.023, 0.402) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Proposed minMax</td>
<td>( (0.570, 0.582, 0.605) )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Proposed simple LGP</td>
<td>( (0.000, 0.156, 0.219) )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Wang and Chin [31] method</td>
<td>( (0.4194, 0.5405, 0.5927) )</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Wang and Chin [31] are shown in Table 5. By using the ranking function we convert the fuzzy data of Table 5, to exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 5. In this example we see that the both of the proposed method and Wang and Chin [31] method produce the fuzzy weights and when we defuzzifize them by ranking function we can see that the results are very close.
5.4. Example 4

Now, consider the following fuzzy comparison matrix which is derived from Ayag and Ozdemir [2].

\[ D = \begin{bmatrix}
(1, 1, 1) & (1, 3, 5) & (5, 7, 9) \\
(5, 7, 9) & (1, 1, 1) & (1, 3, 5) \\
(1, 3, 5) & (5, 7, 9) & (1, 1, 1)
\end{bmatrix} \]

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Ayag and Ozdemir [2] are shown in Table 6.

By using the ranking function we convert the fuzzy data of Table 6, to exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 6. In this example we see that the both of the proposed methods produce the fuzzy weights, but the Ayag and Ozdemir [2] method produce the crisp weights. However, if we defuzzicate the obtained fuzzy weights of proposed methods by ranking function, we can see that the results of three methods are very close.

5.5. Example 5

Consider the following fuzzy comparison matrix which is derived from Taha and Rostam [27].

\[ E = \begin{bmatrix}
(1, 1, 1) & (1, 3, 5) & (5, 7, 9) \\
(1, 3, 5) & (1, 1, 1) & (1, 3, 5) \\
(5, 7, 9) & (1, 3, 5) & (1, 1, 1)
\end{bmatrix} \]

For deriving the fuzzy weights of criteria, we construct the corresponding goal programming model. The results of the proposed models and Taha and Rostam [27] are shown in Table 7.

By using the ranking function we convert the fuzzy data of Table 7, to exact one. Therefore we can compare these criteria. The results are shown in two last columns of Table 7. In this example we see that the both of the proposed methods produce the fuzzy weights, but the Taha and Rostam [27] method produce the crisp weights. However, if we defuzzicate the obtained fuzzy weights of proposed methods by ranking function, we can see that the results of three methods are very close.

---

Table 6

<table>
<thead>
<tr>
<th>Method</th>
<th>The obtained fuzzy weights</th>
<th>( M_{w1} )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed simple LBP method</td>
<td>((0.066, 0.078, 0.701))</td>
<td>0.661</td>
<td>1</td>
</tr>
<tr>
<td>Proposed minMax method</td>
<td>((0.203, 0.226, 0.238))</td>
<td>0.235</td>
<td>2</td>
</tr>
<tr>
<td>Proposed simple LGP method</td>
<td>((0.081, 0.093, 0.116))</td>
<td>0.096</td>
<td>3</td>
</tr>
<tr>
<td>Proposed minMax method</td>
<td>((0.661, 0.673, 0.684))</td>
<td>0.676</td>
<td>1</td>
</tr>
<tr>
<td>Ayag and Ozdemir [2] method</td>
<td>((0.215, 0.238, 0.250))</td>
<td>0.235</td>
<td>2</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.774, 0.080, 0.109))</td>
<td>0.089</td>
<td>3</td>
</tr>
</tbody>
</table>

\( W_1 = 0.666 \)  
\( W_2 = 0.249 \)  
\( W_3 = 0.091 \)

Table 7

<table>
<thead>
<tr>
<th>Method</th>
<th>The obtained fuzzy weights</th>
<th>( M_{w1} )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed simple LBP method</td>
<td>((0.027, 0.039, 0.062))</td>
<td>0.042</td>
<td>4</td>
</tr>
<tr>
<td>Proposed minMax method</td>
<td>((0.651, 0.658, 0.660))</td>
<td>0.651</td>
<td>1</td>
</tr>
<tr>
<td>Proposed simple LGP method</td>
<td>((0.114, 0.126, 0.149))</td>
<td>0.129</td>
<td>3</td>
</tr>
<tr>
<td>Proposed minMax method</td>
<td>((0.000, 0.011, 0.062))</td>
<td>0.179</td>
<td>2</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.581, 0.604, 0.676))</td>
<td>0.639</td>
<td>1</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.119, 0.131, 0.154))</td>
<td>0.134</td>
<td>3</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.000, 0.010, 0.044))</td>
<td>0.047</td>
<td>2</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.052)</td>
<td>0.555</td>
<td>1</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.710)</td>
<td>0.877</td>
<td>3</td>
</tr>
<tr>
<td>Taha and Rostam [27] method</td>
<td>((0.233)</td>
<td>0.723</td>
<td>2</td>
</tr>
</tbody>
</table>
6. Conclusion

In this paper, we investigated deriving the fuzzy weights of criteria of pair-wise comparison matrix with triangular fuzzy elements. We have proposed two new methods. These two methods for driving the fuzzy weights, proposed in this paper have some common characteristics. Both methods:

- derive the fuzzy weights from fuzzy pairwise comparison matrix;
- derive crisp weights in the case of crisp pairwise comparison matrix;
- can easily be applied for group decision-making.

In the presented methods by using the ranking function we construct a goal programming model for driving the fuzzy weights of criteria. In their present formulations, the simple LGP method and MINMAX method are suitable for driving fuzzy weights, where the judgments are represented as triangular fuzzy sets, but they can easily be modified for other types of fuzzy judgment matrices such as trapezoidal numbers. The approaches are illustrated by using some numerical examples.

References


