SETS OF GOOD INDISCERNIBLES AND CHANG CONJECTURES WITHOUT CHOICE

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ABSTRACT. With the help of sets of good indiscernibles above a certain height, we show that Chang conjectures involving four, finitely many, or an ω-sequence of cardinals have a much lower consistency strength with ZF than they do with ZFC. We will prove equiconsistency results for *any* finitely long Chang conjecture that starts with the successor of a regular cardinal. In particular, any Chang conjecture of the form

$$(\kappa_n, \dots, \kappa_0) \twoheadrightarrow (\lambda_n, \dots, \lambda_0),$$

where κ_n is the successor of a regular cardinal, in a model of ZF, is equiconsistent to the existence of $(n-1)$-many Erdős cardinals in a model of ZFC.

For ω-long Chang conjectures we will see that ZF+"$\bigcup \kappa_n = \bigcup \lambda_n$"+ "$\kappa_n > \lambda_n$ for at least one $n \in \omega$" +

$$(\dots, \kappa_n, \dots, \kappa_0) \twoheadrightarrow (\dots, \lambda_n, \dots, \lambda_0),$$

is equiconsistent to the theory ZFC+"a measurable cardinal exists".

We will use symmetric forcing to create models of ZF from models of ZFC and the Dodd-Jensen core model for the other way around. All theorems in this paper are theorems of ZFC, unless otherwise stated.

Since Rowbottom's PhD thesis in 1964, where he connects large cardinal properties with model theoretic transfer properties, there has been extensive research on the connection between these two fields of mathematical logic. The property that lies in the centre of these investigations is that of good indiscernibility.

Definition 0.1. For a structure $\mathcal{A} = \langle A, \dots \rangle$, where A is a set of ordinals, a set $I \subseteq A$ is called a set of indiscernibles if for every $n \in \omega$, every n-ary formula ϕ in the language for $\mathcal{A}$, and every $\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n$ in I, if $\alpha_1 < \dots < \alpha_n$ and $\alpha'_1 < \dots < \alpha'_n$ then

$$\mathcal{A} \models \phi(\alpha_1, \dots, \alpha_n) \text{ iff } \mathcal{A} \models \phi(\alpha'_1, \dots, \alpha'_n).$$

The set I is called a set of *good indiscernibles* iff it is as above and moreover we allow parameters that lie below $\min\{\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n\}$, i.e., if moreover for every $x_1, \dots, x_m \in A$ such that $x_1, \dots, x_m \leq \min\{\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_n\}$, and every $(n+m)$-ary formula ϕ,

$$\mathcal{A} \models \phi(x_1, \dots, x_m, \alpha_1, \dots, \alpha_n) \text{ iff } \mathcal{A} \models \phi(x_1, \dots, x_m, \alpha'_1, \dots, \alpha'_n).$$

The existence of sets of good indiscernibles for first order structures ensures Chang conjectures.

Definition 0.2. For infinite cardinals $\kappa_0 < \kappa_1 < \dots < \kappa_n$ and $\lambda_0 < \lambda_1 < \dots < \lambda_n$, a Chang conjecture is the statement

$$(\kappa_n, \dots, \kappa_0) \twoheadrightarrow (\lambda_n, \dots, \lambda_0),$$

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which we define to mean that for every first order structure $\mathcal{A} = \langle \kappa_n, \ldots \rangle$ with a countable language there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ of cardinality λ_n such that for every $i \leq n$, $|\mathcal{B} \cap \kappa_i| = \lambda_i$.

Since the structures we will consider will always be wellorderable, we will implicitly assume that the have complete sets of Skolem functions. Thus we will always be able to take Skolem hulls, even when the axiom of choice (AC) is not available. According to Vaught [Vau63], the model theoretic relation

$$(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$$

was first considered by Chang, and it is referred to as "the original Chang conjecture".

Under AC there has been extensive research in the connection between Chang conjectures and Erdős cardinals.

Definition 0.3. For ordinals α, β, γ, the partition relation

$$\beta \to (\alpha)^{<\omega}_{\gamma}$$

means that for any partition $f : [\beta]^{<\omega} \to \gamma$ of the set of finite subsets of β into γ many sets, there exists an $X \in [\beta]^{\alpha}$, i.e., a subset of β with ordertype α, that is homogeneous for $f \restriction [\beta]^n$ for each $n \in \omega$, i.e., for every $n \in \omega$, $|f``[X]^n| = 1$.

For an infinite ordinal α, the α-Erdős cardinal $\kappa(\alpha)$ is the least κ such that $\kappa \to (\alpha)^{<\omega}_2$. When there is an infinite ordinal α such that $\kappa = \kappa(\alpha)$ we may call κ an Erdős cardinal.

As we see in [LMS90, 1.8(1)], Silver proved in unpublished work that if the ω_1-Erdős cardinal exists then we can force the original Chang conjecture to be true. Kunen in [Kun78] showed that for every $n \in \omega$, $n \geq 1$, the consistency of the Chang conjecture

$$(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$$

follows from the consistency of the existence of a huge[1] cardinal. Donder, Jensen, and Koppelberg in [DJK79] showed that if the original Chang conjecture is true, then the ω_1-Erdős cardinal exists in an inner model. According to [LMS90], the same proof shows that for any infinite cardinals κ, λ, the Chang conjecture

$$(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$$

implies that there is an inner model in which the μ-Erdős cardinal exists, where $\mu = (\lambda^+)^V$. According to the same source, for many other regular cardinals κ, a Chang conjecture of the form

$$(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$$

is equiconsistent with the existence of the ω_1-Erdős cardinal [LMS90, 1.10].

Chang conjectures involving higher successors on the right hand side have consistency strength stronger than an Erdős cardinal. Donder and Koepke showed in [DK83] that for $\kappa \geq \omega_1$,

$$(\kappa^{++}, \kappa^+) \twoheadrightarrow (\kappa^+, \kappa),$$

then $0^\dagger$ exists, which implies that there is an inner model with a measurable cardinal. A year later Levinski published [Lev84] in which the existence of $0^\dagger$ is derived from each of the following Chang conjectures:

- for any infinite κ and any $\lambda \geq \omega_1$, the Chang conjecture $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$
- for any natural number $m > 1$ and any infinite κ, λ the Chang conjecture $(\kappa^{+m}, \kappa) \twoheadrightarrow (\lambda^{+m}, \lambda)$, and

[1]The definition of a huge cardinal can be found in [Kan03, page 331].

- for any singular cardinal κ, the Chang conjecture $(\kappa^+,\kappa) \twoheadrightarrow (\omega_1,\omega)$.

In 1988, Koepke improved on some of these results by deriving the existence of inner models with sequences of measurable cardinals [Koe88] from Chang conjectures of the form $(\kappa^{++},\kappa^+) \twoheadrightarrow (\kappa^+,\kappa)$ for $\kappa \geq \omega_1$. Since then, much stronger large cardinal lower bounds have been found for Chang conjectures of this form under AC. Schindler showed in [Sch97] that an inner model with a strong cardinal[2] exists, if one assumes the Chang conjecture $(\omega_{n+2},\omega_{n+1}) \twoheadrightarrow (\omega_{n+1},\omega_n)$ plus $2^{\omega_{n-1}} = \omega_n$, for any $1 < n < \omega$, and Cox in [Cox11] got an inner model with a weak repeat measure[3] from the Chang conjecture $(\omega_3,\omega_2) \twoheadrightarrow (\omega_2,\omega_1)$.

Finally, under the axiom of choice we may also get inconsistency from certain Chang conjectures. As we see in [LMS90, 1.6], finite gaps cannot be increased, e.g.,

$$(\omega_5,\omega_4) \twoheadrightarrow (\omega_3,\omega_1)$$

is inconsistent.

If we remove AC from our assumptions this picture changes drastically. In this paper we will get successors of regular cardinals with Erdős-like properties using symmetric forcing, thereby all sorts of Chang conjectures will become ‘accessible’.

The connection between Erdős cardinals and Chang conjectures lies in the existence of sets of good indiscernibles. Koepke, strengthening a result of Silver ([Kan03, Theorem 9.3]), proved in [AK08, Proposition 8] that for limit ordinals α, $\kappa \to (\alpha)^{<\omega}_2$ is equivalent to the existence of a set $X \in [\kappa]^\alpha$ of good indiscernibles for any first order structure $\mathcal{M} = \langle M,\ldots\rangle$ with a countable language and $M \supseteq \kappa$.

This result has been used in connection with core model arguments, and there the ordertype of the set of good indiscernibles is important. For our proofs here it is helpful to also specify at which height does the set of good indiscernibles lie.

Definition 0.4. For a cardinal κ an ordinal $\alpha \leq \kappa$ and an ordinal $\theta < \kappa$ we define the partition property

$$\kappa \to^\theta (\alpha)^{<\omega}_2$$

to mean that for every first order structure $\mathcal{A} = \langle \kappa,\ldots\rangle$ with a countable language, there is a set $I \in [\kappa\setminus\theta]^\alpha$ of good indiscernibles for $\mathcal{A}$. We call such a κ an Erdős-like cardinal with respect to θ,α, or just an Erdős-like cardinal, if there are such θ,α.

We used this notation and called these cardinals Erdős-like, due to the strong connection between Erdős and Erdős-like cardinals.

Lemma 0.5 (ZF)**.** *Let κ,θ be infinite cardinals such that $\kappa > \theta$ and let $\alpha \leq \kappa$ be a limit ordinal. The following are equivalent:*

(a) $\kappa \to^\theta (\alpha)^{<\omega}_2$

(b) *For any partition $f : [\kappa]^{<\omega} \to 2$ there is a homogeneous set $I \in [\kappa \setminus \theta]^\alpha$.*

Consequently, $\kappa \to^\theta (\alpha)^{<\omega}_2$ implies that the Erdős cardinal $\kappa(\alpha)$ exists. Moreover, the existence of $\kappa(\alpha)$ implies that for every $\theta' < \kappa(\alpha)$, $\kappa(\alpha) \to^{\theta'} (\alpha)^{<\omega}_2$.

Proof. Assume (a) and let $f : [\kappa]^{<\omega} \to 2$ be arbitrary. Consider the structure $\mathcal{A} = \langle \kappa, f \restriction [\kappa]^n\rangle_{n\in\omega}$, where each $f \restriction [\kappa]^n$ is considered as a relation. Clearly, any set $I \in [\kappa \setminus \theta]^\alpha$ of good indiscernibles for $\mathcal{A}$ is also a homogeneous set for f, so (b) holds.

Now assume that (b) holds and let $\mathcal{A} = \langle \kappa,\ldots\rangle$ be an arbitrary first order structure with a countable language. Using the same proof as [AK08, Proposition 8] we get a set of good indiscernibles $X \in [\kappa \setminus \theta]^\alpha$ for $\mathcal{A}$. Therefore (a) holds.

[2]For a definition of a strong cardinal see e.g. [Kan03, Page 358].

[3]A definition of a weak repeat measure can be found in [Cox11, Definition 11] and in [Git95].

Now assume that the α-Erdős cardinal $\kappa(\alpha) =: \mu$ exists. Let $\theta' < \mu$ be arbitrary and let $g : [\theta']^{<\omega} \to 2$ be a partition without any homogeneous sets. Let $\mathcal{A} = \langle \kappa, \dots \rangle$ be an arbitrary first order structure with a countable language and consider the structure $\bar{\mathcal{A}} := \mathcal{A}^\frown \langle \theta', g \restriction [\theta']^n \rangle_{n\in\omega}$, where θ' and each $g \restriction [\theta']^n$ are considered as relations. By [AK08, Proposition 8] there is a set $I \in [\mu]^\alpha$ of good indiscernibles for $\bar{\mathcal{A}}$. There must be at least one $x \in I \setminus \theta$ otherwise I would be a homogeneous set for g. By indiscernibility, every element of I is above θ. Therefore $\mu \to^{\theta'} (\alpha)_2^{<\omega}$. qed

The existence of an Erdős-like cardinal implies all sorts of four cardinal Chang conjectures.

Lemma 0.6 (ZF)**.** *If $\kappa \geq \theta, \lambda$ are cardinals such that $\kappa \neq \theta$ and $\kappa \to^\theta (\lambda)_2^{<\omega}$ then for all infinite $\rho \leq \lambda \cap \theta$, the Chang conjecture $(\kappa, \theta) \twoheadrightarrow (\lambda, \rho)$ holds.*

Proof. Let $\mathcal{A} = \langle \kappa, \dots \rangle$ be an arbitrary first order structure with a countable language. Since κ is Erdős-like with respect to θ, λ, there is a set $I \in [\kappa \setminus \theta]^\lambda$ of good indiscernibles for $\mathcal{A}$. Let $\rho \leq \lambda \cap \theta$ be arbitrary and let $\mathsf{Hull}(I \cup \rho)$ be the $\mathcal{A}$-Skolem hull of $I \cup \rho$. By [Hod97, 1.2.3] we have that

$$|\mathsf{Hull}(I \cup \rho)| \leq |I \cup \rho| + |L| = \lambda.$$

But $\lambda = |I \cup \rho| \leq |\mathsf{Hull}(I \cup \rho)|$, thus $\mathsf{Hull}(I \cup \rho)$ has cardinality λ. Because all the indiscernibles lie above θ and because they are good indiscernibles, they are indiscernibles with respect to parameters below θ. So

$$\rho \leq |\mathsf{Hull}(I \cup \rho) \cap \theta| \leq \omega \cdot \rho = \rho.$$

So the elementary substructure $\mathsf{Hull}(I \cup \rho) \prec \mathcal{A}$ is as we wanted and $(\kappa, \theta) \twoheadrightarrow (\lambda, \rho)$ holds. qed

At this point we should note that Chang conjectures do not imply that some cardinal is Erdős (or Erdős-like), and therefore justify our consistency strength investigation. For this we'll need that Chang conjectures are preserved under c.c.c.-forcing. This is a well known fact but we could not find a reference for it, so we attach a proof here. We assume basic knowledge of forcing as presented, e.g., in [Kun80] or [Jec03, Chapter 14].

Proposition 0.7. *Let V be a model of* ZFC *in which for the cardinals $\kappa, \theta, \lambda, \rho$, the Chang conjecture $(\kappa, \lambda) \twoheadrightarrow (\lambda, \rho)$ holds. Assume also that P is a c.c.c.-forcing. If G is a P-generic filter, then $(\kappa, \lambda) \twoheadrightarrow (\lambda, \rho)$ holds in $V[G]$ as well.*

Proof. Let $\mathcal{A} = \langle \kappa, f_i, R_j, c_k \rangle_{i,j,k\in\omega} \in V[G]$ be arbitrary. Since the language of $\mathcal{A}$ is countable, let $\{\exists x \phi_n(x) \ ; \ n \in \omega\}$ enumerate the existential formulas of $\mathcal{A}$'s language in a way such that for every $n \in \omega$, the arity $\mathsf{ar}(\phi_n) = k_n$ of ϕ_n is less than n. For every $n \in \omega$ let g_n be the Skolem function that corresponds to ϕ_n, and let $\dot{g}_n$ be a nice name for g_n as a subset of κ^{k_n}. Since $\dot{g}_n$ is a nice name, it is of the form

$$\dot{g}_n := \bigcup \{\{\check{x}\} \times A_x \ ; \ x \in \kappa^{k_n}\}.$$

Where each A_x is an antichain of P and since P has the c.c.c., each A_x is countable. For each $x \in \kappa^{k_n}$, let $A_x := \{p_{x,0}, p_{x,1}, p_{x,2}, \dots\}$. In V define for each $n \in \omega$ a function $g_n : \kappa^{k_n - 1} \times \omega \to \kappa$ as follows:

$$g_n(\alpha_1, \dots, \alpha_{k_n-1}, \ell) := \begin{cases} \beta & \text{if } p_{\{\alpha_1,\dots,\alpha_{k_n-1},\beta\},\ell} \Vdash \dot{g}_n(\check{\alpha}_1, \dots, \check{\alpha}_{k_n-1}) = \check{\beta} \\ 0 & \text{otherwise.} \end{cases}$$

In V consider the structure $\mathcal{C} := \langle \kappa, g_n \rangle_{n\in\omega}$. Using the Chang conjecture in V take a Chang substructure $\langle B, g_n \rangle_{n\in\omega} \prec \mathcal{C}$. But then in $V[G]$ we have that $\mathcal{B} := \langle B, f_i, R_j, c_k \rangle_{i,j,k\in\omega} \prec \mathcal{A}$ is the elementary substructure we were looking for. qed

Lemma 0.8. *Let $\kappa, \theta, \lambda, \rho$ be infinite cardinals in a model V of* ZFC*, such that $\kappa \geq \lambda$, $\kappa > \theta$, and $\lambda \cap \theta \geq \rho$, and assume that $(\kappa, \theta) \twoheadrightarrow (\lambda, \rho)$. Then there is a generic extension where $(\kappa, \theta) \twoheadrightarrow (\lambda, \rho)$ holds and κ is not the λ-Erdős.*

Proof. If κ is not the λ-Erdős in V then we are done. So assume that $\kappa = \kappa(\lambda)$ in V. Let $\mu \geq \kappa$ and consider the partial order $P := \{p : \mu \times \omega \rightharpoonup 2\ ;\ |p| < \omega\}$, where $\rightharpoonup$ denotes a partial function. This partial order adds μ many Cohen reals and has the c.c.c. so all cardinals are preserved by this forcing. By Proposition 0.7, the Chang conjecture is preserved as well. Now let G be a P-generic filter. We have that $(2^\omega)^{V[G]} \geq \mu > \kappa$. We will show that in $V[G]$, $\kappa \not\rightarrow (\omega_1)^2_2$ so κ is not ξ-Erdős for any $\xi \geq \omega_1$.

Let $\mathbb{R}$ denote the set of reals and let $g : \kappa \to \mathbb{R}$ be injective. Define $F : [\kappa]^2 \to 2$ by

$$F(\{\alpha, \beta\}) := \begin{cases} 1 & \text{if } g(\alpha) <_{\mathbb{R}} g(\beta) \\ 0 & \text{otherwise} \end{cases}$$

If there was an ω_1-sized homogeneous set for F then $\mathbb{R}$ would have an ω_1-long strictly monotonous $<_{\mathbb{R}}$-chain which is a contradiction. qed

This connection between an Erdős-like cardinal and four cardinal Chang conjectures extends also to longer Chang conjectures.

Lemma 0.9 (ZF)**.** *Assume that $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ and $\kappa_0 < \kappa_1 < \cdots < \kappa_n$ are cardinals such that $\kappa_i \rightarrow^{\kappa_{i-1}} (\lambda_i)^{<\omega}_2$. Then the Chang conjecture*

$$(\kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_0) \text{ holds.}$$

Proof. Let $\mathcal{A} = \langle \kappa_n, \ldots \rangle$ be an arbitrary first order structure in a countable language, and let

$$\{f_j\ ;\ j \in \omega\}$$

be a complete set of Skolem functions for $\mathcal{A}$. Since $\kappa_n \rightarrow^{\kappa_{n-1}} (\lambda_n)^{<\omega}_2$ holds, let $I_n \in [\kappa_n \setminus \kappa_{n-1}]^{\lambda_n}$ be a set of good indiscernibles for $\mathcal{A}$. To take the next set of indiscernibles I_{n-1} we must make sure that it is, in a sense, compatible with I_n. That is, the Skolem hull of $I_n \cup I_{n-1}$ must not contain more than λ_{n-2} many elements below κ_{n-2}.

To do this we will enrich the structure $\mathcal{A}$ with functions, e.g.,

$$f_j(e_1, e_2, x_1, x_2),$$

for some f_j with arity $\mathsf{ar}(f_j) = 4$ and some $e_1, e_2 \in I_n$. Since f_j takes ordered tuples as arguments we must consider separately the cases $f_j(e_1, x_1, e_2, x_2)$, $f_j(e_1, x_1, x_2, e_2)$, etc..

Formally, let $\bar{I}_n := \{e_1, e_2, \ldots\}$ be the first ω-many elements of I_n. For every $s < \omega$ let $\{g_{s,t}\ ;\ t < s!\}$ be an enumeration of all the permutations of s, and for every $t \in s!$ let

$$h_{s,t}(x_1, \ldots, x_s) := (x_{g_{s,t}(1)}, \ldots, x_{g_{s,t}}(s)).$$

For every $j < \omega$, every $k < \mathsf{ar}(f_j)$, and every $\ell \in \mathsf{ar}(f_j)!$ define a function $f_{j;k;\ell} : {}^{\mathsf{ar}(f_j)}\kappa_n \to \kappa_n$ by

$$f_{j;k;\ell}(x_1, \ldots, x_{\mathsf{ar}(f_j)-k}) := f_j(h_{\mathsf{ar}(f_j),\ell}(x_1, \ldots, x_{\mathsf{ar}(f_j)-k}, e_1, \ldots, e_k)).$$

Consider the structure

$$\mathcal{A}_{n-1} := \mathcal{A}^\frown \langle f_{i;c;t} \rangle_{j<\omega, k<\mathsf{ar}(f_j), \ell<\mathsf{ar}(f_j)!}.$$

Since $\kappa_{n-1} \rightarrow^{\kappa_{n-2}} (\lambda_{n-1})^{<\omega}_2$, let $I_{n-1} \in [\kappa_{n-1} \setminus \kappa_{n-2}]^{\lambda_{n-1}}$ be a set of good indiscernibles for $\mathcal{A}_{n-1}$.

Claim 1. For any infinite set $Z \subseteq \kappa_{n-2}$ of size λ_{n-2},

$$|\mathsf{Hull}_{\mathcal{A}}(I_n \cup I_{n-1} \cup Z) \cap \kappa_{n-2}| = \lambda_{n-2}.$$

Proof of Claim. Let $\bar{I}_{n-1} := \{e'_1, e'_2, \dots\}$ be the first ω-many elements of I_{n-1}. The domain of the $\mathcal{A}$-Skolem Hull $\mathsf{Hull}_{\mathcal{A}}(I_n \cup I_{n-1} \cup Z)$ is the set

$$X := \{f_j(\alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)}) \ ; \ j < \omega \text{ and } \alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)} \in I_n \cup I_{n-1} \cup Z\}.$$

If for some $x = f_j(\alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)}) \in X \cap \kappa_{n-2}$ there are elements of I_n among $\alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)}$ then since I_n is a set of indiscernibles for $\mathcal{A}$ and $\alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)}$ are finitely many, we can find $\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)} \in \bar{I}_n \cup I_{n-1} \cup Z$ such that

$$x = f_j(\alpha_1, \dots, \alpha_{\mathsf{ar}(f_j)}) = f_j(\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)}).$$

We rewrite the tuple $(\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)})$ so that the elements of $\bar{I}_n$ (if any) appear in ascending order at the end:

$$\{\alpha'_1, \dots, \alpha'_n\} = \{\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}, e_1, \dots, e_k\}.$$

Let $(\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}, e_1, \dots, e_k)$ be a permutation of $(\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)})$, so for some $\ell < \mathsf{ar}(f_j)!$,

$$(\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)}) = h_{\mathsf{ar}(f_j),\ell}(\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}, e_1, \dots, e_k).$$

But then

$$\begin{aligned} x &= f_j(\alpha'_1, \dots, \alpha'_{\mathsf{ar}(f_j)}) \\ &= f_j(h_{\mathsf{ar}(f_j),\ell}(\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}, e_1, \dots, e_k) \\ &= f_{j,k,\ell}(\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}). \end{aligned}$$

Therefore,

$$\begin{aligned} X \cap \kappa_{n-2} = \{f_{j,k,\ell}(\beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k}) <\kappa_{n-2} \ ; \ j < \omega, k < \mathsf{ar}(f_j), \ell \in \mathsf{ar}(f_j)!, \\ \text{and } \beta_1, \dots, \beta_{\mathsf{ar}(f_j)-k} \in I_{n-1} \cup Z\}. \end{aligned}$$

But I_{n-1} is a set of good indiscernibles for $\mathcal{A}_{n-1}$, i.e., it is a set of indiscernibles for formulas with parameters below $\min I_{n-1} > \kappa_{n-2}$, therefore a set of indiscernibles for formulas with parameters from Z as well. Thus in the equation above we may replace I_{n-1} with $\bar{I}_{n-1}$. It's easy to see then that the set $X \cap \kappa_{n-2}$ has size λ_{n-2}. qed claim

Continuing like this we get for each $i = 1, \dots, n$ a set $I_i \in [\kappa_i \setminus \kappa_{i-1}]^{\lambda_i}$ of good indiscernibles for $\mathcal{A}$ with the property that for every infinite $Z \subseteq \kappa_{i-1}$, of size λ_{i-1},

$$|\mathsf{Hull}_{\mathcal{A}}(I_n \cup \dots \cup I_i \cup Z) \cap \kappa_{i-1}| = \lambda_{i-1}.$$

So let $I := \bigcup_{i=1,\dots,n} I_i$ and take $\mathcal{B} := \mathsf{Hull}_{\mathcal{A}}(I \cup \lambda_0)$. By [Hod97, 1.2.3], we have that

$$\lambda_n = |I \cup \lambda_0| \leq |\mathsf{Hull}_{\mathcal{A}}(I \cup \lambda_0)| \leq |I \cup \rho| + \omega = \lambda_n.$$

Because for each $i = 1, \dots, n$ we have that $I_i \in [\kappa_i \setminus \kappa_{i-1}]^{\lambda_i}$ and by the way we defined the I_i, we have that

$$|\mathsf{Hull}(I \cup \lambda_0) \cap \kappa_i| = \lambda_i.$$

So the substructure $\mathsf{Hull}(I \cup \lambda_0) \prec \mathcal{A}$ is such as we wanted for our Chang conjecture to hold. qed

0.1. **Countable coherent sequences of sets of good indiscernibles.** In the proof of Lemma 0.9 we had to construct our sets of good indiscernibles in a way that they are compatible. When we have countably many such sets and the axiom of choice is not available, we need that these sets of good indiscernibles are, in a way, coherent.

Definition 0.10. Let $\langle\kappa_i\ ;\ i<\omega\rangle$ and $\langle\lambda_i\ ;\ 0<i<\omega\rangle$ be strictly increasing sequences of cardinals, let $\kappa := \bigcup_{i<\omega}\kappa_i$, and let $\mathcal{A} = \langle\kappa,\ldots\rangle$ be a first order structure with a countable language. A $\langle\lambda_i\ ;\ 0<i<\omega\rangle$-coherent sequence of good indiscernibles for $\mathcal{A}$ with respect to $\langle\kappa_i\ ;\ i<\omega\rangle$ is a sequence $\langle A_i\ ;\ 0<i<\omega\rangle$ such that

(1) for every $0<i<\omega$, $A_i \in [\kappa_i \setminus \kappa_{i-1}]^{\lambda_i}$, and
(2) if $x, y \in [\kappa]^{<\omega}$ are such that $x = \{x_1,\ldots,x_n\}$, $y = \{y_1,\ldots,y_n\}$, $x, y \subseteq \bigcup_{0<i<\omega} A_i$, and for every $0<i<\omega$ $|x\cap A_i| = |y\cap A_i|$ then for every $(n+\ell)$-ary formula ϕ in the language of $\mathcal{A}$ and every $z_1,\ldots,z_\ell < \min\{x_1,\ldots,x_n,y_1,\ldots,y_n\}$,
$$\mathcal{A}\models\phi(z_1,\ldots,z_\ell,x_1,\ldots,x_n) \iff \mathcal{A}\models\phi(z_1,\ldots,z_\ell,y_1,\ldots,y_n).$$

We say that the sequence $\langle\kappa_i\ ;\ i<\omega\rangle$ is a coherent sequence of the Erdős-like cardinals $\kappa_{i+1}\to^{\kappa_i}(\lambda_{i+1})^{<\omega}_2$ iff for every structure $\mathcal{A}=\langle\kappa,\ldots\rangle$ with a countable language, there is a $\langle\lambda_i\ ;\ 0<i<\omega\rangle$-coherent sequence of good indiscernibles for $\mathcal{A}$ with respect to $\langle\kappa_i\ ;\ i<\omega\rangle$.

Similarly to coherent sequences of Erdős-like cardinals, we have coherent sequences of Erdős cardinals.

Definition 0.11. Let $\lambda_1<\cdots<\lambda_i<\ldots$ and $\kappa_0<\cdots<\kappa_i<\ldots$ be cardinals and let $\kappa:=\bigcup_{i<\omega}\kappa_i$. We say that the sequence $\langle\kappa_i\ ;\ i<\omega\rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle\lambda_i\ ;\ 0<i<\omega\rangle$ if for every $\gamma<\kappa_1$ and every $f:[\kappa]^{<\omega}\to\gamma$ there is a sequence $\langle A_i\ ;\ 0<i<\omega\rangle$ such that

(1) for every $0<i<\omega$, $A_i\in[\kappa_i\setminus\kappa_{i-1}]^{\lambda_i}$, and
(2) if $x,y\in[\kappa]^{<\omega}$ are such that $x,y\subseteq\bigcup_{i<\omega}A_i$ and for every $0<i<\omega$ $|x\cap A_i|=|y\cap A_i|$ then $f(x)=f(y)$.

Such a sequence $\langle A_i\ ;\ 0<i<\omega\rangle$ is called a $\langle\lambda_i\ ;\ 0<i<\omega\rangle$-coherent sequence of homogeneous sets for f with respect to $\langle\kappa_i\ ;\ i<\omega\rangle$. Note that the 0th element of a coherent sequence of Erdős cardinals need not be an Erdős cardinal, and indeed, none of the κ_n need satisfy the minimality requirement of the usual Erdős cardinals.

Coherent sequences of Ramsey cardinals are such sequences. In [AK06, Theorem 3] a coherent sequence of Ramsey cardinals in ZF is forced from a model of ZFC with one measurable cardinal.

Similarly to Lemma 0.5 we get that coherent sequences of Erdős and Erdős-like cardinals are equivalent.

Lemma 0.12 (ZF). *Let $\lambda_1<\cdots<\lambda_i<\ldots$ and $\kappa_0<\cdots<\kappa_i<\ldots$ be infinite cardinals. The following are equivalent:*

(a) *The sequence $\langle\kappa_i\ ;\ i<\omega\rangle$ is a coherent sequence of the Erdős-like cardinals $\kappa_{i+1}\to^{\kappa_i}(\lambda_{i+1})^{<\omega}_2$.*
(b) *The sequence $\langle\kappa_i\ ;\ i<\omega\rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle\lambda_i\ ;\ 0<i<\omega\rangle$.*

To show that when $\bigcup_{i\in\omega}\kappa_i=\bigcup_{i\in\omega}\lambda_i$, then such a coherent sequence of Erdős or Erdős-like cardinals in a model of ZF is equiconsistent with a measurable cardinal in a model of ZFC, we will use results that involve the infinitary Chang conjecture.

Definition 0.13. For cardinals $\kappa_0 < \cdots < \kappa_n < ...$ and $\lambda_0 < \cdots < \lambda_n < \ldots$, with $\kappa_n \geq \lambda_n$ for all n, define the infinitary Chang conjecture

$$(\kappa_n)_{n\in\omega} \twoheadrightarrow (\lambda_n)_{n\in\omega}$$

to mean that for every first order structure $\mathcal{A} = \langle \bigcup_{n\in\omega} \kappa_n, f_i, R_j, c_k\rangle_{i,j,k\in\omega}$ there is an elementary substructure $\mathcal{B} \prec \mathcal{A}$ with domain B such that for all $n \in \omega$, $|B \cap \kappa_n| = \lambda_n$.

Sometimes, when this uniform notation is not convenient, we will write the infinitary Chang conjecture as

$$(\ldots, \kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\ldots, \lambda_n, \ldots, \lambda_0).$$

The infinitary Chang conjecture is connected to Jónsson cardinals.

Definition 0.14. A cardinal κ is called Jónsson if for every first order structure with domain κ and a countable language, there is a proper elementary substructure of cardinality κ.

In [For10, §12 (4)] we read:

> Assuming that $2^{\aleph_0} < \aleph_\omega$, Silver showed that the cardinal $\aleph_\omega$ is Jónsson iff there is an infinite subsequence $\langle \kappa_n\ ;\ n \in \omega\rangle$ of the $\aleph_n$'s such that the infinitary Chang conjecture of the form
>
> $$(\ldots, \kappa_n, \kappa_{n-1}, \ldots, \kappa_1) \twoheadrightarrow (\ldots, \kappa_{n-1}, \kappa_{n-2}, \ldots, \kappa_0)$$
>
> holds. It is not known how to get such a sequence of length 4.

Clearly, if $\kappa = \bigcup_{n\in\omega} \kappa_n = \bigcup_{n\in\omega} \lambda_n$ and for some $n \in \omega$ $\kappa_n \neq \lambda_n$, then $(\kappa_n)_{n\in\omega} \twoheadrightarrow (\lambda_n)_{n\in\omega}$ implies that κ is a singular Jónsson cardinal of cofinality ω.

We can get an infinitary Chang conjecture from a coherent sequence of Erdős-like cardinals as in Lemma 0.9, but without having to take care of the "compatibility" of the sets of indiscernibles, since here they are coherent.

Lemma 0.15. (ZF) *Let $\langle \kappa_n\ ;\ n < \omega\rangle$ and $\langle \lambda_n\ ;\ 0 < n < \omega\rangle$ be increasing sequences of cardinals, and let $\kappa = \bigcup_{n<\omega} \kappa_n$. If $\langle \kappa_i\ ;\ i < \omega\rangle$ is a coherent sequence of cardinals with the property $\kappa_{n+1} \rightarrow^{\kappa_n} (\lambda_{n+1})_2^{<\omega}$ then the Chang conjecture*

$$(\kappa_n)_{n\in\omega} \twoheadrightarrow (\lambda_n)_{n\in\omega}$$

holds.

1. Forcing good sets of indiscernibles to lie between regular cardinals and their sucessors.

Here we assume knowledge of symmetric forcing as presented in [Dim11, Sections 2 and 3 of Chapter 2]. This technique is used to produce models of ZF+¬AC, called symmetric models, starting from a model of ZFC. In particular, we will use the generalised Jech model $V(G)$ and its property of satisfying the approximation lemma, i.e., that all sets of ordinals in $V(G)$ are included in some "initial" ZFC model. One may think of this symmetric model as being the closed under set theoretic operations union of generic extensions (the aforementioned initial ZFC models) that are made with the Lévy collapses

$$E_\alpha := \{p : \eta \rightharpoonup \alpha\ ;\ |p| < \eta\}$$

for $\alpha < \kappa$ and η a fixed regular cardinal. This "union" does not contain a generic object for the entire Lévy collapse $\mathbb{P} := \{p : \eta \rightharpoonup \kappa\ ;\ |p| < \eta\}$, so that κ becomes the sucessor of η in the symmetric model $V(G)$.

1.1. **Finitely many sets of good indiscernibles.** First, let us take a look at the case of just one set of good indiscernibles being forced to lie between a regular cardinal θ and its successor θ^+.

One approach would be to take a cardinal κ with the property $\kappa \to^\theta (\alpha)_2^{<\omega}$ and, using the generalised Jech model, symmetrically collapse κ to become θ^+. But to use the theorems for this model and preserve the property $\kappa \to^\theta (\alpha)_2^{<\omega}$, κ would have to satisfy certain large cardinal properties, e.g., inaccessibility. Erdős-like cardinals such as κ are far from inaccessible. In fact, for every $\kappa' \geq \kappa$, $\kappa' \to^\theta (\alpha)_2^{<\omega}$ holds. So we construct a model of $\mathsf{ZF} + \neg\mathsf{AC}$ by starting with an Erdős cardinal. This is not too bad since, by Lemma 0.5, Erdős cardinals and Erdős-like cardinals are mutually existent.

First we will show that an Erdős cardinal is Erdős-like after small forcing.

Lemma 1.1. *If V is a model of* ZFC+ *"$\kappa = \kappa(\alpha)$ exists" for some limit ordinal α, if $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < \kappa$, and G is a generic filter, then in $V[G]$, for any $\theta < \kappa$ the property $\kappa \to^\theta (\alpha)_2^{<\omega}$ holds.*

Proof. Let $\mathcal{A} = \langle \kappa, \ldots \rangle \in V[G]$ be an arbitrary structure in a countable language and $\theta < \kappa$ be arbitrary. Let $g : [\theta]^{<\omega} \to 2$ be a function in the ground model that has no homogeneous sets (in the ground model) of ordertype λ, and consider the structure

$$\bar{\mathcal{A}} = \mathcal{A}^\frown \langle \theta, g \restriction [\theta]^n \rangle_{n\in\omega},$$

where θ, and each $g \restriction [\theta]^n$ is considered as a relation. Let $\{\phi_n \ ; \ n < \omega\}$ enumerate the formulas of the language of $\bar{\mathcal{A}}$ so that each ϕ_n has $k(n) < n$ many free variables. Define $f : [\kappa]^{<\omega} \to 2$ by $f(\xi_1, \ldots, \xi_n) = 1$ iff $\mathcal{A} \models \phi_n(\xi_1, \ldots, \xi_{k(n)})$ and $f(\xi_1, \ldots, \xi_n) = 0$ otherwise. We call this f the function that describes truth in $\mathcal{A}$. Let $\dot{f}$ be a $\mathbb{P}$-name for f. Since κ is inaccessible in V, $|\mathcal{P}(\mathbb{P})| < \kappa$ in V. In V define the function $h : [\kappa]^{<\omega} \to \mathcal{P}(\mathbb{P})$ by

$$h(x) := \{p \in \mathbb{P} \ ; \ p \Vdash \dot{f}(\check{x}) = 0\}.$$

By [Kan03, Proposition 7.15] let $A \in [\kappa]^\lambda$ be homogeneous for h. Note that since we have attached g to $\mathcal{A}$, $A \subseteq \kappa \setminus \theta$. We will show that A is homogeneous for f in $V[G]$, and therefore a set of good indiscernibles for $\mathcal{A}$.

Let $n \in \omega$ and $x \in [A]^n$ be arbitrary.

- If $h(x) = \emptyset$ then for all $p \in \mathbb{P}$, $p \not\Vdash \dot{f}(\check{x}) = \check{0}$. So for some $p \in G \cap E_\gamma$, $p \Vdash \dot{f}(\check{x}) = \check{1}$ and so the colour of $[A]^n$ is 1.
- If $h(x) \neq \emptyset$ and $h(x) \cap G \neq \emptyset$ then the colour of $[A]^n$ is 0.
- If $h(x) \neq \emptyset$ and $h(x) \cap G = \emptyset$ then assume for a contradiction that for some $y \in [A]^n$, $f(x) \neq f(y)$. Without loss of generality say $f(y) = 0$. But then there is $p \in G$ such that $p \Vdash \dot{f}(\check{y}) = \check{0}$ so $\emptyset \neq h(y) \cap G = h(x) \cap G$, contradiction.

So in $V[G]$, $\kappa \to^\theta (\lambda)_2^{<\omega}$ holds. qed

We can use this to get the following.

Lemma 1.2. *If V is a model of* ZFC+ *"$\kappa = \kappa(\lambda)$ exists", then for any regular cardinal $\eta < \kappa$, there is a symmetric model $V(G)$ of* ZF *in which for every $\theta < \kappa$, $\eta^+ \to^\theta (\lambda)_2^{<\omega}$ holds.*

Proof. Let $\eta < \kappa$ be a regular cardinal, and construct the generalised Jech model $V(G)$ (see [Dim11, Section 3 of Chapter 1] and the beginning of this section) that makes $\kappa = \eta^+$. The approximation lemma holds in this model. Let $\theta < \kappa$ be arbitrary. Let $\mathcal{A} = \langle \kappa, \ldots \rangle$ be an arbitrary first order structure with a countable language and let $\dot{\mathcal{A}} \in \mathsf{HS}$ be a name for $\mathcal{A}$ with support E_γ for some $\eta < \gamma < \kappa$. By the approximation lemma

$$\mathcal{A} \in V[G \cap E_\gamma].$$

Note that $|E_\gamma| < \kappa$ therefore by Lemma 1.1, for every $\theta < \kappa$ the property $\kappa \to^\theta (\lambda)_2^{<\omega}$ holds in $V(G)$. Therefore the structure $\mathcal{A}$ has a set of indiscernibles $A \in [\kappa \setminus \theta]^\lambda$ and $A \in V[G \cap E_\gamma] \subseteq V(G)$. qed

By Corollary 0.6 we get the following.

Corollary 1.3. *If V is a model of* ZFC *with a cardinal κ that is the λ-Erdős cardinal then for any $\eta < \kappa$ regular cardinal there is a symmetric model $V(G)$ in which for every $\theta < \kappa$ with $\theta \leq \eta$, and $\rho \leq \lambda \cap \theta$*

$$(\eta^+, \theta) \twoheadrightarrow (\lambda, \rho) \text{ holds.}$$

Note that as with many of our forcing constructions here, this η could be *any* predefined regular ordinal of V. So we get an infinity of consistency strength results, some of them looking very strange for someone accustomed to the theory ZFC, such as the following.

Corollary 1.4. *If $V \models$ ZFC+ "$\kappa(\omega_{12})$ exists" then there is a symmetric model $V(G) \models \mathsf{ZF} + (\omega_{13}, \omega_{12}) \twoheadrightarrow (\omega_{12}, \omega_5)$.*

Or even stranger:

Corollary 1.5. *If $V \models$ ZFC+ "$\kappa(\omega_\omega)$ exists" then there is a symmetric model $V(G) \models \mathsf{ZF} + (\omega_{\omega+3}, \omega_\omega) \twoheadrightarrow (\omega_\omega, \omega_2)$.*

To get Chang conjectures that involve more than four cardinals we will have to collapse the Erdős cardinals simultaneously. We will give an example in which the Chang conjecture

$$(\omega_4, \omega_2, \omega_1) \twoheadrightarrow (\omega_3, \omega_1, \omega)$$

is forced from a model of ZFC with two Erdős cardinals. Before we do that let us see a very useful proposition.

Proposition 1.6. *Assume that $V \models$ ZFC+ "$\kappa = \kappa(\lambda)$ exists", $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < \kappa$, and $\mathbb{Q}$ is a partial order that doesn't add subsets to κ. If G is $\mathbb{P} \times \mathbb{Q}$-generic then for every $\theta < \kappa$,*

$$V[G] \models \kappa \to^\theta (\lambda)_2^{<\omega}.$$

Proof. Let $\mathcal{A} = \langle \kappa, \ldots \rangle \in V[G]$ be an arbitrary structure with a countable language. By [Kun80, Chapter VII, Lemma 1.3], $G = G_1 \times G_2$ for some G_1 $\mathbb{P}$-generic and some G_2 $\mathbb{Q}$-generic. Since $\mathbb{Q}$ does not add subsets to κ, we have that $\mathcal{A} \in V[G_1]$. By Lemma 1.1 we get that $\kappa \to^\theta (\lambda)_2^{<\omega}$ in $V[G_1] \subset V[G]$ and from that we get a set $H \in [\kappa \setminus \theta]^\lambda$ of indiscernibles for $\mathcal{A}$ with respect to parameters below θ, and $H \in V[G]$,. Therefore $V[G] \models \kappa \to^\theta (\lambda)_2^{<\omega}$. qed

To get the desired Chang conjecture we will construct a symmetric model that can also be used to create a model of ZF with sucessive alternating measurable and non-measurable cardinals (see [Dim11, Section 4 of Chapter 1] for $\rho = 2$).

Lemma 1.7. (ZFC) *Assume that $\kappa_1 = \kappa(\omega_1)$, and $\kappa_2 = \kappa(\kappa_1^+)$ exist. Then there is a symmetric extension of V in which* ZF $+ \omega_4 \to^{\omega_2} (\omega_3)_2^{<\omega} + \omega_2 \to^{\omega_1} (\omega_1)_2^{<\omega}$.

Consequently,

$$(\omega_4, \omega_2, \omega_1) \twoheadrightarrow (\omega_3, \omega_1, \omega)$$

holds in V as well.

Proof. Let $\kappa_1' = (\kappa_1^+)^V$ and define

$$\mathbb{P} := \{p : \omega_1 \rightharpoonup \kappa_1 \ ; \ |p| < \omega_1\} \times \{p : \kappa_1' \rightharpoonup \kappa_2 \ ; \ |p| < \kappa_1'\}.$$

Let $\mathcal{G}_1$ be the full permutation group of κ_1 and $\mathcal{G}_2$ the full permutation group of κ_2. We define an automorphism group $\mathcal{G}$ of $\mathbb{P}$ by letting $a \in \mathcal{G}$ iff for some $a_1 \in \mathcal{G}_1$ and $a_2 \in \mathcal{G}_2$,

$$a((p_1, p_2)) := (\{(\xi_1, a_1(\beta_1)) \ ; \ (\xi_1, \beta_1) \in p_1\}, \{(\xi_2, a_2(\beta_2)) \ ; \ (\xi_2, \beta_2) \in p_2\}).$$

Let I be the symmetry generator that is induced by the ordinals in the product of intervals $(\omega_1, \kappa_1) \times (\kappa_1', \kappa_2)$, i.e.,

$$I := \{E_{\alpha,\beta} \ ; \ \alpha \in (\omega_1, \kappa_1) \text{ and } \beta \in (\kappa_1', \kappa_2)\},$$

where

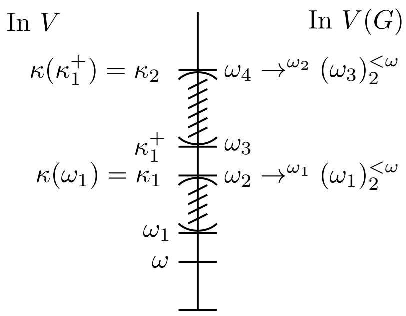
$$E_{\alpha,\beta} := \{(p_1 \cap (\omega_1 \times \alpha), p_2 \cap (\kappa_1' \times \beta)) \ ; \ (p_1, p_2) \in \mathbb{P}\}.$$

This I is a projectable symmetry generator with projections

$$(p_1, p_2) \restriction^* E_{\alpha,\beta} = (p_1 \cap (\omega_1 \times \alpha), p_2 \cap (\kappa_1' \times \beta)).$$

Take the symmetric model $V(G) = V(G)^{\mathcal{F}_I}$. It's easy to see that the approximation lemma holds for this model.

This construction can be illustrated as below.



With the standard arguments we can show that in $V(G)$ we have that $\kappa_1 = \omega_2$ and $\kappa_2 = \omega_4$. We want to show that moreover $\kappa_2 \to^{\kappa_1} (\kappa_1')_2^{<\omega}$ and $\kappa_1 \to^{\omega_1} (\omega_1)_2^{<\omega}$.

For the first partition property let $\mathcal{A} = \langle \kappa_2, \ldots \rangle$ be an arbitrary structure in a countable language and let $\dot{\mathcal{A}} \in \mathsf{HS}$ be a name for $\mathcal{A}$ with support $E_{\alpha,\beta}$. By the approximation lemma we have that $\mathcal{A} \in V[G \cap E_{\alpha,\beta}]$. Since $|E_{\alpha,\beta}| < \kappa_2$, by Lemma 1.1 we have that $V[G \cap E_{\alpha,\beta}] \models \kappa_2 \to^{\kappa_1} (\kappa_1')_2^{<\omega}$ therefore there is a set $A \in [\kappa_2 \setminus \kappa_1]^{\kappa_1'}$ of indiscernibles for $\mathcal{A}$ with respect to parameters below κ_1, and $A \in V[G \cap E_{\alpha,\beta}] \subseteq V(G)$.

For the second partition property let $\mathcal{B} = \langle \kappa_1, \ldots \rangle$ be and arbitrary structure in a countable language and let $\dot{\mathcal{B}} \in \mathsf{HS}$ be a name for $\mathcal{B}$ with support $E_{\gamma,\delta}$. We have that $E_{\gamma,\delta} = \{p : \omega_1 \rightharpoonup \gamma \ ; \ |p| < \omega_1\} \times \{p : \kappa_1' \rightharpoonup \delta \ ; \ |p| < \kappa_1'\}$, $|\{p : \omega_1 \rightharpoonup \gamma \ ; \ |p| < \omega_1\}| < \kappa_1$, and $\{p : \kappa_1' \rightharpoonup \delta \ ; \ |p| < \kappa_1'\}$ does not add subsets to κ_1. Therefore by Proposition 1.6 we get that $V[G \cap E_{\gamma,\delta}] \models \kappa_1 \to^{\omega_1} (\omega_1)_2^{<\omega}$ so there is a set $B \in [\kappa_1 \setminus \omega_1]^{\omega_1}$ of indiscernibles for $\mathcal{B}$ with respect to parameters below ω_1, and $B \in V[G \cap E_{\gamma,\delta}] \subseteq V(G)$.

So in $V(G)$ we have that $\omega_4 \to^{\omega_2} (\omega_3)_2^{<\omega}$ and $\omega_2 \to^{\omega_1} (\omega_1)_2^{<\omega}$ thus by Lemma 0.9 we have that in $V(G)$ the Chang conjecture $(\omega_4, \omega_2, \omega_1) \twoheadrightarrow (\omega_3, \omega_1, \omega)$ holds. qed

Note that, the gap in these cardinals is necessary for this method to work. Collapsing further would destroy their partition properties. Keeping this in mind it is easy to see how to modify this proof to get any desired Chang conjecture

$$(\kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_0)$$

with the κ_i and the λ_i being any predefined successor cardinals, as long as we mind the gaps.

1.2. **Coherent sequences of sets of good indiscernibles.** We can do the above for the infinitary version as well, using a finite support product forcing of such collapses, for a coherent sequence of Erdős cardinals $\langle \kappa_n \; ; \; n \in \omega \rangle$ with respect to $\langle \kappa_n^+ \; ; \; n < \omega \rangle$, and with $\kappa_0 = \omega_1$. In that case we will end up with a model of

$$\mathsf{ZF} + \neg \mathsf{AC}_\omega + (\omega_{2n+1})_{n<\omega} \twoheadrightarrow (\omega_{2n})_{n<\omega}.$$

Lemma 1.8. *Let $\langle \kappa_n \; ; \; n < \omega \rangle$ and $\langle \lambda_n \; ; \; 0 < n < \omega \rangle$ be increasing sequences of cardinals such that $\langle \kappa_n \; ; \; n < \omega \rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle \lambda_n \; ; \; n < \omega \rangle$. If $\mathbb{P}$ is a partial order of cardinality $< \kappa_1$ and G is $\mathbb{P}$-generic then in $V[G]$, $\langle \kappa_n \; ; \; n < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{n+1} \rightarrow^{\kappa_n} (\lambda_{n+1})_2^{<\omega}$.*

Proof. Let $\kappa = \bigcup_{n\in\omega}$ and let $\mathcal{A} = \langle \kappa, \ldots \rangle \in V[G]$ be an arbitrary structure in a countable language. Let $\{\phi_n \; ; \; n < \omega\}$ enumerate the formulas of the language of $\mathcal{A}$ so that each ϕ_n has $k(n) < n$ many free variables. Define $f : [\kappa]^{<\omega} \to 2$ by $f(\xi_1, \ldots, \xi_n) = 1$ iff $\mathcal{A} \models \phi_n(\xi_1, \ldots, \xi_{k(n)})$ and $f(\xi_1, \ldots, \xi_n) = 0$ otherwise. Let $\dot{f}$ be a $\mathbb{P}$-name for f. In V define a function $g : [\kappa]^{<\omega} \to \mathcal{P}(\mathbb{P})$ by

$$g(x) = \{p \in \mathbb{P} \; ; \; p \Vdash \dot{f}(\check{x}) = \check{0}\}.$$

Since $|\mathbb{P}| < \kappa_1$ and κ_1 is inaccessible in V, $|\mathcal{P}(\mathbb{P})| < \kappa_1$. So there is a $\langle \lambda_n \; ; \; 0 < n < \omega \rangle$-coherent sequence of homogeneous sets for g with respect to $\langle \kappa_n \; ; \; n \in \omega \rangle$. The standard arguments show that this is a $\langle \lambda_n \; ; \; 0 < n < \omega \rangle$-coherent sequence of homogeneous sets for f with respect to $\langle \kappa_n \; ; \; n \in \omega \rangle$, therefore a $\langle \lambda_n \; ; \; 0 < n < \omega \rangle$-coherent sequence of indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_n \; ; \; n \in \omega \rangle$. qed

The model used for this following proof is again the model of [Dim11, Section 4 of Chapter 1], this time for $\rho = \omega$.

Lemma 1.9. (ZFC) *Let $\langle \kappa_n \; ; \; n \in \omega \rangle$ be a coherent sequence of Erdős cardinals with respect to $\langle \lambda_n \; ; \; 0 < n \in \omega \rangle$, where $\kappa_0 = \omega_1$. Then there is a symmetric model $V(G)$ in which $\langle \omega_{2n} \; ; \; n \in \omega \rangle$ is a coherent sequence of cardinals with the property $\omega_{2n+2} \rightarrow^{\omega_{2n}} (\omega_{2n+1})_2^{<\omega}$.*

Consequently, in $V(G)$

$$(\ldots, \omega_{2n}, \ldots, \omega_4, \omega_2, \omega_1) \twoheadrightarrow (\ldots, \omega_{2n-1}, \ldots, \omega_3, \omega_1, \omega)$$

holds as well, and $\aleph_\omega$ is a Jónsson cardinal.

Proof. Let $\kappa = \bigcup_{0<n<\omega} \kappa_n$, for every $0 < n < \omega$ let $\kappa_n' = \kappa_n^+$, and let $\kappa_0' = \omega_1$. For every $0 < n < \omega$ let

$$\mathbb{P}_n := \{p : \kappa_{n-1}' \rightharpoonup \kappa_n \; ; \; |p| < \kappa_{n-1}'\},$$

and take the finite support product of these forcings

$$\mathbb{P} := \prod_{0<n<\omega}^{\mathsf{fin}} \mathbb{P}_n.$$

For each $0 < n < \omega$ let G_n be the full permutation group of κ_n and define an automorphism group $\mathcal{G}$ of $\mathbb{P}$ by $a \in \mathcal{G}$ iff for every $n \in \omega$ there exists $a_n \in \mathcal{G}_n$ such that

$$a(\langle p_n \ ; \ n \in \omega\rangle) := \langle\{(\xi, a_n(\beta)) \ ; \ (\xi, \beta) \in p_n\} \ ; \ n \in \omega\rangle.$$

For every finite sequence of ordinals $e = \langle\alpha_1, \ldots, \alpha_m\rangle$ such that for every $i = 1, \ldots, m$ there is a distinct $0 < n_i < \omega$ such that $\alpha_i \in (\kappa'_{n_i-1}, \kappa_{n_i})$, define

$$E_e := \{\langle p_{n_i} \cap (\kappa'_{n_i-1}, \alpha_i) \ ; \ \alpha_i \in e\rangle \ ; \ \langle p_{n_i} \ ; \ i = 1, \ldots, m\rangle \in \mathbb{P}\},$$

and take the symmetry generator

$$I := \{E_e \ ; \ e \in \prod_{0<n<\omega}^{\mathsf{fin}} (\kappa'_{n-1}, \kappa_n)\}.$$

This is a projectable symmetry generator with projections

$$\langle p_j \ ; \ 0 < j < \omega\rangle \restriction^* E_e = \langle p_{n_i} \cap (\kappa'_{n_i-1}, \alpha_i) \ ; \ \alpha_i \in e\rangle.$$

Take the symmetric model $V(G) = V(G)^{\mathcal{F}_I}$. The approximation lemma holds for $V(G)$.

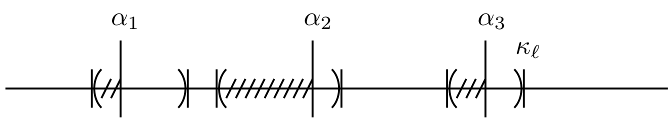
As usual we can show that in $V(G)$, for each $0 < n < \omega$ we have that $\kappa_n = \kappa'^{+}_{n-1}$, i.e., for every $0 < n < \omega$, $\kappa_n = \omega_{2n}$ and $\kappa'_n = \omega_{2n+1}$.

It remains to show that $\langle\kappa_i \ ; \ i \in \omega\rangle$ is a coherent sequence of cardinals with the property $\kappa_{n+1} \to^{\kappa_n} (\lambda_{n+1})^{<\omega}_2$.

Let $\mathcal{A} = \langle\kappa, \ldots\rangle$ be an arbitrary structure in a countable language and let the function $f : [\kappa]^{<\omega} \to 2$ describe the truth in $\mathcal{A}$, as in the proofs of Lemma 1.1 and Lemma 1.8. Let $\dot{f} \in \mathsf{HS}$ be a name for f with support E_e. Let $e = \{\alpha_1, \ldots, \alpha_m\}$ and for each $i = 1, \ldots, m$ let n_i be such that $\alpha \in (\kappa'_{n_i-1}, \kappa_{n_i})$. By the approximation lemma,

$$f \in V[G \cap E_e],$$

i.e., f is forced via $\bar{\mathbb{P}} = \prod_{i=1}^m \{p : \kappa'_{n_i-1} \rightharpoonup \kappa_{n_i} \ ; \ |p| < \kappa'_{n_i-1}\}$.



Let $\ell := \max\{n_i \ ; \ \alpha_i \in e\}$. We're in a situation as in the image above, which is an example for $m = 3$. Since $|\mathbb{P}| < \kappa_\ell$, by Lemma 1.8 there is a $\langle\lambda_n \ ; \ \ell \le n < \omega\rangle$-coherent sequence of indiscernibles for $\mathcal{A}$ with respect to $\langle\kappa_n \ ; \ \ell - 1 \le n \in \omega\rangle$, i.e., a sequence $\langle A_n \ ; \ \ell \le n < \omega\rangle$ such that

- for every $\ell \le n < \omega$, $A_n \subseteq \kappa_n \setminus \kappa_{n-1}$ is of ordertype λ_n, and
- if $x, y \in [\kappa]^{<\omega}$ are such that $x = \{x_1, \ldots, x_m\}$, $y = \{y_1, \ldots, y_m\}$, $x, y \in \bigcup_{\ell \le n < \omega} A_n$, and for every $\ell \le n < \omega$, $|x \cap A_n| = |y \cap A_n|$, then for every $m + k$-ary formula ϕ in the language of $\mathcal{A}$, and every $z_1, \ldots, z_k$ less than $\min \bigcup_{\ell \le n < \omega} A_n$,

$$\mathcal{A} \models \phi(z_1, \ldots, z_k, x_1, \ldots, x_m) \iff \mathcal{A} \models \phi(z_1, \ldots, z_k, y_1, \ldots, y_m)$$

Now we will get sets of indiscernibles from the remaining cardinals $\kappa_1, \ldots, \kappa_{\ell-1}$ step by step, making them coherent as we go along. Before we get the rest of the A_n, note that by Proposition 1.6 we have that for every $0 < n < \ell$,

$$V[G \cap E_e] \models \kappa_n \to^{\kappa_{n-1}} (\lambda_n)^{<\omega}_2.$$

Let's see how to get $A_{\ell-1}$. For every $\ell \le n < \omega$, let $\bar{A}_n$ be the first ω-many elements of A_n. There are only countably many $x \in [\kappa]^{<\omega}$ such that $x \subseteq \bigcup_{\ell \le n<\omega} \bar{A}_n$. For every $i, j \in \omega$, and every $x \in [\kappa]^{<\omega}$ such that $x = \{x_1, \ldots, x_m\} \subseteq \bigcup_{\ell \le n<\omega} \bar{A}_n$ and $m < i, j$, let

$$f_{i,x}(v_1, \ldots, v_{i-m}) := f_i(v_1, \ldots, v_{i-m}, x_1, \ldots, x_m), \text{ and}$$
$$R_{j,x}(v_1, \ldots, v_{j-m}) := R_j(v_1, \ldots, v_{j-m}, x_1, \ldots, x_m).$$

Consider the structure

$$\mathcal{A}' := \mathcal{A}^\frown \langle f_{i,x}, R_{j,x} \rangle_{i,j<\omega, x\in[\kappa]^{<\omega}, x=\{x_1,\ldots,x_m\}\subseteq \bigcup_{\ell\le n<\omega} \bar{A}_n, m<i,j}.$$

Since $\kappa_{\ell-1} \to^{\kappa_{\ell-2}} (\lambda_{\ell-1})^{<\omega}_2$, there is a set $A_{\ell-1} \in [\kappa_{\ell-1} \setminus \kappa_{\ell-2}]^{\lambda_{\ell-1}}$ of indiscernibles for $\mathcal{A}'$ with respect to parameters below $\kappa_{\ell-2}$. By the way we defined $\mathcal{A}'$, the sequence $\langle A_n \ ; \ \ell - 1 \le n < \omega \rangle$ is a $\langle \lambda_n \ ; \ \ell - 1 \le n < \omega \rangle$-coherent sequence of indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_n \ ; \ \ell - 2 \le n < \omega \rangle$.

Continuing in this manner we get a sequence $\langle A_n \ ; \ 0 < n < \omega \rangle$ that is a $\langle \lambda_n \ ; \ 0 < n < \omega \rangle$-coherent sequence of indiscernibles for $\mathcal{A}$ with respect to $\langle \kappa_n \ ; \ n < \omega \rangle$, and such that $\langle A_n \ ; \ 0 < n < \omega \rangle \in V[G \cap E_e] \subseteq V(G)$.

Therefore we have that in $V(G)$, $\langle \omega_{2n} \ ; \ n \in \omega \rangle$ is a coherent sequence of cardinals with the property $\omega_{2n+2} \to^{\omega_{2n}} (\omega_{2n+1})^{<\omega}_2$. By Corollary 0.15 we have that in $V(G)$

$$(\ldots, \omega_{2n}, \ldots, \omega_4, \omega_2, \omega_1) \twoheadrightarrow (\ldots, \omega_{2n-1}, \ldots, \omega_3, \omega_1, \omega)$$

holds. Consequently, $\aleph_\omega$ is a Jónsson cardinal. qed

Note that in this model the axiom of choice fails badly. In particular, by [Dim11, Lemma 1.37] $\mathsf{AC}_{\omega_2}(\mathcal{P}(\omega_1))$ is false. As mentioned after the proof of that lemma, one can get the infinitary Chang conjecture plus the axiom of dependent choice with the construction in the proof of [AK06, Theorem 5]. With that we get the following.

Lemma 1.10. *Let $V_0 \models \mathsf{ZFC}+$ "there exists a measurable cardinal κ". Let $n < \omega$ be fixed but arbitrary. There is a generic extension V of V_0, a forcing notion $\mathbb{P}$, and a symmetric model N such that*

$$N \models \mathsf{ZF} + \mathsf{DC} + (\omega_n)_{0<n<\omega} \twoheadrightarrow (\omega_n)_{n<\omega} + \aleph_\omega \textit{ is Jónsson}$$

2. Getting good sets of indiscernibles from Chang conjectures for successors of regular cardinals in ZF

In this section we will work with the Dodd-Jensen core model K^{DJ} to get strength from the principles we are looking at. We will start from a model of ZF with a Chang conjecture, and we will get an Erdős cardinal in K^{DJ}. To be able to use the known theorems about K^{DJ}, which involve AC, we will build K^{DJ} inside HOD. We will then use the Chang conjecture to get a certain elementary substructure $K' \subseteq \mathsf{HOD}$. With a clever manoeuvre, found in [AK06, Proposition 8], which says $(K^{\mathsf{DJ}})^{\mathsf{HOD}} = (K^{\mathsf{DJ}})^{\mathsf{HOD}[K']}$ we can get this structure into $(K^{\mathsf{DJ}})^{\mathsf{HOD}}$ and use the core model's structured nature to get a set of good indiscernibles.

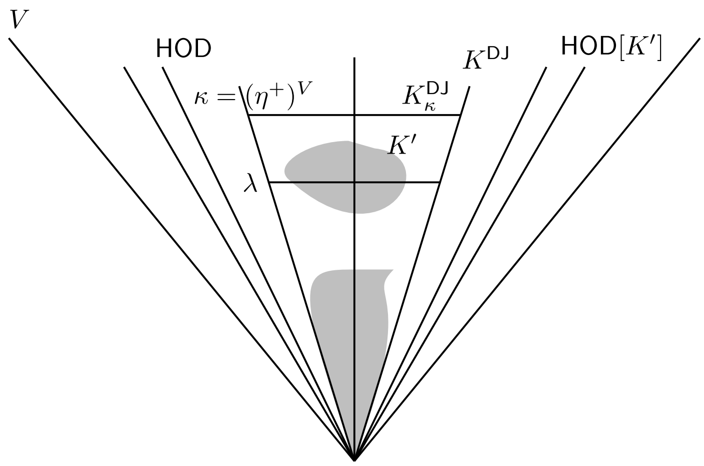
We assume basic knowledge of this core model, as presented in [DJK79], [DJ81], and [DK83], or in the short collection of the relevant results from these papers found in [Dim11, Section 2 of Chapter 3]. We will start by looking at four cardinal Chang conjectures.

Theorem 2.1. *Assume ZF and let η be a regular cardinal. If for some infinite cardinals θ, λ, and ρ such that $\eta^+ > \theta, \lambda > \rho$ and $\mathsf{cf}\lambda > \omega$ the Chang conjecture $(\eta^+, \theta) \twoheadrightarrow (\lambda, \rho)$ holds, then $\kappa(\lambda)$ exists in the Dodd-Jensen core model $(K^{\mathsf{DJ}})^{\mathsf{HOD}}$, and $(K^{\mathsf{DJ}})^{\mathsf{HOD}} \models (\eta^+)^V \to (\lambda)^{<\omega}_2$.*

Proof. Let $\kappa = (\eta^+)^V$ and in K^{DJ} let $g : [\kappa]^{<\omega} \to 2$ be arbitrary and consider the structure $\langle K^{\mathsf{DJ}}_\kappa, \in, D \cap K^{\mathsf{DJ}}_\kappa, g\rangle$, where D is a class[4] such that $K^{\mathsf{DJ}} = L[D]$. This is included in our structure in order to use the results in [DJ81] and [DJK79]. We want to find a set of indiscernibles for this structure in K^{DJ}. Using our Chang conjecture in V we get an elementary substructure

$$\mathcal{K}' = \langle K', \in, D \cap K', g'\rangle \prec \langle K^{\mathsf{DJ}}_\kappa, \dots\rangle$$

such that $|K'| = \lambda$ and $|K' \cap \theta| = \rho$. Since K' is wellorderable it can be seen as a set of ordinals. We attach K' to HOD, getting $\mathsf{HOD}[K']$. By [AK06, Proposition 8], $(K^{\mathsf{DJ}})^{\mathsf{HOD}} = (K^{\mathsf{DJ}})^{\mathsf{HOD}[K']}$.

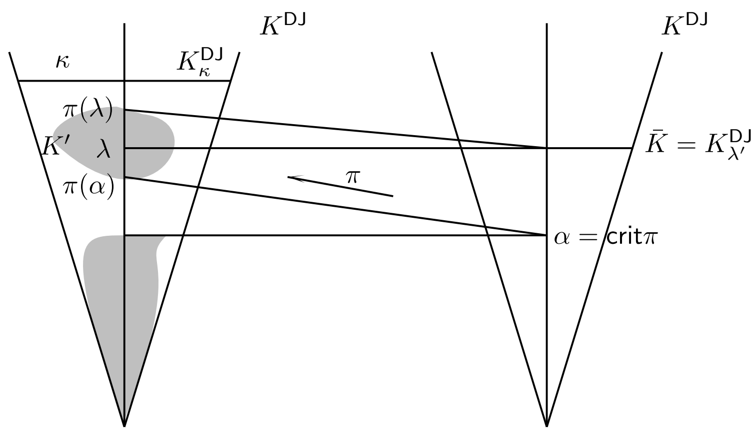


We are now, and for the rest of this proof, working in $\mathsf{HOD}[K']$.

Let $\langle \bar{K}, \in, A'\rangle$ be the Mostowski collapse of $\mathcal{K}'$, with $\pi\colon \bar{K} \to K'$ being an elementary embedding.

We distinguish two cases.

Case 1. If $\bar{K} = K^{\mathsf{DJ}}_{\lambda'}$ for some λ'. Then the map $\pi : K^{\mathsf{DJ}}_{\lambda'} \to K^{\mathsf{DJ}}_\kappa$ is elementary.



[4]For details see [DJ81, Definition 6.3].

Since $\lambda \geq \omega_1$, by [DK83, Lemma 2.9], there is a non trivial elementary embedding of K^{DJ} to K^{DJ} with critical point α. By [DK83, 1.5] this means that there is an inner model with a measurable cardinal β, such that if $\alpha < \omega_1$ then $\beta \leq \omega_1$ and if $\alpha \geq \omega_1$ then $\beta < \alpha^+$. Because $\alpha = \mathsf{crit}\pi$ and $|\bar{K}| = \lambda$, $\alpha < \lambda^+$.

Let's take a closer look at this inner model. Let U be a normal measure for β in the inner model M, define $\bar{U} := U \cap L[U]$ and build $L[\bar{U}]$. It is known that then $L[\bar{U}] \models$"$\bar{U}$ is a normal ultrafilter over β" (see [Kan03, Exercise 20.1]). We also have that $L[\bar{U}] \models \mathsf{ZFC}$ which by [Kan03, Lemma 20.5] means that $\langle L[\bar{U}], \in, \bar{U}\rangle$ is iterable. Recall that such a structure $\langle L[\bar{U}], \in, \bar{U}\rangle$ is called a β-model and that for a regular cardinal ν, C_ν is the club filter over ν. According to [Kan03, Corollary 20.7], if there is a β-model, if ν is a regular cardinal above β^+, and if $\bar{C}_\nu = C_\nu \cap L[C_\nu]$, then $L[\bar{C}_\nu]$ is a ν-model.

Now, we have that if $\alpha < \omega$ then $\beta \leq \omega_1$, so $\beta \leq \lambda$. If $\alpha \geq \omega_1$ then $\beta < \alpha^+ \leq \lambda^+$, so again $\beta \leq \lambda$. If $\beta = \lambda$ then $L[\bar{U}] \models$"λ is Ramsey". By [DK83, 1.6], $\mathcal{P}(\lambda) \cap L[\bar{U}] = \mathcal{P}(\lambda) \cap K^{\mathsf{DJ}}$. So λ is Ramsey in K and we're done.

So assume that $\beta < \lambda$. Then $\beta^+ < \kappa$. We need a regular cardinal $\nu > \beta^+$ such that $\lambda \leq \nu \leq \kappa$. If κ is regular, let $\nu = \kappa$. If κ is singular then κ is a limit cardinal so there is such a regular cardinal ν (e.g., $\nu = \lambda^{++}$).

Then by [DK83, 1.6] we have that $L[\bar{C}_\nu] \models$"ν is Ramsey". Because [DK83, 1.6] says that $\mathcal{P}(\nu) \cap L[\bar{C}_\nu] = \mathcal{P}(\nu) \cap K^{\mathsf{DJ}}$, this ν is Ramsey in K^{DJ}. But this implies that in K^{DJ}, $\kappa \to (\lambda)_2^{<\omega}$.

Case 2. If $\bar{K} \neq K_\lambda^{\mathsf{DJ}}$ for any λ'. By [DK83, Lemma 2.1] $K_\kappa^{\mathsf{DJ}} \models$"$V = K^{\mathsf{DJ}}$". Since $\bar{K}$ is elementary with K_κ^{DJ}, $\bar{K} \models$"$V = K^{\mathsf{DJ}}$". This is because being the lower part of a premouse is a property describable by a formula. Let $x \in \bar{K}$. Since $\bar{K} \models$"$V = K^{\mathsf{DJ}}$", there must be some M such that

$$\bar{K} \models \text{“}M \text{ is an iterable premouse and } x \in \mathsf{lp}(M).$$

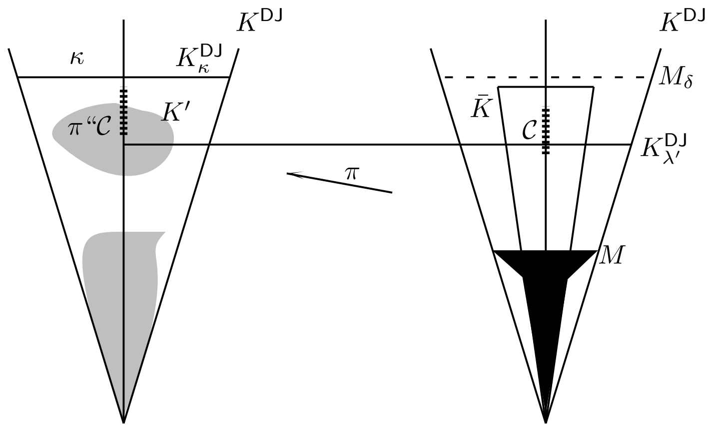
So $K_\kappa^{\mathsf{DJ}} \models$ "M is an iterable premouse" and by [DK83, Lemma 1.16], $\pi(M)$ is an iterable premouse in $\mathsf{HOD}[K']$. Since $\pi \restriction M \to \pi(M)$ is elementary and $\pi(M)$ is an iterable premouse, by [DK83, Lemma 1.17], M is an iterable premouse in $\mathsf{HOD}[K']$. Thus $x \in K^{\mathsf{DJ}}$, so $\bar{K} \subseteq K^{\mathsf{DJ}}$. But then, since $K_\lambda^{\mathsf{DJ}} \neq \bar{K}$ for any λ', and $\bar{K}$ has cardinality λ, there must be an iterable premouse $M \notin \bar{K}$ and a $z \in K_\lambda^{\mathsf{DJ}} \setminus \bar{K}$ such that $\mathsf{lp}(M) \cap (K_\lambda \setminus \bar{K}) \neq \emptyset$, $z \in \mathsf{lp}(M)$, and $M \in K_\lambda^{\mathsf{DJ}}$. Fix M.

Claim 1. If $\delta > \lambda$ is a regular cardinal then for every iterable premouse $N \in \bar{K}$, $N_\delta \in M_\delta$.

Proof of Claim. Since $M \in K_\lambda$ and $|\bar{K}| = \lambda$ by [DK83, Lemma 1.13] we have that for every regular cardinal $\delta > \lambda$ and every iterable premouse $N \in \bar{K}$, N_δ and M_δ are comparable. Assume for a contradiction that for some $N \in \bar{K}$, $M_\delta \subseteq N_\delta$. Then $z \in \mathsf{lp}(N_\delta)$ and since $z \in K_\lambda^{\mathsf{DJ}}$, for some $\xi < \lambda$, $z \in \mathsf{lp}(N_\xi)$. But since $N_\xi \in \bar{K}$, $z \in N_\xi \in \bar{K}$ which is transitive so $z \in \bar{K}$, contradiction. qed claim

We want such a $\delta \leq \kappa$. As before, if κ is regular then take $\delta = \kappa$, and if κ is singular then take $\delta = \lambda^+$. Look at M_δ. By Claim 1 we have that $\bar{K} \subseteq M_\delta$. So $g' \in M_\delta$.

Let $\langle M_i, \pi_{ij}, \gamma_i, U_i\rangle_{i \leq j < \delta}$ be the δ-iteration of M. By [DK83, Lemma 1.14] there is some $x \in M$ and $\vec{\rho} \in {}^{<\omega}\{\gamma_i \,;\, i < \delta\}$ such that $g' = \pi_{0,\delta}(x)(\vec{\rho})$. Let $\mathcal{C} = \{\gamma_i \,;\, i < \lambda\}$. By the same lemma there is a sequence $\langle k_n \,;\, n < \omega\rangle \in {}^{\omega}2$ such that for every $n < \omega$, g'“$[\mathcal{C}]^n = \{k_n\}$.



By elementarity, π"$\mathcal{C}$ is a homogeneous set for g in $\mathsf{HOD}[K']$ and π"$\mathcal{C}$ is a good set of indiscernibles for $\langle K^{\mathsf{DJ}}_\kappa, \in, D \cap K^{\mathsf{DJ}}_\kappa, g\rangle$ of ordertype $\mathsf{cf}\lambda \geq \omega_1$. By Jensen's indiscernibility lemma [DJK79, Lemma 1.3] there is a homogeneous set for g of ordertype λ in K^{DJ}. qed

Therefore we have the following.

Theorem 2.2. *The theory* $\mathsf{ZF} + (\kappa, \theta) \twoheadrightarrow (\lambda, \rho)+$ "$\mathsf{cf}\lambda > \omega$" *is equiconsistent with the theory* $\mathsf{ZFC}+$ "$\kappa(\lambda)$ *exists".*

In the proof of Lemma 0.9 we see how to combine finitely many sets of indiscernibles to make them coherent. Using this we get the following.

Lemma 2.3. *Assume* ZF *and let* $\kappa_n > \cdots > \kappa_0$, $\lambda_n > \cdots > \lambda_0$ *be regular cardinals, such that the Chang conjecture* $(\kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_0)$ *holds, then for each* $i = 1, \ldots, n$, $\kappa(\lambda_i)$ *exists in the Dodd-Jensen core model* $(K^{\mathsf{DJ}})^{\mathsf{HOD}}$ *and* $(K^{\mathsf{DJ}})^{\mathsf{HOD}} \models \forall i = 1, \ldots, n(\kappa_i \to (\lambda_i)^{<\omega}_2)$.

Theorem 2.4. *For every finite* n*, the theory* $\mathsf{ZF}+$ "$(\kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_0)$" *is equiconsistent with the theory* $\mathsf{ZFC}+$ "$\kappa(\lambda_n^{+(n-1)})$ *exists.", where* λ_0 *is the last cardinal appearing on the Chang conjecture.*

2.1. **The infinitary case.** For the infinitary version, recall that if $\bigcup_{n\in\omega} \kappa_n = \bigcup_{n\in\omega} \lambda_n$ and $\kappa_n > \lambda_n$ for at least one $n \in \omega$ then

$$(\kappa_n)_{n<\omega} \twoheadrightarrow (\lambda_n)_{n<\omega}$$

implies that $\kappa := \bigcup_{n\in\omega} \kappa_n$ is a singular Jónsson cardinal. In [AK06, Theorem 6] it is proved that if κ is a singular Jónsson cardinal in a model of ZF then κ is measurable in some inner model. As a corollary to that we get the following.

Corollary 2.5. *If* $\langle \kappa_n \ ; \ n \in \omega\rangle$ *and* $\langle \lambda_n \ ; \ n \in \omega\rangle$ *are increasing sequences of cardinals such that* $\bigcup_{n\in\omega} \kappa_n = \bigcup_{n\in\omega} \lambda_n$ *and* $\kappa_n > \lambda_n$ *for at least one* $n \in \omega$*, then the infinitary Chang conjecture* $(\kappa_n)_{n<\omega} \twoheadrightarrow (\lambda_n)_{n<\omega}$ *implies that there is an inner model in which* $\kappa = \bigcup_{n\in\omega} \kappa_n$ *is measurable.*

Since we can force such a coherent sequence of Erdős cardinals by starting with a measurable cardinal (see [AK06, Theorem 3]) we have the following.

Theorem 2.6. *The theory* $\mathsf{ZF}+$ *"an infinitary Chang conjecture holds with the supremum of the left hand side cardinals being the same as the supremum of the right hand side cardinals" is equiconsistent with the theory* $\mathsf{ZFC}+$ *"a measurable cardinal exists".*

We conjecture that if the supremum of the κ_n is strictly bigger than the supremum of the λ_n then the consistency strength of such an infinitary Chang conjecture in ZF is weaker.

To prove lower bounds for the consistency strength of the existence of a set of good indiscernibles between a singular cardinal and its successor, more complex core models, for stronger large cardinal axioms, must be employed. Some results on this direction can be found in the authors PhD thesis [Dim11, Section 4 of Chapter 3], and a paper on the subject is under preparation.

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