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Article

SOME PROPERTIES OF THE HERMITE POLYNOMIALS AND THEIR SQUARES AND GENERATING FUNCTIONS

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ABSTRACT. In the paper, the authors consider the generating functions of the Hermite polynomials and their squares, present explicit formulas for higher order derivatives of the generating functions of the Hermite polynomials and their squares, which can be viewed as ordinary differential equations or derivative polynomials, find differential equations that the generating functions of the Hermite polynomials and their squares satisfy, and derive explicit formulas and recurrence relations for the Hermite polynomials and their squares.

1. INTRODUCTION

It is well known that the Hermite polynomials $H_n(x)$ can be generated by

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}. \tag{1}$$

The first six Hermite polynomials $H_n(x)$ for $0\le n\le 5$ are

$$1,\quad 2x,\quad 2\bigl(2x^2-1\bigr),\quad 4x\bigl(2x^2-3\bigr),\quad 4\bigl(4x^4-12x^2+3\bigr),\quad 8x\bigl(4x^4-20x^2+15\bigr).$$

In [3, p. 250], it was given that the squares $H_n^2(x)$ for $n\ge0$ of the Hermite polynomials $H_n(x)$ can be generated by

$$\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}=\sum_{n=0}^{\infty}\frac{H_n^2(x)}{2^n}\frac{t^n}{n!}. \tag{2}$$

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In [5], the equation (2) was reformulated as

$$\frac{1}{\sqrt{1-t^2}}\exp\frac{xt}{1+t}=\sum_{n=0}^\infty H_n^2\big(\sqrt{x}\,\big)\frac{t^n}{n!}.$$

Indeed, this is a typo and the corrected one should be

$$\frac{1}{\sqrt{1-t^2}}\exp\frac{xt}{1+t}=\sum_{n=0}^\infty \frac{H_n^2\big(\sqrt{x/2}\,\big)}{2^n}\frac{t^n}{n!}. \tag{3}$$

After inductively arguing for nine pages, it was obtained in [5, Theorem 1] that the ordinary differential equations

$$F^{(n)}(t)=\left[\sum_{i=0}^{n}\sum_{j=n-i}^{2(n-i)}\frac{a_{i,j}(n,x)}{(1-t)^i(1+t)^j}\right]F(t)$$

for $n\ge0$ have the same solution

$$F(t)=F(t,x)=\frac{1}{\sqrt{1-t^2}}\exp\frac{xt}{1+t}, \tag{4}$$

where

$$a_{0,0}(0,x)=1,\quad a_{1,0}(1,x)=\frac12,\quad a_{0,1}(1,x)=-\frac12,$$
$$a_{0,2}(1,x)=x,\quad a_{0,n}(n,x)=\left(-\frac12\right)^n(2n-1)!!,$$

and

$$\begin{aligned} a_{i,j}(n,x)={}&\sum_{k=0}^{2n-j-2i}\left(-\frac12\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\ &\times\left[\frac{2i-1}{2}a_{i-1,j-k}(n-k-1,x)+xa_{i,j-k-2}(n-k-1,x)\right]. \end{aligned} \tag{5}$$

From [5, Theorem 1] mentioned above, Theorems 2 and 3 in [5], which can be corrected as

$$\frac{H_{k+n}^2\big(\sqrt{x/2}\,\big)}{2^{k+n}}=\sum_{i=0}^{n}\sum_{j=n-i}^{2(n-i)}\sum_{p+q+r=k}(-1)^q\binom{k}{p,q,r}(i+p-1)_p(j+q-1)_q a_{i,j}(n,x)\frac{H_r^2\big(\sqrt{x/2}\,\big)}{2^r}$$

and

$$\frac{H_n^2\big(\sqrt{x/2}\,\big)}{2^n}=\sum_{i=0}^{n}\sum_{j=n-i}^{2(n-i)}a_{i,j}(n,x)$$

for $k,n\ge0$, were derived, where

$$(x)_n=\begin{cases}x(x+1)(x+2)\dots(x+n-1), & n\ge1\\ 1, & n=0\end{cases}$$

denotes the rising factorial and

$$\binom{n}{k_1,k_2,\dots,k_m}=\frac{n!}{k_1!k_2!\cdots k_m!}$$

is the multinomial coefficients.

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It is clear that the quantities $a_{i,j}(n,x)$ in [5] were expressed by a recurrent relation and can not be computed easily by hand and by computer softwares. We observe that, when $k=2n-j-2i$ and $i+j=n$, the quantity $a_{i,j-k-2}(n-k-1,x)$ in the recurrence relation (5) becomes

$$\begin{aligned} a_{i,j-k-2}(n-k-1,x) &= a_{i,j-(2n-j-2i)-2}(n-(2n-j-2i)-1,x) \\ &= a_{i,2(i+j-n-1)}(2i+j-n-1,x) = a_{i,-2}(i-1,x) \end{aligned}$$

which implies that Theorem 1, consequently Theorems 2 and 3, in [5], are wrong.

In this paper, we will reconsider the generating functions e^{2tx-t^2} and $F(t)=F(t,x)$ defined in (4), present explicit formulas for the nth derivatives of the functions $F(t)$ and e^{2tx-t^2}, which can be viewed as ordinary differential equations or derivative polynomials [7], find more differential equations that the functions $F(t)$ and e^{2tx-t^2} satisfy, and derive explicit formulas and recurrence relations for the Hermite polynomials $H_n(x)$ and their squares $H_n^2(x)$.

The main results of this paper can be stated as the following theorems.

Theorem 1.1. *For $n\ge0$, the nth derivative of the function $F(t)=F(t,x)$ defined in (4) can be computed by*

$$\begin{aligned} \frac{\mathrm{d}^n F(t)}{\mathrm{d} t^n} = \Biggl\{ \frac{(-1)^n n!}{(1+t)^n} \sum_{m=0}^{n} \frac{(-1)^m}{m!} \frac{1}{(1+t)^m} \Biggl(\sum_{k=0}^{n-m} \frac{(-1)^k(1+t)^k}{2^k} \binom{n-k-1}{m-1} \\ \times \Biggl[\frac{1}{t^k} \sum_{\ell=0}^{k} \frac{(2\ell-1)!!2^\ell}{\ell!} \binom{\ell}{k-\ell} \frac{t^{2\ell}}{(1-t^2)^\ell} \Biggr] \Biggr) x^m \Biggr\} F(t), \end{aligned} \tag{6}$$

where $\binom{0}{0}=1$ and $\binom{p}{q}=0$ for $q>p\ge0$.

Theorem 1.2. *For $n\ge0$, the squares $H_n^2(x)$ of the Hermite polynomials $H_n(x)$ can be computed by*

$$H_n^2(x) = (-1)^n 2^n n! \sum_{k=0}^{n} (-1)^k \frac{2^k}{k!} \Biggl[\sum_{\ell=0}^{n-k} \frac{1+(-1)^\ell}{2} \frac{(\ell-1)!!}{\ell!!} \binom{n-\ell-1}{k-1} \Biggr] x^{2k}. \tag{7}$$

Theorem 1.3. *For $n\ge0$, the Hermite polynomials $H_n(x)$ can be computed by*

$$H_n(x) = (-1)^n \frac{n!}{2^n} \sum_{k=0}^{n} (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} x^{2k-n} \tag{8}$$

and the nth derivative of their generating function e^{2xt-t^2} can be computed by

$$\frac{\mathrm{d}^n e^{2xt-t^2}}{\mathrm{d} t^n} = e^{2xt-t^2} \frac{n!}{2^n} \sum_{k=0}^{n} (-1)^k \frac{2^{2k}}{k!} \binom{k}{n-k} (t-x)^{2k-n}.$$

Theorem 1.4. *For $n\ge0$, the Hermite polynomials $H_n(x)$ and their derivatives $H_n'(x)$ satisfy $H_0'(x)=0$,*

$$H_n'(x) = 2nH_{n-1}(x), \tag{9}$$

and

$$H_n(x) = 2xH_{n-1}(x) - H_{n-1}'(x) \tag{10}$$

for $n\in\mathbb{N}$. Consequently,

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \tag{11}$$

for $n\ge2$.

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Theorem 1.5. *For $n\ge0$, the Hermite polynomials $H_n(x)$ satisfy the recurrence relations*

$$\sum_{k=0}^{n}\frac{1+(-1)^{n-k}}{2}\frac{2^{(n-k)/2}}{(n-k)!!k!}H_k(x)=\frac{(2x)^n}{n!} \tag{12}$$

and

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(2x)^{n-k}H_k(x)=\frac{1+(-1)^n}{2}(-2)^{n/2}\frac{n!}{n!!}. \tag{13}$$

For $n\ge0$, the squares $H_n^2(x)$ of the Hermite polynomials $H_n(x)$ satisfy the recurrence relations

$$\sum_{k=0}^{n}\frac{1+(-1)^{n-k}}{2}\frac{(n-k-3)!!}{(n-k)!!(2k)!!}H_k^2(x)=(-1)^{n+1}\sum_{\ell=0}^{n}\frac{(-1)^\ell}{\ell!}\binom{n-1}{\ell-1}\bigl(2x^2\bigr)^\ell \tag{14}$$

and

$$\sum_{k=0}^{n}\frac{(-1)^k}{2^kk!}\Biggl[\sum_{\ell=0}^{n-k}\frac{2^\ell}{\ell!}\binom{n-k-1}{\ell-1}x^{2\ell}\Biggr]H_k^2(x)=\frac{1+(-1)^n}{2}\frac{(n-1)!!}{n!!}. \tag{15}$$

2. Lemmas

In order to prove our main results, we need several lemmas below.

Lemma 2.1 ([2, p. 134, Theorem A] and [2, p. 139, Theorem C])**.** *For $n\ge k\ge0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\Bigl(\frac{x_i}{i!}\Bigr)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr). \tag{16}$$

Lemma 2.2 ([2, p. 135])**.** *For complex numbers a and b, we have*

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}). \tag{17}$$

Lemma 2.3 ([4, Theorem 4.1], [9, Eq. (2.8)], and [10, Section 3])**.** *For $0\le k\le n$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0)=\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}. \tag{18}$$

Lemma 2.4 ([2, p. 135, Theorem B] and [6, Theorem 1.1])**.** *For $n\ge k\ge0$, we have*

$$\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!)=\binom{n-1}{k-1}\frac{n!}{k!}. \tag{19}$$

Lemma 2.5. *For $n \ge k \ge 0$, the Bell polynomials of the second kind*

$$\mathrm{B}_{n,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (n-k+1)![1-(-1)^{n-k+2}]\bigr)$$

satisfy

$$\mathrm{B}_{2j+1,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (2j-k+2)![1-(-1)^{2j-k+3}]\bigr) = 0, \quad 2j+1 \ge k, \tag{20}$$

$$\mathrm{B}_{2j,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (2j-k+1)![1-(-1)^{2j-k+2}]\bigr) = 0, \quad 2j \ge k > j \ge 0, \tag{21}$$

and

$$\mathrm{B}_{2j,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (2j-k+1)![1-(-1)^{2j-k+2}]\bigr) = \frac{2^k(2j)!}{k!}\binom{j-1}{k-1} \tag{22}$$

for $j \ge k \ge 0$. Equivalently and unifiedly,

$$\mathrm{B}_{n,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (n-k+1)![1-(-1)^{n-k+2}]\bigr) = [1+(-1)^n]\frac{2^{k-1}n!}{k!}\binom{\frac{n}{2}-1}{k-1} \tag{23}$$

or

$$\mathrm{B}_{n,k}\left(0, 2!, 0, 4!, 0, 6!, 0, 8!, 0, \dots, \frac{1-(-1)^{n-k+2}}{2}(n-k+1)!\right) = \frac{1+(-1)^n}{2}\frac{n!}{k!}\binom{\frac{n}{2}-1}{k-1}, \tag{24}$$

where

$$\binom{\alpha}{k} = \frac{\langle\alpha\rangle_k}{k!} = \begin{cases} \dfrac{1}{k!}\displaystyle\prod_{\ell=0}^{k-1}(\alpha-\ell+1), & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}$$

for arbitrary $a \in \mathbb{C}$ and $k \ge 0$ and $\langle\alpha\rangle_k$ is called the falling factorial.

Proof. In [2, p. 133], it was listed that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!}$$

for $k \ge 0$. From this, it follows that

$$\begin{aligned}
&\sum_{n=k}^{\infty} \mathrm{B}_{n,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (n-k+1)![1-(-1)^{n-k+2}]\bigr)\frac{t^n}{n!} \\
&= \frac{1}{k!}\left[\sum_{m=1}^{\infty} 2\cdot(2m)!\frac{t^{2m}}{(2m)!}\right]^k = \frac{2^k}{k!}\left(\frac{t^2}{1-t^2}\right)^k = \frac{2^k}{k!}\left(\frac{1}{1-t^2}-1\right)^k \\
&= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\left(\frac{1}{1-t^2}\right)^\ell = \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\frac{1}{(1-t^2)^\ell}.
\end{aligned}$$

Further differentiating $m \ge k$ times and making use of (16), (17), and (18) yield

$$\begin{aligned}
&\sum_{n=m}^{\infty} \mathrm{B}_{n,k}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (n-k+1)![1-(-1)^{n-k+2}]\bigr)\langle n\rangle_m\frac{t^{n-m}}{n!} \\
&= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\left[\frac{1}{(1-t^2)^\ell}\right]^{(m)}
\end{aligned}$$

$$= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\sum_{p=0}^{m}\left(\frac{1}{u^\ell}\right)^{(p)}\mathrm{B}_{m,p}(-2t,-2,0,\dots,0)$$
$$= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\sum_{p=0}^{m}\frac{(-1)^p\langle-\ell\rangle_p}{u^{\ell+p}}(-2)^p\mathrm{B}_{m,p}(t,1,0,\dots,0)$$
$$= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\sum_{p=0}^{m}\frac{2^p\langle-\ell\rangle_p}{(1-t^2)^{\ell+p}}\frac{1}{2^{m-p}}\frac{m!}{p!}\binom{p}{m-p}t^{2p-m}.$$

Taking $t\to0$ gives

$$\mathrm{B}_{m,k}\big(1![1-(-1)^2],2![1-(-1)^3],\dots,(m-k+1)![1-(-1)^{m-k+2}]\big)$$
$$= \frac{2^k}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\lim_{t\to0}\sum_{p=0}^{m}\frac{2^p\langle-\ell\rangle_p}{2^{m-p}}\frac{m!}{p!}\binom{p}{m-p}t^{2p-m}$$
$$= \begin{cases}0, & m=2j+1\\ \dfrac{2^k}{k!}\displaystyle\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\langle-\ell\rangle_j\frac{(2j)!}{j!}, & m=2j\end{cases}$$
$$= \begin{cases}0, & m=2j+1\\ (-1)^k\dfrac{2^k}{k!}(2j)!\displaystyle\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\ell+j-1}{j}, & m=2j\end{cases}$$

which is equivalent to (20) and

$$\mathrm{B}_{2j,k}\big(1![1-(-1)^2],2![1-(-1)^3],\dots,(2j-k+1)![1-(-1)^{2j-k+2}]\big)$$
$$= (-1)^k\frac{2^k}{k!}(2j)!\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\ell+j-1}{j}$$
$$= (-1)^k\frac{2^k}{k!}(2j)!(-1)^k\binom{j-1}{k-1} = \frac{2^k(2j)!}{k!}\binom{j-1}{k-1}$$

for $j,k\ge0$. The formulas (21) and (22) are thus proved.

It is straightforward to rewrite (20), (21), and (22) as either (23) or (24). The proof of Lemma 2.5 is complete. □

Remark 2.1. By the formula

$$\mathrm{B}_{n,k}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1})$$
$$= \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})\mathrm{B}_{m,s}(y_1,y_2,\dots,y_{m-s+1})$$

in [1, Example 2.6], [2, p. 136, Eq. [3n]], and [8, Lemma 5] and by the formulas (17) and (19), it follows that

$$\mathrm{B}_{n,k}\big(1![1-(-1)^2],2![1-(-1)^3],\dots,(n-k+1)![1-(-1)^{n-k+2}]\big)$$
$$= \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(1!,2!,\dots,(\ell-r+1)!)\mathrm{B}_{m,s}(-1!,2!,\dots,(-1)^{m-s+1}(m-s+1)!)$$

$$
\begin{aligned}
&= \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\binom{\ell-1}{r-1}\frac{\ell!}{r!}(-1)^m \mathrm{B}_{m,s}(1!,2!,\dots,(m-s+1)!)\\
&= \sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}\binom{\ell-1}{r-1}\frac{\ell!}{r!}\binom{m-1}{s-1}\frac{m!}{s!}\\
&= \sum_{r=0}^{k}\sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{n}{\ell}\binom{\ell-1}{r-1}\binom{n-\ell-1}{k-r-1}\frac{\ell!(n-\ell)!}{r!(k-r)!}\\
&= \frac{n!}{k!}\sum_{r=0}^{k}\sum_{\ell=0}^{n}(-1)^{n-\ell}\binom{k}{r}\binom{\ell-1}{r-1}\binom{n-\ell-1}{k-r-1}
\end{aligned}
$$

which is not simpler than the nice expression (23).

3. Proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.1. By the formulas (16), (17), and (18), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\biggl(\frac{1}{\sqrt{1-t^2}}\biggr) &= \sum_{\ell=0}^{k}\frac{\mathrm{d}^\ell}{\mathrm{d}\,u^\ell}\biggl(\frac{1}{\sqrt{u}}\biggr)\mathrm{B}_{k,\ell}(-2t,-2,0,\dots,0)\\
&= \sum_{\ell=0}^{k}\biggl\langle-\frac{1}{2}\biggr\rangle_\ell\frac{1}{u^{\ell+1/2}}(-2)^\ell \mathrm{B}_{k,\ell}(t,1,0,\dots,0)\\
&= \sum_{\ell=0}^{k}\biggl\langle-\frac{1}{2}\biggr\rangle_\ell\frac{1}{(1-t^2)^{\ell+1/2}}(-2)^\ell\frac{1}{2^{k-\ell}}\frac{k!}{\ell!}\binom{\ell}{k-\ell}t^{2\ell-k}\\
&= \sum_{\ell=0}^{k}\frac{(2\ell-1)!!}{2^\ell}\frac{1}{(1-t^2)^{\ell+1/2}}2^\ell\frac{1}{2^{k-\ell}}\frac{k!}{\ell!}\binom{\ell}{k-\ell}t^{2\ell-k}\\
&= \frac{1}{\sqrt{1-t^2}}\frac{k!}{(2t)^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell},
\end{aligned}
\tag{25}
$$

where $u=u(t)=1-t^2$.

Similarly, by the formulas (16), (17), and (19), we acquire

$$
\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\biggl(\exp\frac{xt}{1+t}\biggr) &= \sum_{\ell=0}^{k}x^\ell e^{xv}\mathrm{B}_{k,\ell}\biggl(\frac{1!}{(1+t)^2},\frac{-2!}{(1+t)^3},\dots,\frac{(-1)^{k-\ell}(k-\ell+1)!}{(1+t)^{k-\ell+2}}\biggr)\\
&= \sum_{\ell=0}^{k}x^\ell e^{xt/(1+t)}\frac{(-1)^{k+\ell}}{(1+t)^{k+\ell}}\mathrm{B}_{k,\ell}(1!,2!,\dots,(k-\ell+1)!)\\
&= e^{xt/(1+t)}\frac{(-1)^k k!}{(1+t)^k}\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}\binom{k-1}{\ell-1}\frac{x^\ell}{(1+t)^\ell},
\end{aligned}
\tag{26}
$$

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where $v=v(t)=\frac{t}{1+t}$. Making use of the above two results and employing the Leibniz rule yield

$$\begin{aligned}
\frac{\mathrm{d}^n F(t)}{\mathrm{d}\, t^n} &= \sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}\, t^k}\biggl(\frac{1}{\sqrt{1-t^2}}\biggr)\frac{\mathrm{d}^{n-k}}{\mathrm{d}\, t^{n-k}}\biggl(\exp\frac{xt}{1+t}\biggr)\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{\sqrt{1-t^2}}\frac{k!}{(2t)^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\\
&\quad\times e^{xt/(1+t)}\frac{(-1)^{n-k}(n-k)!}{(1+t)^{n-k}}\sum_{\ell=0}^{n-k}\frac{(-1)^\ell}{\ell!}\binom{n-k-1}{\ell-1}\frac{x^\ell}{(1+t)^\ell}\\
&= \frac{e^{xt/(1+t)}}{\sqrt{1-t^2}}\frac{(-1)^n n!}{(1+t)^n}\sum_{k=0}^{n}\frac{(-1)^k(1+t)^k}{(2t)^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\\
&\quad\times\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\sum_{m=0}^{n-k}\frac{(-1)^m}{m!}\binom{n-k-1}{m-1}\frac{x^m}{(1+t)^m}\\
&= F(t)\frac{(-1)^n n!}{(1+t)^n}\sum_{k=0}^{n}\frac{(-1)^k(1+t)^k}{2^k t^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\\
&\quad\times\frac{t^{2\ell}}{(1-t^2)^\ell}\sum_{m=0}^{n-k}\frac{(-1)^m}{m!}\binom{n-k-1}{m-1}\frac{x^m}{(1+t)^m}\\
&= F(t)\frac{(-1)^n n!}{(1+t)^n}\sum_{m=0}^{n}\frac{(-1)^m}{m!}\frac{1}{(1+t)^m}\Biggl(\sum_{k=0}^{n-m}\frac{(-1)^k(1+t)^k}{2^k}\\
&\quad\times\binom{n-k-1}{m-1}\Biggl[\frac{1}{t^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\Biggr]\Biggr)x^m.
\end{aligned}$$

The formula (6) is thus proved. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Since

$$\begin{aligned}
\biggl[\frac{t^{2\ell}}{(1-t^2)^\ell}\biggr]^{(k)} &= \biggl[\biggl(\frac{1}{2(t+1)}-\frac{1}{2(t-1)}-1\biggr)^\ell\biggr]^{(k)}\\
&= \sum_{p=0}^{k}\bigl(w^\ell\bigr)^{(p)}\mathrm{B}_{k,p}\biggl(-\frac{1!}{2}\biggl[\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\biggr],\frac{2!}{2}\biggl[\frac{1}{(t+1)^3}-\frac{1}{(t-1)^3}\biggr],\\
&\quad\dots,(-1)^{k-p+1}\frac{(k-p+1)!}{2}\biggl[\frac{1}{(t+1)^{k-p+2}}-\frac{1}{(t-1)^{k-p+2}}\biggr]\biggr)\\
&= \sum_{p=0}^{k}\langle\ell\rangle_p w^{\ell-p}\frac{(-1)^k}{2^p}\mathrm{B}_{k,p}\biggl(1!\biggl[\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\biggr],2!\biggl[\frac{1}{(t+1)^3}-\frac{1}{(t-1)^3}\biggr],\\
&\quad\dots,(k-p+1)!\biggl[\frac{1}{(t+1)^{k-p+2}}-\frac{1}{(t-1)^{k-p+2}}\biggr]\biggr)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{p=0}^{k} \langle \ell \rangle_p \left(\frac{t^2}{1-t^2}\right)^{\ell-p} \frac{(-1)^k}{2^p} \mathrm{B}_{k,p}\Biggl(1!\left[\frac{1}{(t+1)^2} - \frac{1}{(t-1)^2}\right], \\
&\quad 2!\left[\frac{1}{(t+1)^3} - \frac{1}{(t-1)^3}\right], \dots, (k-p+1)!\left[\frac{1}{(t+1)^{k-p+2}} - \frac{1}{(t-1)^{k-p+2}}\right]\Biggr) \\
&\to \langle \ell \rangle_\ell \frac{(-1)^k}{2^\ell} \mathrm{B}_{k,\ell}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (k-\ell+1)![1-(-1)^{k-\ell+2}]\bigr) \\
&= (-1)^k \frac{\ell!}{2^\ell} \mathrm{B}_{k,\ell}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (k-\ell+1)![1-(-1)^{k-\ell+2}]\bigr)
\end{aligned}
$$

as $t \to 0$, where $w = w(t) = \frac{t^2}{1-t^2} = \frac{1}{2(t+1)} - \frac{1}{2(t-1)} - 1$, employing the L'Hôspital rule and the formula (23) leads to

$$
\begin{aligned}
&\lim_{t\to 0}\left[\frac{1}{t^k}\sum_{\ell=0}^{k} \frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\right] = \frac{1}{k!}\sum_{\ell=0}^{k} \frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\lim_{t\to 0}\left[\frac{t^{2\ell}}{(1-t^2)^\ell}\right]^{(k)} \\
&= \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(2\ell-1)!!\binom{\ell}{k-\ell}\mathrm{B}_{k,\ell}\bigl(1![1-(-1)^2], 2![1-(-1)^3], \dots, (k-\ell+1)![1-(-1)^{k-\ell+2}]\bigr) \\
&= \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(2\ell-1)!!\binom{\ell}{k-\ell}[1+(-1)^k]\frac{2^{\ell-1}k!}{\ell!}\binom{\frac{k}{2}-1}{\ell-1} \\
&= (-1)^k[1+(-1)^k]\sum_{\ell=0}^{k}\frac{2^{\ell-1}(2\ell-1)!!}{\ell!}\binom{\ell}{k-\ell}\binom{\frac{k}{2}-1}{\ell-1} \\
&= \frac{1+(-1)^k}{2}\sum_{\ell=0}^{k}\frac{2^{2\ell}(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{k-\ell}\binom{\frac{k}{2}-1}{\ell-1} = \frac{1+(-1)^k}{2}\frac{2^k(k-1)!!}{k!!}. \qquad (27)
\end{aligned}
$$

Therefore, taking the limit $t \to$ on both sides of (6) yields

$$
\begin{aligned}
\lim_{t\to 0}\frac{\mathrm{d}^n F(t)}{\mathrm{d}\, t^n} &= (-1)^n n! \sum_{m=0}^{n}\frac{(-1)^m}{m!}\sum_{k=0}^{n-m}\frac{(-1)^k}{2^k}\binom{n-k-1}{m-1} \\
&\quad \times \lim_{t\to 0}\left[\frac{1}{t^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\right]x^m \\
&= (-1)^n n! \sum_{m=0}^{n}\frac{(-1)^m}{m!}\sum_{k=0}^{n-m}\frac{(-1)^k}{2^k}\binom{n-k-1}{m-1}\frac{1+(-1)^k}{2}\frac{2^k(k-1)!!}{k!!}x^m \\
&= (-1)^n n! \sum_{m=0}^{n}\frac{(-1)^m}{m!}\left[\sum_{k=0}^{n-m}\binom{n-k-1}{m-1}\frac{1+(-1)^k}{2}\frac{(k-1)!!}{k!!}\right]x^m
\end{aligned}
$$

which means by (3) that

$$
\frac{H_n^2\bigl(\sqrt{x/2}\bigr)}{2^n} = (-1)^n n! \sum_{m=0}^{n}\frac{(-1)^m}{m!}\left[\sum_{k=0}^{n-m}\frac{1+(-1)^k}{2}\frac{(k-1)!!}{k!!}\binom{n-k-1}{m-1}\right]x^m, \quad n \ge 0.
$$

This can be rearranged as (7). The proof of Theorem 1.2 is complete. ☐

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Proof of Theorem 1.3. By the formulas (16), (17), and (18), we obtain

$$\begin{aligned}\frac{\mathrm{d}^n e^{2xt-t^2}}{\mathrm{d}\,t^n}&=\sum_{k=0}^{n}(e^u)^{(k)}\mathrm{B}_{n,k}(2x-2t,-2,0,\dots,0)\\&=\sum_{k=0}^{n}e^{2xt-t^2}(-2)^k\mathrm{B}_{n,k}(t-x,1,0,\dots,0)\\&=e^{2xt-t^2}\sum_{k=0}^{n}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}(t-x)^{2k-n}\\&=\frac{n!}{2^n}\frac{e^{2xt-t^2}}{(t-x)^n}\sum_{k=0}^{n}(-1)^k\frac{2^{2k}}{k!}\binom{k}{n-k}(t-x)^{2k},\end{aligned}$$

where $u=u(t)=2xt-t^2$. Hence, we acquire

$$\begin{aligned}H_n(x)&=\lim_{t\to0}\frac{\mathrm{d}^n e^{2xt-t^2}}{\mathrm{d}\,t^n}\\&=\frac{n!}{2^n}\lim_{t\to0}\frac{e^{2xt-t^2}}{(t-x)^n}\sum_{k=0}^{n}(-1)^k\frac{2^{2k}}{k!}\binom{k}{n-k}(t-x)^{2k}\\&=\frac{n!}{2^n}\frac{1}{(-x)^n}\sum_{k=0}^{n}(-1)^k\frac{2^{2k}}{k!}\binom{k}{n-k}(-x)^{2k}\\&=(-1)^n\frac{n!}{2^n}\sum_{k=0}^{n}(-1)^k\frac{2^{2k}}{k!}\binom{k}{n-k}x^{2k-n}.\end{aligned}$$

The formula (8) follows. This formula can also be derived similarly by considering

$$H_n(x)=(-1)^ne^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}e^{-x^2}\quad\text{or}\quad H_n(x)=e^{x^2/2}\biggl(x-\frac{\mathrm{d}}{\mathrm{d}\,x}\biggr)^ne^{-x^2/2}.$$

The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. Differentiating with respect to x on both sides of (1) yields

$$\begin{aligned}2te^{2xt-t^2}&=\sum_{n=0}^{\infty}H_n'(x)\frac{t^n}{n!},\\2t\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}H_n'(x)\frac{t^n}{n!},\\\sum_{n=0}^{\infty}2H_n(x)\frac{t^{n+1}}{n!}&=\sum_{n=0}^{\infty}H_n'(x)\frac{t^n}{n!},\\\sum_{n=1}^{\infty}2H_{n-1}(x)\frac{t^n}{(n-1)!}&=\sum_{n=0}^{\infty}H_n'(x)\frac{t^n}{n!}.\end{aligned}$$

Hence, it follows that $H_0'(x)=0$ and the formula (9) is valid.

Differentiating with respect to x on both sides of (2) gives

$$\frac{4xt}{1+t}\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}=\sum_{n=0}^{\infty}\frac{2H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!},$$

$$\frac{2xt}{1+t}\sum_{n=0}^{\infty}\frac{H_n^2(x)}{2^n}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!},$$

$$2xt\sum_{n=0}^{\infty}\frac{H_n^2(x)}{2^n}\frac{t^n}{n!}=(1+t)\sum_{n=0}^{\infty}\frac{H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{2xH_n^2(x)}{2^n}\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\frac{H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!}+\sum_{n=0}^{\infty}\frac{H_n(x)H_n'(x)}{2^n}\frac{t^{n+1}}{n!},$$

$$\sum_{n=1}^{\infty}\frac{2xH_{n-1}^2(x)}{2^{n-1}}\frac{t^n}{(n-1)!}=\sum_{n=0}^{\infty}\frac{H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!}+\sum_{n=1}^{\infty}\frac{H_{n-1}(x)H_{n-1}'(x)}{2^{n-1}}\frac{t^n}{(n-1)!}.$$

This means that $H_0'(x)=0$ and

$$\frac{2xH_{n-1}^2(x)}{2^{n-1}}\frac{t^n}{(n-1)!}=\frac{H_n(x)H_n'(x)}{2^n}\frac{t^n}{n!}+\frac{H_{n-1}(x)H_{n-1}'(x)}{2^{n-1}}\frac{t^n}{(n-1)!}$$

for $n\in\mathbb{N}$, which can be simplified as

$$H_n(x)H_n'(x)=2nH_{n-1}(x)\big[2xH_{n-1}(x)-H_{n-1}'(x)\big]$$

Combining this with (9) derives the formula (10).

Substituting (9) into (10) results in (11) readily. The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. It is easy to see that $e^{t^2}e^{2xt-t^2}=e^{2xt}$. Differentiating with respect to t on both sides of this equation and utilizing (16), (17), and (18) give

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}e^{t^2}}{\mathrm{d}t^{n-k}}\frac{\mathrm{d}^k e^{2xt-t^2}}{\mathrm{d}t^k}=(2x)^ne^{2xt},$$

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell=0}^{n-k}e^{t^2}\mathrm{B}_{n-k,\ell}(2t,2,0,\dots,0)\frac{\mathrm{d}^k e^{2xt-t^2}}{\mathrm{d}t^k}=(2x)^ne^{2xt},$$

$$e^{t^2}\sum_{k=0}^{n}\binom{n}{k}\sum_{\ell=0}^{n-k}2^\ell\frac{1}{2^{n-k-\ell}}\frac{(n-k)!}{\ell!}\binom{\ell}{n-k-\ell}t^{2\ell-n+k}\frac{\mathrm{d}^k e^{2xt-t^2}}{\mathrm{d}t^k}=(2x)^ne^{2xt},$$

$$e^{t^2}\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!}{2^{n-k}}\sum_{\ell=0}^{n-k}\frac{2^{2\ell}}{\ell!}\binom{\ell}{n-k-\ell}t^{2\ell-n+k}\frac{\mathrm{d}^k e^{2xt-t^2}}{\mathrm{d}t^k}=(2x)^ne^{2xt}.$$

Further taking the limit $t\to0$ yields

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!}{2^{n-k}}\frac{1+(-1)^{n-k}}{2}\frac{2^{3(n-k)/2}}{(n-k)!!}\lim_{t\to0}\frac{\mathrm{d}^k e^{2xt-t^2}}{\mathrm{d}t^k}=(2x)^n,$$

$$n!\sum_{k=0}^{n}\frac{1+(-1)^{n-k}}{2}\frac{2^{(n-k)/2}}{(n-k)!!k!}H_k(x)=(2x)^n.$$

The recurrence relation (12) is thus proved.

Similarly, from $e^{-2xt}e^{2xt-t^2}=e^{-t^2}$, it follows that

$$\sum_{k=0}^{n}\binom{n}{k}(-2x)^{n-k}e^{-2xt}\frac{\mathrm{d}^k\,e^{2xt-t^2}}{\mathrm{d}\,t^k}=\sum_{k=0}^{n}e^{-t^2}\mathrm{B}_{n,k}(-2t,-2,0,\dots,0),$$

$$\sum_{k=0}^{n}\binom{n}{k}(-2x)^{n-k}e^{-2xt}\frac{\mathrm{d}^k\,e^{2xt-t^2}}{\mathrm{d}\,t^k}=\sum_{k=0}^{n}e^{-t^2}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}t^{2k-n},$$

and, as $t\to0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-2x)^{n-k}H_k(x)=\lim_{t\to0}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\frac{2^{3k}n!}{(2k)!!}\binom{k}{n-k}t^{2k-n}.$$

The recurrence relation (13) is thus proved.

Similarly, since

$$\sqrt{1-t^2}\,\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}=\exp\frac{2x^2t}{1+t}$$

and

$$\exp\frac{-2x^2t}{1+t}\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}=\frac{1}{\sqrt{1-t^2}},$$

by (25), (26), (18), the formula

$$\begin{aligned}\left(\sqrt{1-t^2}\right)^{(k)}&=\sum_{\ell=0}^{k}\left\langle\frac12\right\rangle_\ell u^{1/2-\ell}\mathrm{B}_{k,\ell}(-2t,-2,0,\dots,0)\\&=\sum_{\ell=0}^{k}\frac{(-1)^{\ell-1}(2\ell-3)!!}{2^\ell}\left(1-t^2\right)^{1/2-\ell}(-2)^\ell\mathrm{B}_{k,\ell}(t,1,0,\dots,0)\\&=-\sum_{\ell=0}^{k}(2\ell-3)!!\left(1-t^2\right)^{1/2-\ell}\frac{1}{2^{k-\ell}}\frac{k!}{\ell!}\binom{\ell}{k-\ell}t^{2\ell-k}\\&=-\frac{k!}{2^k}\frac{(1-t^2)^{1/2}}{t^k}\sum_{\ell=0}^{k}(2\ell-3)!!\frac{2^\ell}{\ell!}\binom{\ell}{k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell},\end{aligned}$$

and by the Leibniz rule for differentiation, it follows that

$$\sum_{k=0}^{n}\binom{n}{k}\left(\sqrt{1-t^2}\right)^{(n-k)}\left(\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}\right)^{(k)}=\left(\exp\frac{2x^2t}{1+t}\right)^{(n)},$$

$$\begin{aligned}-\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!}{2^{n-k}}\frac{(1-t^2)^{1/2}}{t^{n-k}}\sum_{\ell=0}^{n-k}(2\ell-3)!!\frac{2^\ell}{\ell!}\binom{\ell}{n-k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\left(\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}\right)^{(k)}\\=e^{2x^2t/(1+t)}\frac{(-1)^nn!}{(1+t)^n}\sum_{\ell=0}^{n}\frac{(-1)^\ell}{\ell!}\binom{n-1}{\ell-1}\frac{(2x^2)^\ell}{(1+t)^\ell}\end{aligned}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\left(\exp\frac{-2x^2t}{1+t}\right)^{(n-k)}\left(\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}\right)^{(k)}=\left(\frac{1}{\sqrt{1-t^2}}\right)^{(n)},$$

$$\sum_{k=0}^{n}\binom{n}{k}e^{-2x^2t/(1+t)}\frac{(-1)^{n-k}(n-k)!}{(1+t)^{n-k}}\sum_{\ell=0}^{n-k}\frac{(-1)^\ell}{\ell!}\binom{n-k-1}{\ell-1}\frac{(-2x^2)^\ell}{(1+t)^\ell}\biggl(\frac{1}{\sqrt{1-t^2}}\exp\frac{2x^2t}{1+t}\biggr)^{(k)} \\ =\frac{1}{\sqrt{1-t^2}}\frac{n!}{(2t)^n}\sum_{\ell=0}^{n}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{n-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}.$$

Further taking the limit $t\to0$ results in

$$-\sum_{k=0}^{n}\binom{n}{k}\frac{(n-k)!}{2^{n-k}}\lim_{t\to0}\Biggl[\frac{1}{t^{n-k}}\sum_{\ell=0}^{n-k}(2\ell-3)!!\frac{2^\ell}{\ell!}\binom{\ell}{n-k-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\Biggr]\frac{H_k^2(x)}{2^k} \\ =(-1)^n n!\sum_{\ell=0}^{n}\frac{(-1)^\ell}{\ell!}\binom{n-1}{\ell-1}\bigl(2x^2\bigr)^\ell \quad (28)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(n-k)!\sum_{\ell=0}^{n-k}\frac{(-1)^\ell}{\ell!}\binom{n-k-1}{\ell-1}\bigl(-2x^2\bigr)^\ell\frac{H_k^2(x)}{2^k} \\ =n!\lim_{t\to0}\Biggl[\frac{1}{(2t)^n}\sum_{\ell=0}^{n}\frac{(2\ell-1)!!2^\ell}{\ell!}\binom{\ell}{n-\ell}\frac{t^{2\ell}}{(1-t^2)^\ell}\Biggr]. \quad (29)$$

Substituting (27) into (28) and (29) acquires the recurrence relations (14) and (15). The proof of Theorem 1.5 is complete. □

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