Proof of two Maurer's conjectures on basis graphs of matroids

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Abstract. We characterize 2–dimensional complexes associated canonically with basis graphs of matroids as simply connected triangle-square complexes satisfying some local conditions. This proves a version of a (disproved) conjecture by Stephen Maurer (Conjecture 3 of S. Maurer, Matroid basis graphs I, JCTB 14 (1973), 216–240). We also establish Conjecture 1 from the same paper about the redundancy of the conditions in the characterization of basis graphs. We indicate positive-curvature-like aspects of the local properties of the studied complexes. We characterize similarly the corresponding 2–complexes of even Δ–matroids.

1. Introduction

Matroids constitute an important unifying structure in combinatorics, algorithmics, and combinatorial optimization — cf. e.g. [Oxl11] and references therein. A *matroid* on a finite set of elements I is a collection $\mathcal{B}$ of subsets of I, called *bases,* which satisfy the following exchange property: for all $A, B \in \mathcal{B}$ and $a \in A \setminus B$ there exists $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathcal{B}$ (the base $A \setminus \{a\} \cup \{b\}$ is obtained from the base A by an *elementary exchange*). The *basis graph* $G = G(\mathcal{B})$ of a matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by an elementary exchange, i.e., $|A\Delta B| = 2$. Basis graphs faithfully represent their matroids [HNT73, Mau73], thus studying the basis graph amounts to studying the matroid itself.

By the exchange property, basis graphs are connected. For any two bases A and B at distance 2 there exist at most four bases adjacent to A and B: if $A \setminus B = \{a_1, a_2\}$ and $B \setminus A = \{b_1, b_2\}$, then these bases have the form $A \setminus \{a_i\} \cup \{b_j\} = B \setminus \{b_j\} \cup \{a_i\}$ for $i, j \in \{1, 2\}$. On the other hand, the exchange property ensures that at least one of the pairs $A \setminus \{a_1\} \cup \{b_1\}, A \setminus \{a_2\} \cup \{b_2\}$ or $A \setminus \{a_1\} \cup \{b_2\}, A \setminus \{a_2\} \cup \{b_1\}$ must be bases. Together with A and B, this pair of bases C, C' induce a square in the basis graph. Therefore, A and B, together with their common neighbors induce a square, a pyramid, or an octahedron, i.e., basis graphs satisfy what we call the *interval condition.* The exchange property of bases also shows that if A, C, B, C' induce a square in the basis graph, then for any other base

$D \in \mathcal{B}$, the equality $d(D, A) + d(D, B) = d(D, C) + d(D, C')$ holds (i.e., the total number of elementary exchanges to transform D to A and B equals to the total number of exchanges to transform D to C and C'). Following [Mau73], we call this property of basis graphs the *positioning condition.* Finally, by Lemma 1.8 of [Mau73], the subgraph induced by all bases adjacent to a given base is the line graph of a finite bipartite graph; we will call it the *link condition.*

In [Mau73, Theorem 2.1] Maurer characterized the basis graphs of matroids as connected graphs satisfying the three conditions above — see Theorem 3 in Section 2.3 below for the precise statement and for a stronger version of this characterization provided in [Mau73, Theorem 3.1]. Furthermore, in [Mau73, Theorem 3.5], it is established that under some additional conditions the link condition is redundant and Conjecture 1 of [Mau73] asks if this is the case in general. Our first result provides a positive answer to this conjecture. (Note that for a finite graph G the finiteness assumptions on a link are trivially satisfied.)

Theorem 1. *The link condition is redundant for all basis graphs, in the following sense. A graph G is the basis graph of a matroid if and only if G is connected, satisfies the interval and the positioning conditions, and has at least one vertex with finitely many neighbors.*

According to [Bjö95] (and implicitly introduced on pp. 237–239 in [Mau73]), the *basis complex* $X = X(\mathcal{B})$ of a matroid $\mathcal{B}$ is the 2–dimensional cell complex whose 1–skeleton is the basis graph G, and whose 2–cells are the triangles and the squares of the basis graph. We call this complex also the *triangle-square complex* of G, and denote it by $X(G)$.

From the characterization of basis graphs, Maurer deduced in [Mau73, Theorem 5.1] that all basis complexes of matroids are simply connected. Consequently, he proposed (a natural from the topological viewpoint) Conjecture 3 of [Mau73], stating that in the characterization of basis graphs the global (metric) positioning condition on G can be replaced by the topological condition of simply connectedness of the triangle-square complex $X(G)$ of G. Donald, Holzmann, and Tobey [DHT77] (as well as Maurer in the personal communication to the authors of [DHT77]) presented counterexamples to this conjecture (as well as to Conjecture 2 of [Mau73] about the eventual redundancy of the positioning condition), i.e., simply connected triangle-square complexes, satisfying the interval and the link conditions, but not being basis complexes — cf. Section 5 below. Nevertheless, the main result of our paper shows that a general form of Maurer's Conjecture 3 — saying that triangle-square complexes of basis graphs of matroids may be characterized as simply connected complexes satisfying some local conditions — is true.

Theorem 2. *For a graph G the following conditions are equivalent:*

(i) G is the basis graph of a matroid;

(ii) the triangle-square complex $X(G)$ is simply connected and every ball of radius 3 in G is isomorphic to a ball of radius 3 in the basis graph of a matroid;

(iii) the triangle-square complex $X(G)$ is simply connected, G satisfies the interval and the local positioning conditions, and G contains at least one vertex with finitely many neighbors.

A formal definition of the local positioning condition is given in the next section. This condition, as well as the interval condition, are local because they concern at most quintets of vertices at distance ≤ 3 from each other.

Simple connectivity of basis complexes of matroids was used several times in the theory of ordinary and oriented matroids, in particular, in the proof of Las Vergnas's theorem [LV78, BLVS⁺93] about the characterization of basis orientations of ordinary matroids. This result was generalized in [BKL85] to basis complexes of 3–connected interval greedoids and in [Wen95] to even Δ–matroids. From this result also follows that the 2–dimensional faces of the basis matroid polyhedron are equilateral triangles or squares, i.e., the 2–skeleton of the basis matroid polyhedron is a simply connected subcomplex of the basis complex, namely, it comprises all triangles and a part of squares of this complex; cf. also [BGW97] (a *basis matroid polyhedron* [GGMS87] is the convex hull of the characteristic vectors of bases of a matroid). Moreover, Gelfand et al. [GGMS87] showed that the 1–skeleton of a basis matroid polyhedron coincides with the basis graph of the matroid.

Characterizing spaces by requiring they are simply connected and satisfy some local conditions is natural and appears often in the setting of a (very general) nonpositive curvature. In particular, in a simple but fundamental result, Gromov [Gro87] characterized the CAT(0) cubical complexes (i.e., cubical complexes with global nonpositive curvature) as simply connected cubical complexes in which the links of vertices are flag. Many similar characterizations concerning widely understood nonpositive curvature appeared — cf. e.g. [BCC⁺11] for an example and for further references. Such characterizations are very useful, since they allow to construct objects out of just local conditions: Having a space satisfying given local conditions, its universal cover (whose existence and uniqueness follows from a basic algebraic topology) is a simply connected space satisfying the same collection of local conditions. Note (compare also Corollaries 1&2 below) that constructing a complex satisfying our local conditions will finish after finitely many steps. Then either this complex or its finitely sheeted (universal) cover is the basis complex of a matroid. As a matter of fact, building the universal cover of a triangle-square complex with a prescribed local behavior is our way to prove Theorem 2 — see Theorem 5 in Section 4. Note however that our setting is opposite to the case of nonpositive curvature. Since basis graphs of matroids are finite (unlike universal covers of homotopically nontrivial spaces with nonpositive curvature), Theorem 2 implies immediately the following. (Note that the conditions in the statements below are local.)

Corollary 1. *Let G be a connected graph satisfying the interval and the local positioning conditions, and having at least one vertex with finitely many neighbors. Then the 1–skeleton of the universal cover $\widetilde{X(G)}$ of its triangle-square complex $X(G)$ is the basis graph of a matroid. In particular, $\widetilde{X(G)}$ is a finite complex.*

Corollary 2. *Let G be a connected graph satisfying the interval and the local positioning conditions, and having at least one vertex with finitely many neighbors. Then the fundamental group $\pi_1(X(G))$ of its triangle-square complex $X(G)$ is finite.*

Thus the collection of our local conditions may be treated as a kind of a positive curvature. Our characterization might be seen as an analogue of e.g. the classical result of Myers [Mye41] characterizing spheres by means of positive curvature. However there are not many similar results in a combinatorial settings, suggesting that there is possibly a wide field of research — parallel to the nonpositive curvature world.

Our construction can be used to obtain a similar characterization of basis graphs of even Δ–matroids, for which an analogue of Maurer's characterization is provided in [Che07] — see Theorem 6 in Section 5. This construction may also be useful to obtain similar characterizations in other cases.

Article's structure. In the next section, we define the local conditions employed in the formulation of Theorem 2 and we prove several auxiliary results. We also provide a slight enhancement of the original Maurer's characterization. Theorem 1 is proved in Section 3. Section 4 is devoted to the proof of Theorem 2. We conclude in Section 5 with some examples, in particular we analyze examples of non-basis graphs described in [DHT77], and we extend Theorem 2 to even Δ–matroids.

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2. Preliminaries

2.1. **Graphs.** All graphs $G = (V, E)$ occurring in this paper are undirected, connected, without loops or multiple edges, and not necessarily finite (unless stated otherwise). The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)–path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. If $d(u, v) = 2$, then we will call $I(u, v)$ a 2–*interval*. For two vertices u and v of a graph G, we will write $u \sim v$ if u and v are adjacent and $u \nsim v$, otherwise. Having vertices $u, v_1, v_2, \dots, v_k$, we will write $u \sim v_1, v_2, \dots, v_k$ (respectively, $u \nsim v_1, v_2, \dots, v_k$) if $u \sim v_i$ (respectively, $u \nsim v_i$), for every i. For a vertex v of a graph G and an integer $r \geq 1$, we will denote by $B_r(v, G)$ the *ball* in G (and the subgraph induced by this ball) of radius r centered at v, i.e., $B_r(v, G) = \{x \in V : d(v, x) \leq r\}$. As usual, $N(v) = B_1(v, G) \setminus \{v\}$ denotes the set of neighbors of a vertex v in G. The *link* of $v \in V(G)$ is the subgraph of G induced by $N(v)$.

A *wheel* W_k is a graph obtained by connecting a single vertex — the *central vertex* — to all vertices of the k–cycle; the *almost wheel* W_4^- is the graph obtained from W_4 by deleting a spoke (i.e., an edge between the central vertex and a vertex of the 4–cycle). A *pyramid* is the 4–wheel W_4. A *triangle* and a *square* of G are subgraphs of G which are induced 3– and 4–cycles. An *octahedron* is the 1–skeleton of the 3–dimensional octahedron, i.e., it is the complete graph K_6 minus a perfect matching. The following two graphs were

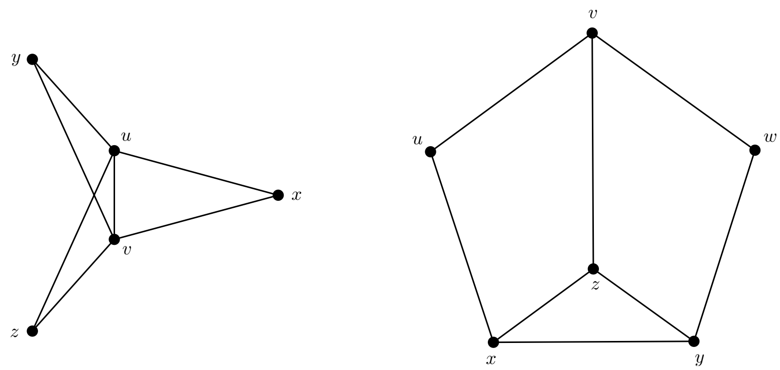

Figure 1. A propeller (on the left) and a half open book (on the right).

shown in [Mau73] to be forbidden (as induced subgraphs) in basis graphs of matroids. A *propeller* with *shaft* uv and *tips* x, y, z is the graph P defined by $V(P) = \{u, v, x, y, z\}$ and $E(P) = \{uv, ux, uy, uz, vx, vy, vz\}$ (see Figure 1, left). A *half open book* is the graph B defined by $V(B) = \{u, v, w, x, y, z\}$ and $E(B) = \{uv, ux, vw, vz, wy, xy, xz, yz\}$ (see Figure 1, right).

We continue with definitions of local and global conditions used in Maurer's and our characterizations of basis graphs. A graph G satisfies the *interval condition* if each 2–interval induces a square, a pyramid, or an octahedron. A graph G satisfies the *link condition at vertex* v, denoted LC(v) if the link of v in G is the line graph of a finite bipartite graph. A graph G satisfies the *link condition* if G satisfies LC(v) for all vertices v. Next, we introduce three global metric conditions with respect to a given basepoint v:

Triangle condition TC(v): for any two adjacent vertices u, w of G with $d(v, u) = d(v, w) = k \geq 2$ there exists $x \sim u, w$ such that $d(v, x) = k - 1$.

Square-pyramid condition SPC(v): for any three vertices u, w, w' of G with $u \sim w, w'$ and $2 = d(w, w') \leq d(v, u) = d(v, w') + 1 = d(v, w) + 1 = k + 1$, either there exists $x \sim w, w'$ such that $d(v, x) = k - 1$, or there exists $x \sim u, w, w'$ and $x' \sim u, w, w'$ such that $x \nsim x'$, and $d(x, v) = d(x', v) = k$.

Positioning condition PC(v): for each square $u_1u_2u_3u_4$ of G, we have $d(v, u_1) + d(v, u_3) = d(v, u_2) + d(v, u_4)$.

A graph G satisfies the *triangle*, the *square-pyramid*, or the *positioning conditions* if G satisfies TC(v), SPC(v), or PC(v), respectively, for all vertices v. A graph G satisfies the *local triangle condition* if for every v, u, w with $u \sim w$ and $d(v, u) = d(v, w) = 2$ there exists $x \sim v, u, w$. A graph G satisfies the *local positioning condition* if for each square $u_1u_2u_3u_4$ and each vertex v such that $d(v, u_1) = d(v, u_3) = 2$, we have $d(v, u_2) + d(v, u_4) = 4$.

Lemma 2.1. *If G satisfies the interval and the local positioning conditions, then G satisfies the local triangle condition, and G does not contain propellers and half open books as induced subgraphs.*

Proof. Consider three vertices u, v, w such that $u \sim w$ and $d(u, v) = d(w, v) = 2$. By the interval condition, there exist $x, x' \sim u, v$ such that $x \nsim x'$. If $w \nsim x, x'$, then $d(w, x) = d(w, x') = 2$ and $d(w, u) + d(w, v) = 3$, contradicting the local positioning condition. Consequently, either $x \sim u, v, w$ or $x' \sim u, v, w$ and thus G satisfies the local triangle condition.

Consider three triangles uvx, uvy, uvz, all three sharing the common edge uv. Suppose that $x \nsim y$ (see Figure 1, left). By the interval condition, there exists $t \sim x, y$ such that $t \sim u, t \nsim v$ or $t \sim v, t \nsim u$, say $t \sim u, t \nsim v$. Since $d(z, t) \leq 2$, $d(z, v) + d(z, t) \leq 3$. By the local positioning condition applied to the square $vxty$ and the basepoint z, we get that $d(x, z) + d(z, y) \leq 3$, and consequently, z is adjacent to x or y. Thus, G does not contain propellers.

Suppose now that G contains a half open book where xyz is a triangle and $uxzv$ and $vzyw$ are squares (see Figure 1, right). Then, considering the square $vwyz$, we have $d(u, w) = d(u, z) = 2$ and $d(u, v) + d(u, y) = 3$, contradicting the local positioning condition with respect to u. □

2.2. **Triangle-square complexes.** In this paper, we consider only triangle-square complexes, a particular class of 2–dimensional cell complexes. Although most of the notions presented below can be defined for all cell complexes and some of them for topological spaces, we will introduce them only for triangle-square complexes.

A *triangle-square complex* is a 2–dimensional cell complex X in which all 2–cells are triangles or squares. For a triangle-square complex X, denote by $V(X)$ and $E(X)$ the set of all 0–dimensional and 1–dimensional cells of X and call the pair $G(X) = (V(X), E(X))$ the 1*–skeleton* of X, or the *underlying graph.* Conversely, for a graph G one can derive a triangle-square complex $X(G)$ by taking all vertices of G as 0–cells, all edges of G as 1–cells, and all triangles and squares of G as 2–cells of $X(G)$. Then G is the 1–skeleton of $X(G)$. A triangle-square complex X is a *flag complex* if the triangular and the square cells of X are exactly the triangles and the squares of its 1–skeleton $G(X)$; a triangle-square flag complex X can therefore be recovered from its underlying graph $G(X)$. The *star* $\mathrm{St}(v, X)$ of a vertex v in a triangle-square complex X is the subcomplex consisting of the union of all cells in X containing v.

As morphisms between triangle-square complexes we consider all *cellular maps*, i.e., maps sending (linearly) cells to cells. An *isomorphism* is a bijective cellular map being a linear isomorphism (isometry) on each cell. A *covering (map)* of a cell complex X is a cellular surjection $p\colon \widetilde{X} \to X$ such that $p|_{\mathrm{St}(\widetilde{v}, \widetilde{X})}$ is an isomorphism onto its image for every vertex v in X; compare [Hat02, Section 1.3]. The space $\widetilde{X}$ is then called a *covering space*. A *universal cover* of X is a simply connected covering space $\widetilde{X}$. It is unique up to an isomorphism; cf. [Hat02, page 67]. In particular, if X is simply connected, then its universal cover is X itself. (Note that X is connected iff $G(X)$ is connected, and X is *simply connected* if every continuous map $S^1 \to X$ is null-homotopic).

The following lemma, that is important in the proof of Theorem 2 presented in Section 4, also provides an alternative proof of Maurer's Theorem 5.1 from [Mau73], establishing simple connectedness of basis complexes.

Lemma 2.2. *Let X be a triangle-square flag complex such that $G(X)$ satisfies the triangle and the square-pyramid conditions TC(v) and SPC(v), for some basepoint v. Then X is simply connected.*

Proof. A *loop* in X is a sequence $(w_1, w_2, \ldots, w_k, w_1)$ of vertices of X consecutively joined by edges in $G(X)$. To prove the lemma it is enough to show that every loop in X can be freely homotoped to a constant loop v. By contradiction, let A be the set of loops in $G(X)$, which are not freely homotopic to v, and assume that A is non-empty. For a loop $\alpha \in A$ let $r(\alpha)$ denote the maximal distance $d(w, v)$ of a vertex w of α from the basepoint v. Clearly $r(\alpha) \geq 2$ for any loop $\alpha \in A$ (otherwise α would be null-homotopic). Let $B \subseteq A$ be the set of loops α with minimal $r(\alpha)$ among loops in A. Let $r := r(\alpha)$ for some $\alpha \in B$. Let $C \subseteq B$ be the set of loops having minimal number e of edges in the r–sphere around v, i.e., with both endpoints at distance r from v. Further, let $D \subseteq C$ be the set of loops with the minimal number m of vertices at distance r from v.

Consider a loop $\alpha = (w_1, w_2, \ldots, w_k, w_1) \in D$. We can assume without loss of generality that $d(w_2, v) = r$. We treat separately the three following cases.

Case 1: $d(w_1, v) = r$ or $d(w_3, v) = r$. Assume without loss of generality that $d(w_1, v) = r$. Then, by the triangle condition $TC(v)$, there exists a vertex $w \sim w_1, w_2$ with $d(w, v) = r - 1$. Observe that the loop $\alpha' = (w_1, w, w_2, \ldots, w_k, w_1)$ belongs to B — in X it is freely homotopic to α by a homotopy going through the triangle ww_1w_2. The number of edges of α' lying on the r–sphere around v is less than e (we removed the edge w_1w_2). This contradicts the choice of the number e.

Case 2: $d(w_1, v) = d(w_3, v) = r - 1$ and $w_1 \sim w_3$. Then the loop $\alpha' = (w_1, w_3, \ldots, w_k, w_1)$ is homotopic to α via the triangle $w_1w_2w_3$. Thus α' belongs to C and the number of its vertices at distance r from v is $m - 1$. This contradicts the choice of the number m.

Case 3: $d(w_1, v) = d(w_3, v) = r - 1$ and $d(w_1, w_3) = 2$. By the square-pyramid condition $SPC(v)$, there exists a vertex $w \sim w_1, w_3$ with $d(w, v) \leq r - 1$. Again, the loop $\alpha' = (w_1, w, w_3, \ldots, w_k, w_1)$ is freely homotopic to α (via the square $w_1w_2w_3w$, or the triangles ww_1w_2 and ww_2w_3). Thus α' belongs to C and the number of its vertices at distance r from v is equal to $m - 1$. This contradicts the choice of the number m.

In all cases above we get a contradiction. It follows that the set A is empty and hence the lemma is proved. □

2.3. **A note on Maurer's characterizations.** Now we formulate the main characterizations of basis graphs presented in Theorems 2.1 and 3.1 of [Mau73]. Both these results were proved in [Mau73] for finite graphs. However, the analysis of the proof shows that one does not need to assume that the graphs are finite. Indeed, the result shows that if a graph G satisfies Maurer's conditions, then G is necessarily finite.

Theorem 3. [Mau73, Theorems 2.1&3.1] *For a graph G the following statements are equivalent:*

(i) G is the basis graph of a matroid;
(ii) G is connected, satisfies the interval and the positioning conditions, and some vertex of G satisfies the link condition (in particular, has finite degree);
(iii) G is connected, satisfies the interval condition, does not contain propellers and half-open books, and for some vertex v, the graph G satisfies the link condition $LC(v)$ (in particular, v has finite degree) and the positioning condition $PC(v)$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear and are showed in [Mau73]. For the implication $(iii) \Rightarrow (i)$, the main part of the proof of Theorems 2.1 and 3.1 of [Mau73] is to encode the vertices $x \in V$ of G with pairwise distinct, equicardinal sets (labels) S_x such that two vertices x and y are adjacent in G if and only if $|S_x \Delta S_y| = 2$. The only place where the finiteness assumption is used is the beginning of the encoding, namely, to find an encoding for the basepoint v and its neighbors. By the link condition, the link of v is the line graph of a finite bipartite graph $H = (B_0 \cup B, E)$. Then, set $S_v := B_0$ and for each vertex $x \in N(v)$, if x corresponds to the edge $b_0 b$ of H with $b_0 \in B_0$ and $b \in B$, then set $S_x := B_0 \setminus \{b_0\} \cup \{b\}$. The encoding is then propagated level-by-level to the whole graph G using the interval condition, the positioning condition PC(v), and the fact that G does not contain propellers and half-open books. Each vertex x of G is encoded with a subset S_x of $B_0 \cup B$ of size $|B_0|$. Since there exists only a finite number of such subsets, we conclude that G is finite and is the basis graph of a matroid. □

3. Proof of Theorem 1

In this section we prove Theorem 1, which establishes Conjecture 1 of Maurer [Mau73]. The proof is a direct consequence of Theorem 3 above and the following result.

Theorem 4. *Let G be a graph, and let v be a vertex adjacent to finitely many vertices in G. If G satisfies the interval condition and G does not contain propellers and half open books, then G satisfies the link condition $LC(v)$ at vertex v.*

Proof. Let G' be the link of v in G. It is well known that G' is the line graph of a bipartite graph if and only if G' does not contain induced claws $K_{1,3}$, diamonds $K_4 - e$, and odd induced cycles $C_{2k+1}, k \geq 2$.

If G' contains a claw, then this $K_{1,3}$ together with v induces in G a propeller. Analogously, if G' contains a diamond with vertices a, b, c, d such that $c \nsim d$, then in G the interval $I(c, d)$ contains a triangle abv, which is impossible by the interval condition. If G' contains an induced odd cycle, then this cycle together with v induces in G an odd wheel W_{2k+1} with $k \geq 2$, which is impossible by the next Proposition 3.1. □

Proposition 3.1. *If G satisfies the interval condition and G does not contain propellers and half open books, then G does not contain any odd wheel W_{2k+1}, $k \geq 2$.*

In the rest of this section, we will prove Proposition 3.1. Consider the smallest $k \geq 2$ such that G contains an induced odd wheel W_{2k+1}. Let c be the center of the wheel, and let $v_0, \ldots, v_{2k}$ be the vertices of the cycle of the wheel such that for every i, $v_i \sim v_{i+1}$ (here and in the rest of this section all additions are performed modulo $2k$).

Lemma 3.2. *For every i, j such that $v_i \nsim v_j$, there exists a unique $x_{i,j} \sim v_i, v_j$ such that $x_{i,j} \notin \{c, v_0, \ldots, v_{2k}\}$. Moreover, the following properties are satisfied:*

(1) $x_{i,j} \nsim c$;

(2) $x_{i,j} \sim v_k$ with $v_k \notin \{v_i, v_j\}$ if and only if $v_k \sim v_i, v_j$.

Proof. By symmetry, we can assume that $i = 1$ and $3 \leq j \leq k+1$. If $j = 3$, by the interval condition there exists $x \notin \{c, v_2\}$ such that $x \sim v_1, v_3$. If $j \geq 4$, by the interval condition between v_1 and v_j, there exists $x \sim v_1, v_j$ with $x \neq c$. In both cases, $x \notin \{v_0, \ldots, v_{2k}\}$. We first show that $x \nsim c$.

Claim 3.3. *$x \nsim c$.*

Proof. Suppose that there exists $x \sim v_1, v_j, c$. Let $x \sim v_m$ for some $m \notin \{1, j\}$. Consider the three triangles cxv_1, cxv_j, and cxv_m, all three sharing the common edge cx. Since G does not contain propellers, v_m is a neighbor of v_1 or v_j. Note that if $v_m \sim v_1, v_j$, then $m = 2$ and $j = 3$, but then the interval $I(v_1, v_3)$ contains a triangle cxv_2. Consequently, either $v_m \sim v_1$ and $v_m \nsim v_j$, or $v_m \sim v_j$ and $v_m \nsim v_1$.

Consider the triangles cv_1x, cv_1v_2, and cv_1v_0. Since G has no propellers, either $x \sim v_2$, or $x \sim v_0$. By the previous remark, x cannot be adjacent to both v_0 and v_2. Up to renaming the vertices, we can assume that $x \sim v_2$. For the same reasons, we can assume that $x \sim v_{j+1}$.

Consequently, $x \sim c, v_1, v_2, v_j, v_{j+1}$ and x is not adjacent to any other vertex of the wheel. Thus c and the cycle $v_2v_3 \ldots v_jx$ form the wheel W_j, while c and the cycle $v_{j+1} \ldots v_{2k}v_0v_1x$ form the wheel W_{2k+3-j}. Since j or $2k+3-j$ is odd and strictly smaller than $2k+1$, we get a contradiction with the choice of k, except if $j = 3$. In the latter case, the interval $I(v_1, v_3)$ contains a triangle cxv_2, a contradiction. This establishes Claim 3.3. □

Hence, if $x \sim v_1, v_j$, then $x \nsim c$. Then the interval condition for v_1 and v_j ensures that x is unique. Suppose now that $x \sim v_m$ for some $m \notin \{1, j\}$. By the interval condition between c and x, and since $v_1 \nsim v_j$, we get that $v_m \sim v_1, v_j$, i.e., $m = 2$ and $j = 3$ (since we assumed that $3 \leq j \leq k+1$). Conversely, assume that $v_m \sim v_1, v_j$, i.e., $m = 2$ and $j = 3$. Then c, x, v_2 belong to the interval $I(v_1, v_3)$ and, by the interval condition, $x \sim v_2$ since $x \nsim c$. This finishes the proof of the lemma. □

In the following, for any $v_i \nsim v_j$, let $x_{i,j}$ be the unique vertex $x_{i,j} \sim v_i, v_j$ such that $x_{i,j} \notin \{c, v_0, \ldots, v_{2k}\}$. From Lemma 3.2(2), we know that for every i, j, i', j' such that $\{i, j\} \neq \{i', j'\}$, we have $x_{i,j} \neq x_{i',j'}$.

Lemma 3.4. *For any v_i, v_j, v_m, we have $x_{i,j} \sim x_{i,m}$ if and only if $v_j \sim v_m$.*

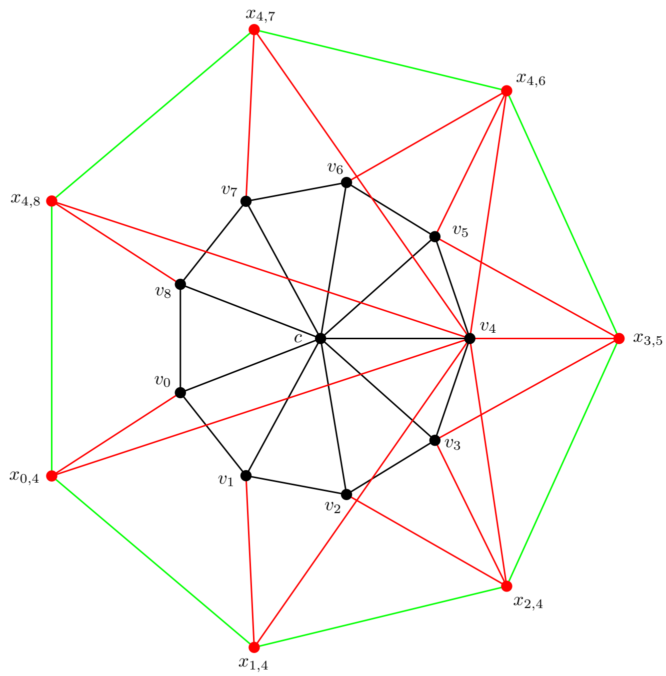

FIGURE 2. The wheel W_7 around v_4 obtained from W_9 by Lemma 3.7.

Proof. Note that since $x_{i,j}, x_{i,m}$ are defined, $v_i \nsim v_j, v_m$. Consider the subgraph of G induced by $c, v_i, v_j, v_m, x_{i,j}, x_{i,m}$. Observe that $cv_ix_{i,j}v_j$ and $cv_ix_{i,m}v_m$ are squares, since $v_i \nsim v_j, v_m$, and from Lemma 3.2 we have $c \nsim x_{i,j}, x_{i,m}$, $v_j \nsim x_{i,m}$, and $v_m \nsim x_{i,j}$. Since G does not contain half open books, $v_j \sim v_m$ if and only if $x_{i,j} \sim x_{i,m}$. □

Lemma 3.5. *For any v_i, v_j, $x_{i-1,i+1} \sim x_{i,j}$ if and only if $j = i-2$, or $j = i+2$.*

Proof. By symmetry, we can assume that $i = 1$ and $2 \le j \le k+1$. Note that since $x_{i,j}$ exists, $j \ge 3$. Recall that $x_{0,2} \sim v_0, v_1, v_2$. First assume that $j = 3$; recall that $x_{1,3} \sim v_1, v_2, v_3$. Consider the triangles v_1v_2c, $v_1v_2x_{0,2}$, $v_1v_2x_{1,3}$, all three sharing the common edge v_1v_2. Since G does not contain propellers and $c \nsim x_{0,2}, x_{1,3}$, we get that $x_{0,2} \sim x_{1,3}$.

Suppose now that there exists an index j such that $x_{1,j} \sim x_{0,2}$. Consider the triangles $v_1x_{0,2}v_0$, $v_1x_{0,2}v_2$, $v_1x_{0,2}x_{1,j}$, all three sharing the common edge $v_1x_{0,2}$. Since $v_0 \nsim v_2$ and G does not contain propellers, either $x_{1,j} \sim v_0$ or $x_{1,j} \sim v_2$. Since $3 \le j \le k+1$, by Lemma 3.2, the only possibility is $j = 3$. □

Lemma 3.6. *G does not contain any W_5, i.e., $k > 2$.*

Proof. Suppose by way of contradiction that $k = 2$. By Lemma 3.2, $v_2 \nsim x_{1,4}$. Consider the interval $I(v_2, x_{1,4})$. By Lemma 3.2, $v_2 \sim x_{0,2}, x_{1,3}, x_{2,4}$. Lemma 3.4 implies that $x_{1,4} \sim x_{2,4}, x_{1,3}$ and $x_{0,2} \sim x_{2,4}$. By Lemma 3.5, $x_{0,2} \sim x_{1,3}, x_{1,4}$ and $x_{1,3} \sim x_{2,4}$. Consequently, the pairwise adjacent vertices $x_{0,2}, x_{1,3}, x_{2,4}$ belong to the interval $I(v_2, x_{1,4})$, contrary to the interval condition. □

Lemma 3.7. *The vertices $x_{0,k}, x_{1,k}, \dots, x_{k-2,k}, x_{k-1,k+1}, x_{k,k+2}, x_{k,k+3} \cdots, x_{k,2k-1}, x_{k,2k}$ form an induced cycle C of length $2k-1$ of G such that v_k is adjacent to all vertices of C (see Figure 2).*

Proof. By Lemma 3.4, $x_{i,k} \sim x_{i+1,k}$ for every $0 \leq i \leq k-3$ and $k+2 \leq i \leq 2k$. By Lemma 3.5, $x_{k-1,k+1} \sim x_{k-2,k}, x_{k,k+2}$. Hence, $x_{0,k}x_{1,k}\dots x_{k-2,k}x_{k-1,k+1}x_{k,k+2}x_{k,k+3}\dots x_{k,2k-1}x_{k,2k}$ is a cycle C of G. By Lemma 3.4, $x_{i,k}$ is adjacent to $x_{j,k}$ if and only if $j \in \{i-1, i+1\}$; consequently, C does not contain chords of the form $x_{i,k}x_{j,k}$. Since, by Lemma 3.5, we have $x_{k-1,k+1} \nsim x_{k,j}$ when $j \notin \{k-2, k+2\}$, we conclude that C is an induced cycle of G. By the definition of $x_{i,k}$, we have $v_k \sim x_{i,k}$ for every i, and $v_k \sim x_{k-1,k+1}$, by Lemma 3.2 (2). □

By Lemma 3.7, we have constructed a wheel W_{2k-1}, contrary to our choice of k, except if $2k+1=5$; this latter case is impossible by Lemma 3.6. This finishes the proof of Proposition 3.1.

4. Proof of Theorem 2

In this section, we present the proof of Theorem 2 — the main result of our paper. Theorem 2 presents a topological characterization of basis complexes of matroids and shows that a general form of Conjecture 3 of [Mau73] is true. Note that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear and follows from [Mau73]. Thus in what follows we focus on proving the implication $(iii) \Rightarrow (i)$.

Consider a graph G that satisfies the interval and local positioning conditions, that has a vertex with finitely many neighbors, and such that its triangle-square complex $X(G)$ is simply connected. From the following result, G satisfies the positioning condition. Consequently, from Lemma 2.1, Theorem 4, and Theorem 3, the graph G is the basis graph of a matroid.

Theorem 5. *Let G be a connected graph satisfying the interval and the local positioning conditions. Then the 1–skeleton of the universal cover $\widetilde{X(G)}$ of the triangle-square complex $X(G)$ of G satisfies the interval and the positioning conditions.*

The rest of the current section is devoted to the proof of the above theorem. In the following, we consider a connected graph G that satisfies the interval and the local positioning conditions. By Lemma 2.1, G satisfies the local triangle condition and G does not contain propellers and half open books as induced subgraphs. We construct inductively the universal cover, simultaneously exhibiting its various properties.

Remark 4.1. Our proof follows closely (including much of notations) the proof of an analogous result from [BCC+11]. Note however that the overall setting is totally different — positive versus nonpositive curvature (see the introduction for more background). Thus, consequences of the two constructions (of the universal cover) are very different — finite versus infinite (see Corollaries 1 and 2). Moreover, as for technical details, the current proof is much more involved.

4.1. **Structure of the construction.** In this subsection we describe our inductive construction of the universal cover and we set the basis for the induction.

We construct the universal cover $\widetilde{X} := \widetilde{X(G)}$ of $X := X(G)$ as an increasing union $\bigcup_{i\geq 1} \widetilde{X}_i$ of triangle-square complexes. The complexes $\widetilde{X}_i$ are in fact spanned by concentric combinatorial balls $\widetilde{B}_i$ in $\widetilde{X}$. The covering map f is then the union $\bigcup_{i\geq 1} f_i$, where $f_i : \widetilde{X}_i \to X$ is a locally injective cellular map such that $f_i|_{\widetilde{X}_j} = f_j$, for every $j \leq i$. We denote by $\widetilde{G}_i = G(\widetilde{X}_i)$ the underlying graph of $\widetilde{X}_i$. We denote by $\widetilde{S}_i$ the set of vertices $\widetilde{B}_i \setminus \widetilde{B}_{i-1}$.

Pick any vertex v of X as the basepoint. Define $\widetilde{B}_0 = \{\widetilde{v}\} := \{v\}, \widetilde{B}_1 := B_1(v, G)$, and $f_1 := \mathrm{Id}_{B_1(v,G)}$. Let $\widetilde{X}_1$ be the triangle-square complex spanned by $B_1(v, G)$. Assume that, for $i \geq 1$, we have constructed the vertex sets $\widetilde{B}_1, \ldots, \widetilde{B}_i$, and we have defined the triangle-square complexes $\widetilde{X}_1, \ldots, \widetilde{X}_i$ and the corresponding cellular maps $f_1, \ldots, f_i$ from, respectively, $\widetilde{X}_1, \ldots, \widetilde{X}_i$ to X so that the graph $\widetilde{G}_i = G(\widetilde{X}_i)$ and the complex $\widetilde{X}_i$ satisfy the following conditions:

(P_i) $B_j(\widetilde{v}, \widetilde{G}_i) = \widetilde{B}_j$ for any $j \leq i$;

(Q_i) $\widetilde{G}_i$ satisfies the triangle and the square-pyramid conditions with respect to $\widetilde{v}$, i.e., $\mathrm{TC}(v)$ and $\mathrm{SPC}(v)$.

(R_i) for any $\widetilde{u} \in \widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraph of $\widetilde{G}_i$ induced by $B_1(\widetilde{u}, \widetilde{G}_i)$ and the subgraph of G induced by $B_1(f_i(\widetilde{u}), G)$;

(S_i) for any $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_{i-1}$ such that the vertices $w = f_i(\widetilde{w}), w' = f_i(\widetilde{w}')$ belong to a square $ww'uu'$ of X, there exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_i$ such that $f_i(\widetilde{u}) = u, f_i(\widetilde{u}') = u'$ and $\widetilde{w}\widetilde{w}'\widetilde{u}\widetilde{u}'$ is a square of $\widetilde{X}_i$.

(T_i) for any $\widetilde{w} \in \widetilde{S}_i := \widetilde{B}_i \setminus \widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraphs of $\widetilde{G}_i$ and of G induced by $B_1(\widetilde{w}, \widetilde{G}_i)$ and $f_i(B_1(\widetilde{w}, \widetilde{G}_i))$.

(U_i) $\widetilde{G}_i$ satisfies the positioning condition with respect to $\widetilde{v}$.

It can be easily checked that, $\widetilde{B}_1, \widetilde{G}_1, \widetilde{X}_1$ and f_1 satisfy the conditions (P_1) through (U_1). Now we construct the set $\widetilde{B}_{i+1}$, the graph $\widetilde{G}_{i+1}$ having $\widetilde{B}_{i+1}$ as the vertex-set, the triangle-square complex $\widetilde{X}_{i+1}$ having $\widetilde{G}_{i+1}$ as its 1–skeleton, and the map $f_{i+1} : \widetilde{X}_{i+1} \to X$. Let

$$Z = \{(\widetilde{w}, z) : \widetilde{w} \in \widetilde{S}_i \text{ and } z \in B_1(f_i(\widetilde{w}), G) \setminus f_i(B_1(\widetilde{w}, \widetilde{G}_i))\}.$$

On Z we define a binary relation $\equiv$ by setting $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ if and only if $z = z'$ and one of the following three conditions is satisfied:

(Z1) $\widetilde{w}$ and $\widetilde{w}'$ are the same or adjacent in $\widetilde{G}_i$;

(Z2) there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}$ and $\widetilde{w}'$ and such that $f_i(\widetilde{u})f_i(\widetilde{w})zf_i(\widetilde{w}')$ is a square in G;

(Z3) there exists a square in $\widetilde{S}_i$ containing $\widetilde{w}$ and $\widetilde{w}'$ such that its image under f_i together with z induces a pyramid in G.

In what follows, the above relation will be used in the inductive step to construct $\widetilde{X}_{i+1}$, f_{i+1}, and all related objects.

4.2. **Definition of $\widetilde{G}_{i+1}$.** In this subsection, performing the inductive step, we define $\widetilde{G}_{i+1}$ and f_{i+1}. First however we show that the relation $\equiv$ defined in the previous subsection is an equivalence relation. The set of vertices of the graph $\widetilde{G}_{i+1}$ will be then defined as the union of the set of vertices of the previously constructed graph $\widetilde{G}_i$ and the set of equivalence classes of $\equiv$.

Convention: In what follows, for any vertex $\widetilde{w} \in \widetilde{B}_i$, we will denote by w its image $f_i(\widetilde{w})$ in X.

We now aim at showing that the relation $\equiv$ is an equivalence relation (Proposition 4.4). First we prove two auxiliary results.

Lemma 4.2. *For any couple $(\widetilde{w}, z) \in Z$ the following properties hold:*
(A_1) there is no neighbor $\widetilde{z} \in \widetilde{B}_i$ of $\widetilde{w}$ such that $f_i(\widetilde{z}) = z$;
(A_2) there is no neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ of $\widetilde{w}$ such that $u \sim z$;
(A_3) there are no $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$.

Proof. If $\widetilde{w}$ has a neighbor $\widetilde{z} \in \widetilde{B}_{i-1}$ such that $f_i(\widetilde{z}) = z$, then $(\widetilde{w}, z) \notin Z$, a contradiction. This establishes (A_1).

If $\widetilde{w}$ has a neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $u \sim z$, then by (R_i) applied to $\widetilde{u}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{u}, \widetilde{w}$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, establishing (A_2).

If there exist $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$, then $yxwz$ is a square in G. From (S_i) applied to $\widetilde{y}, \widetilde{x}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{y}, \widetilde{w}$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, and therefore (A_3) holds as well. □

Lemma 4.3. *Let $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i-1}$ and $\widetilde{w}, \widetilde{w}', \widetilde{w}'' \in \widetilde{S}_i$ be such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}'$ and $\widetilde{u}' \sim \widetilde{w}', \widetilde{w}''$. If $\widetilde{w} \sim \widetilde{w}'$, then there exist $\widetilde{y} \in \widetilde{B}_{i-1}$ and $\widetilde{x} \in \widetilde{B}_{i-2}$ such that $\widetilde{y} \sim \widetilde{w}, \widetilde{w}'$ and $\widetilde{x} \sim \widetilde{u}', \widetilde{y}$.*

Proof. If there exists $\widetilde{x} \in \widetilde{B}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$, we are done by setting $\widetilde{y} = \widetilde{u}$. Assume in the following that it is not the case. By the square-pyramid condition (Q_i), there exist $\widetilde{y}, \widetilde{y}' \in \widetilde{B}_{i-1}$ such that $\widetilde{w}' \sim \widetilde{y}, \widetilde{y}'$ and $\widetilde{u}, \widetilde{y}, \widetilde{u}', \widetilde{y}'$ is a square. By (T_i) applied to $\widetilde{w}'$, $uyu'y'$ is a square. Consider the triangles $uw'y, uw'y'$, and $uw'w$, all three sharing the common edge uw'. Since G does not contain propellers (cf. Lemma 2.1), either $w \sim y$ or $w \sim y'$, say $w \sim y$. By (R_i) applied to $\widetilde{u}$, we get $\widetilde{w} \sim \widetilde{y}$. Using the triangle condition (Q_i), we get a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ such that $\widetilde{x} \sim \widetilde{y}, \widetilde{u}'$. □

Proposition 4.4. *The relation $\equiv$ is an equivalence relation on Z.*

Proof. Since the binary relation $\equiv$ is reflexive and symmetric, it suffices to show that $\equiv$ is transitive. Let $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$. We will prove that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z'')$. By the definition of $\equiv$, we conclude that $z = z' = z''$. By the definition of $\equiv$, we have $z \sim w, w', w''$.

If $\widetilde{w} \sim \widetilde{w}''$ (in $\widetilde{G}_i$), then by the definition of $\equiv$, we have $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ and we are done. If $\widetilde{w} \nsim \widetilde{w}''$ and if there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$, then by (R_i) applied to $\widetilde{u}$, we obtain that $u \sim w, w''$ and $w \nsim w''$. Since $(\widetilde{w}, z), (\widetilde{w}'', z) \in Z$, we have $z \sim w, w''$. By (A_2) (cf. Lemma 4.2) we have that $z \nsim u$. Thus $uwzw''$ is a square in G, and by condition (Z2), we are done. Therefore, in the rest of the proof, we will assume the following:

(A_4) $\widetilde{w} \nsim \widetilde{w}''$;

(A_5) there is no $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$.

Observe that it implies in particular that $i \geq 2$.

Claim 4.5. *Let $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i-1}$ be two vertices with $\widetilde{u} \sim \widetilde{w}, \widetilde{w}'$ and $\widetilde{u}' \sim \widetilde{w}', \widetilde{w}''$. If $\widetilde{x} \in \widetilde{B}_{i-2}$ is adjacent to both $\widetilde{u}$ and $\widetilde{u}'$, then $d(x,w) = d(x,w') = d(x,w'') = 2$ and $d(x,z) = 3$.*

Proof. By the condition (R_i) applied to $\widetilde{u}$ (respectively, $\widetilde{u}'$) we have that $d(x,w) = d(x,w') = 2$ (respectively, $d(x,w'') = d(x,w') = 2$). We show now that $d(x,z) = 3$. By (A_2) we have that $x \neq z$, and by (A_3) we have that $x \nsim z$. Assume that $d(x,z) = 2$. By the local triangle condition, there exists a vertex $x' \sim z, w, x$. If $x' \sim u$ then, by (R_i), there exists a vertex $\widetilde{x}' \in \widetilde{B}_{i-1}$ such that $\widetilde{x}' \sim \widetilde{w}, \widetilde{u}, \widetilde{x}$ and $f_i(\widetilde{x}') = x'$. This however contradicts (A_2). If $x' \nsim u$, then consider the square $x'wux$. By (S_i) applied to vertices x, u, there exists a square $\widetilde{x}'\widetilde{w}_0\widetilde{u}\widetilde{x}$ in $\widetilde{X}_i$ with $f_i(\widetilde{x}') = x'$ and $f_i(\widetilde{w}_0) = w_0$. By ($\mathrm{R}_i$) applied to $\widetilde{u}$, we have that $\widetilde{w}_0 = \widetilde{w}$. Again we obtain $\widetilde{x}' \sim \widetilde{w}, \widetilde{x}$ and $x' = f_i(\widetilde{x}') \sim z$, which contradicts ($A_2$). In any case we get a contradiction, thus we must have $d(x,z) = 3$. □

We distinguish six cases depending on which of the conditions (Z1), (Z2), or (Z3) are satisfied by the pairs $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$.

Case (Z1)(Z1): $\widetilde{w}'$ is adjacent in $\widetilde{G}_i$ to both $\widetilde{w}$ and $\widetilde{w}''$.

By (T_i), we have that $w \neq w''$ and $w \nsim w''$. By (Q_i), the graph $\widetilde{G}_i$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$, thus there exist two vertices $\widetilde{u}, \widetilde{u}' \in \widetilde{S}_{i-1}$ such that $\widetilde{u}$ is adjacent to $\widetilde{w}, \widetilde{w}'$ and $\widetilde{u}'$ is adjacent to $\widetilde{w}', \widetilde{w}''$. By ($A_5$), we have $\widetilde{u} \nsim \widetilde{w}''$, $\widetilde{u}' \nsim \widetilde{w}$, in particular $\widetilde{u} \neq \widetilde{u}'$. By ($\mathrm{R}_i$) applied to $\widetilde{w}'$, it implies that $u \nsim w''$ and $u' \nsim w$.

By Lemma 4.3, we can assume that there exists a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ adjacent to both $\widetilde{u}$ and $\widetilde{u}'$. By Claim 4.5, we have $d(x,w) = d(x,w') = d(x,w'') = 2$ and $d(x,z) = 3$. By the interval condition applied to $I(w,w'')$, either there exists a vertex $u_0 \sim w, w', w''$ with $u_0 \nsim z$, or there exists a vertex $w''' \sim z, w, w''$ with $w''' \nsim w'$. In the first case, by the local positioning condition we have that $u_0 \sim x$. Observe that $u_0 \sim u$ (respectively, $u_0 \sim u'$), since otherwise x, w, w' (respectively, x, w', w'') belong to the interval $I(u, u_0)$ (respectively, to $I(u', u_0)$), and $x \nsim w, w'$ (respectively, $x \nsim w, w'$), contradicting the interval condition. By (R_i) applied to $\widetilde{u}$, there is a vertex $\widetilde{u}_0 \sim \widetilde{w}, \widetilde{w}', \widetilde{u}, \widetilde{x}$ with $f_i(\widetilde{u}_0) = u_0$. By ($\mathrm{R}_i$) applied to $\widetilde{x}$, $\widetilde{u}_0 \sim \widetilde{u}'$ and by (R_i) applied to $\widetilde{u}'$, $\widetilde{u}_0 \sim \widetilde{w}''$. This contradicts ($A_5$).

Thus, there exists a vertex $w''' \sim z, w, w''$ with $w''' \nsim w'$. By the local positioning condition, we have $d(x, w''') = 2$. By the local triangle condition applied to the edge $w''w'''$, there is a vertex $u'' \sim w'', w''', x$. Since $u \nsim w''$, $u \neq u''$. If $u'' = u'$, then by the interval condition applied to $I(w', w''')$, we have $u'' \sim w$. Thus, by (T_i) applied to $\widetilde{w}'$, we get $\widetilde{u}' \sim \widetilde{w}$, contradicting ($A_5$). If $u'' \sim u'$, by (R_i) applied to $\widetilde{u}'$ there is a vertex $\widetilde{u}'' \in \widetilde{B}_i$ with $\widetilde{u}'' \sim \widetilde{x}, \widetilde{u}', \widetilde{w}''$, and $f_i(\widetilde{u}'') = u''$. If $u'' \nsim u'$, by (S_i) applied to the square $xu'w''u''$ and to the vertices $\widetilde{u}', \widetilde{x}$, there is a vertex $\widetilde{u}'' \in \widetilde{B}_i$ with $\widetilde{u}'' \sim \widetilde{x}, \widetilde{w}''$ and $f_i(\widetilde{u}'') = u''$. In both cases, applying (R_i) to $\widetilde{u}''$ we get a vertex $\widetilde{w}''' \sim \widetilde{w}'', \widetilde{u}''$ such that $f_i(\widetilde{w}''') = w'''$ and $d(\widetilde{x}, \widetilde{w}''') = 2$.

Proceeding analogically for the edge ww''' (instead of $w''w'''$) we obtain a vertex $\widetilde{u}''' \sim \widetilde{w}, \widetilde{x}$ such that $u''' = f_i(\widetilde{u}''') \sim w, w''', x$. If $u'' \sim u'''$, by (R$_i$) applied to $\widetilde{x}$, we have $\widetilde{u}'' \sim \widetilde{u}'''$, and by (R$_i$) applied to $\widetilde{u}''$ we get that $\widetilde{u}''' \sim \widetilde{w}'''$. If $u'' \nsim u'''$, then again we obtain that $\widetilde{u}''' \sim \widetilde{w}'''$, by using (S$_i$) for the square $xu''w'''u'''$ and (R$_i$) for $\widetilde{x}$ and $\widetilde{u}''$. By (R$_i$) applied to $\widetilde{u}'''$, we have $\widetilde{w} \sim \widetilde{w}'''$, and by (T$_i$) applied to $\widetilde{w}$, we have $\widetilde{w}''' \nsim \widetilde{w}'$. Consequently, the condition (Z3) holds for $\widetilde{w}, \widetilde{w}''$ and the pyramid $zww'w''w'''$. Hence, $(\widetilde{w}, z) \equiv (\widetilde{w}'', z'')$. This finishes the proof in Case (Z1)(Z1).

Case (Z1)(Z2): $\widetilde{w}$ is adjacent in $\widetilde{G}_i$ to $\widetilde{w}'$, and there exists $\widetilde{u}' \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}'$ and $\widetilde{w}''$ and such that $w'u'w''z$ is a square in G.

By (R$_i$), we have $w \neq w''$. By (Q$_i$), there exists a vertex $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}'$. By ($A_5$), we have $\widetilde{u} \nsim \widetilde{w}''$, $\widetilde{u}' \nsim \widetilde{w}$, in particular, $\widetilde{u} \neq \widetilde{u}'$.

By Lemma 4.3, we can assume that there exists a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ adjacent to both $\widetilde{u}$ and $\widetilde{u}'$. By Claim 4.5, we have $d(x,w) = d(x,w') = d(x,w'') = 2$ and $d(x,z) = 3$. Note that $w \nsim w''$, otherwise the square $ww'u'w''$ would falsify the local positioning condition for x. By the local triangle condition there is a vertex $y \sim w'', u', w$.

Assume first that $y \sim z$. Then $d(x,y) = 2$, and consequently $y \sim w'$ (otherwise the square $yww'u'$ falsifies the local positioning condition for x). The condition (R$_i$) applied to $\widetilde{u}'$ shows that there exists a vertex $\widetilde{y} \sim \widetilde{u}', \widetilde{w}', \widetilde{w}''$ with $f_i(\widetilde{y}) = y$. By (A_2), we have $\widetilde{y} \in \widetilde{S}_i$, and by (T$_i$) applied to $\widetilde{w}'$, we have that $\widetilde{y} \sim \widetilde{w}$. Hence the situation is the same as in Case (Z1)(Z1), with $\widetilde{y}$ playing the role of $\widetilde{w}'$.

Assume now that $y \nsim z$. By the local positioning condition applied to the square $wyw''z$ and the basepoint x, we have that $y \sim x$. By (R$_i$) applied to $\widetilde{u}'$, there is a vertex $\widetilde{y} \sim \widetilde{x}, \widetilde{w}'', \widetilde{u}'$ in $\widetilde{S}_{i-1}$ with $f_i(\widetilde{y}) = y$. If $y \sim u$, by (R$_i$) (applied to $\widetilde{x}$ and then to $\widetilde{u}$), we have that $\widetilde{y} \sim \widetilde{w}$. If $y \nsim u$, by (S$_i$) applied to the square $xywu$ and to the vertices $\widetilde{x}, \widetilde{u}$, we also get $\widetilde{y} \sim \widetilde{w}$. Since, $\widetilde{y} \in \widetilde{S}_{i-1}$, applying (Z2) to the square $wzw''y$, we obtain that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z'')$. This finishes the proof in Case (Z1)(Z2).

Case (Z1)(Z3): $\widetilde{w}$ is adjacent in $\widetilde{G}_i$ to $\widetilde{w}'$ and there exist $\widetilde{u}', \widetilde{u}'' \in \widetilde{S}_i$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}', \widetilde{w}''$ and such that the vertices u', u'', w', w'', z induce a pyramid in G.

By (T$_i$), $w \neq u', w \neq u''$ and $w \neq w''$. By the no-propeller Lemma 2.1 applied to the triangles $w'zw, w'zu'$, and $w'zu''$, either $w \sim u'$ or $w \sim u''$, say $w \sim u'$. By the condition (T$_i$), $\widetilde{w} \sim \widetilde{u}'$. Then replacing $\widetilde{w}'$ by $\widetilde{u}'$, since $(\widetilde{w}, z) \equiv (\widetilde{u}', z), (\widetilde{u}', z) \equiv (\widetilde{w}'', z)$ and $\widetilde{w} \sim \widetilde{u}' \sim \widetilde{w}''$, we are in conditions of Case (Z1)(Z1), thus $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ and we are done.

Case (Z2)(Z2): There exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}$ and $\widetilde{w}'$ and there exists $\widetilde{u}' \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}'$ and $\widetilde{w}''$ such that $wuw'z$ and $w'u'w''z$ are squares in G.

By (A_5), $\widetilde{u} \neq \widetilde{u}'$ and $\widetilde{u} \nsim \widetilde{w}'', \widetilde{u}' \nsim \widetilde{w}$. By (T$_i$), we have $u \neq u'$ and $w \nsim w'$, $w' \nsim w''$. If $w = w''$, then u, u', z belong to the interval $I(w, w')$ and consequently, $z \sim u$ or $z \sim u'$, contradicting (A_2). If $u \sim w''$, then w, w', w'' belong to the interval $I(u, z)$; consequently, either $w' \sim w$ or $w' \sim w''$, a contradiction. For the same reasons, $u' \nsim w$.

If $u \sim u'$, since G does not contain half open books (Lemma 2.1), the previous constraints imply that $w \sim w''$. Then (S_i) applied to the square $wuu'w''$ implies that $\widetilde{w} \sim \widetilde{w}''$ and we are done because $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ by (Z1). So, further we will suppose that $u \nsim u'$. By (T_i), $\widetilde{u} \nsim \widetilde{u}'$. Below, we consider separately two cases, (i) and (ii).

(i) There exists a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ adjacent to both $\widetilde{u}$ and $\widetilde{u}'$.

By Claim 4.5, we have $d(x, w) = d(x, w') = d(x, w'') = 2$ and $d(x, z) = 3$. By the interval condition, the vertices w and w'' belong to a square of G. If this square contains z, then the fourth vertex of this square, denote it y, will be adjacent to x by the local positioning condition. By (R_i) applied to $\widetilde{x}$, there exists a vertex $\widetilde{y} \in \widetilde{B}_{i-1}$ with $\widetilde{y} \sim \widetilde{x}$ and $f_i(\widetilde{y}) = y$. If $y \sim u$, then by (R_i) applied to $\widetilde{x}$ and then to $\widetilde{u}$ we obtain that $\widetilde{u} \sim \widetilde{y}$ and $\widetilde{y} \sim \widetilde{w}$. On the other hand, if $u \nsim y$, then (S_i) applied to the square $xywu$ also implies that $\widetilde{y} \sim \widetilde{w}$. Analogously, we can conclude that $\widetilde{y} \sim \widetilde{w}''$. By (Z2) applied to the square $wyw''z$, we deduce that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$, as required.

Now suppose that any square of G containing w and w'' has the form $wyw''y'$, where $y, y' \neq z$ and $z \sim y, y'$. By the local positioning condition and since $d(x, z) = 3$, we have $d(x, y) = d(x, y') = 2$. By the local triangle condition applied to vertices w, y, x (respectively, w'', y, x) there exists a vertex $s \sim w, y, x$ (respectively, $s' \sim w'', y, x$). By (R_i) applied to $\widetilde{x}$, there exist vertices $\widetilde{s}, \widetilde{s}' \in \widetilde{B}_i$ adjacent to $\widetilde{x}$ and such that $f_i(\widetilde{s}) = s, f_i(\widetilde{s}') = s'$. By (R_i) applied to $\widetilde{x}$ and then to $\widetilde{u}$ (if $s \sim u$) or by (S_i) (applied to the square $uxsw$, if $s \nsim u$), we have $\widetilde{s} \sim \widetilde{w}$. Similarly, $\widetilde{s}' \sim \widetilde{w}''$. Again, by (R_i) applied to $\widetilde{s}$, there is a vertex $\widetilde{y} \sim \widetilde{s}, \widetilde{w}$ with $f_i(\widetilde{y}) = y$. By (R_i) applied to $\widetilde{s}$ (if $s \sim s'$) or by (S_i) (applied to the square $ysxs'$ if $s \nsim s'$), we have that $\widetilde{y} \sim \widetilde{s}'$. Then, by (R_i) applied to $\widetilde{s}'$, we obtain that $\widetilde{y} \sim \widetilde{w}''$. Analogously, we show that there is a vertex $\widetilde{y}' \sim \widetilde{w}, \widetilde{w}''$ with $f_i(\widetilde{y}') = y'$. Since $z \sim y, y'$, by (A_2) we have that $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_i$. As a consequence, $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ by (Z3).

(ii) There is no vertex in $\widetilde{B}_{i-2}$ adjacent to both $\widetilde{u}$ and $\widetilde{u}'$.

By the square-pyramid condition (Q_i), there exist two distinct vertices $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_{i-1}$ with $f_i(\widetilde{y}) = y, f_i(\widetilde{y}') = y'$, such that $\widetilde{w}' \sim \widetilde{y}, \widetilde{y}'$ and $\widetilde{u}\widetilde{y}\widetilde{u}'\widetilde{y}'$ is a square. By (R_i), the vertices y, y' are both adjacent to u, u', w', and $y \nsim y'$. By the triangle condition (Q_i) there is a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ adjacent to $\widetilde{y}$ and $\widetilde{u}'$. If $\widetilde{y} \sim \widetilde{w}$, then replacing $\widetilde{u}$ by $\widetilde{y}$ and applying case (i), we are done.

Suppose now that $\widetilde{y} \nsim \widetilde{w}$. By (A_2), we have $z \nsim y, y'$. By the local triangle condition, in G there exists a common neighbor w''' of y, w, and z. If $w''' \sim u$, by (R_i) applied to $\widetilde{u}$, there exists a vertex $\widetilde{w}''' \sim \widetilde{u}, \widetilde{w}, \widetilde{y}$ with $f_i(\widetilde{w}''') = w'''$. If $w''' \nsim u$, by (S_i) applied to the square $wuyw'''$, there exists $\widetilde{w}''' \sim \widetilde{w}, \widetilde{y}$ with $f_i(\widetilde{w}''') = w'''$. Since $w''' \sim z$, by (A_2), we have that $\widetilde{w}''' \in \widetilde{S}_i$, and by (T_i) and (A_1), we have $(\widetilde{w}''', z) \in Z$. Thus, if $\widetilde{w}''' \nsim \widetilde{w}'$, by the preceding case (i) (since, by the triangle condition, there is a vertex $\widetilde{x} \in \widetilde{B}_{i-2}$ adjacent to $\widetilde{y}$ and $\widetilde{u}'$) we have that $(\widetilde{w}''', z) \equiv (\widetilde{w}'', z)$. If $\widetilde{w}''' \sim \widetilde{w}'$, we get the same conclusion by applying Case (Z1)(Z2). Since $\widetilde{w} \sim \widetilde{w}'''$, we have $(\widetilde{w}, z) \equiv (\widetilde{w}''', z)$ by condition (Z1). Hence, we obtain that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ by applying one of the cases (Z1)(Z1), (Z1)(Z2), or (Z1)(Z3) to the triplet $(\widetilde{w}, z), (\widetilde{w}''', z), (\widetilde{w}'', z)$. This finishes the proof in Case (Z2)(Z2).

Case (Z2)(Z3): There exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}, \widetilde{w}'$, such that $wuw'z$ is a square, and there exist $\widetilde{u}', \widetilde{u}'' \in \widetilde{S}_i$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}', \widetilde{w}''$ and such that the vertices u', u'', w', w'', z induce a pyramid in G.

By (T_i), $u \notin \{u', u'', w''\}$ and, by (R_i) $w \notin \{u', u''\}$. If $w = w''$, then by the interval condition for $I(w, w')$, we have $u \sim u', u''$. Consequently, by (T_i) applied to $\widetilde{w}'$ and to $\widetilde{u}'$, $\widetilde{u} \sim \widetilde{u}'$ and $\widetilde{u} \sim \widetilde{w}''$, which contradicts ($A_5$). Thus $w \neq w''$. Moreover, $w \nsim w'$, $w' \nsim w''$, $u' \nsim u''$, and $u \nsim z$. Hence, by (T_i) and (A_1), we have $(\widetilde{u}', z) \in Z$. Applying Case (Z1)(Z2) to the triplet $(\widetilde{u}', z), (\widetilde{w}', z), (\widetilde{w}, z)$, we obtain that $(\widetilde{w}, z) \equiv (\widetilde{u}', z)$. By applying one of the cases (Z1)(Z1), (Z1)(Z2), or (Z1)(Z3) to the triplet $(\widetilde{w}'', z), (\widetilde{u}', z), (\widetilde{w}, z)$, we conclude that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$. This finishes the proof in Case (Z2)(Z3).

Case (Z3)(Z3): There exist $\widetilde{y}', \widetilde{y}'' \in \widetilde{S}_i$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}, \widetilde{w}'$ and such that the vertices y', y'', w, w', z induce a pyramid in G, and there exist $\widetilde{u}', \widetilde{u}'' \in \widetilde{S}_i$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}', \widetilde{w}''$ and such that the vertices u', u'', w', w'', z induce a pyramid in G.

Again, we first notice that $w \neq w', w' \neq w''$, $w \nsim w'$, and $w' \nsim w''$. Similarly $y' \neq y'', u' \neq u''$, $y' \nsim y''$, and $u' \nsim u''$. By (T_i) and (A_1), we have $(\widetilde{u}, z), (\widetilde{u}', z), (\widetilde{y}, z), (\widetilde{y}', z) \in Z$. First suppose that one of the vertices y', y'' coincides with one of the vertices u', u'', say $y' = u'$. Then, by (T_i) applied to $\widetilde{w}'$, we have $\widetilde{y}' = \widetilde{u}'$ and $(\widetilde{w}, z) \equiv (\widetilde{u}', z)$ and $(\widetilde{u}', z) \equiv (\widetilde{w}'', z)$ by (Z1). Consequently, Case (Z1)(Z1) implies that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$. Thus suppose that the vertices y', y'' and u', u'' are pairwise distinct. By Case (Z1)(Z3) applied to the triplet $(\widetilde{u}', z), (\widetilde{w}', z), (\widetilde{w}, z)$ we deduce that $(\widetilde{w}, z) \equiv (\widetilde{u}', z)$. Then applying one of the cases (Z1)(Z1), (Z1)(Z2), or (Z1)(Z3) to the triplet $(\widetilde{w}'', z), (\widetilde{u}', z), (\widetilde{w}, z)$, we obtain that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$. This finishes the proof of the Case (Z3)(Z3) and completes the proof that $\equiv$ is an equivalence relation on Z. □

Let $\widetilde{S}_{i+1}$ denote the set of equivalence classes of $\equiv$, i.e., $\widetilde{S}_{i+1} = Z/_{\equiv}$. For a couple $(\widetilde{w}, z) \in Z$, we will denote by $[\widetilde{w}, z]$ the equivalence class of $\equiv$ containing $(\widetilde{w}, z)$. Set $\widetilde{B}_{i+1} := \widetilde{B}_i \cup \widetilde{S}_{i+1}$. Let $\widetilde{G}_{i+1}$ be the graph having $\widetilde{B}_{i+1}$ as the vertex set in which two vertices $\widetilde{a}, \widetilde{b}$ are adjacent if and only if one of the following conditions holds:

(1) $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$ and $\widetilde{a}\widetilde{b}$ is an edge of $\widetilde{G}_i$,
(2) $\widetilde{a} \in \widetilde{B}_i$, $\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{b} = [\widetilde{a}, z]$,
(3) $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$, $\widetilde{a} = [\widetilde{w}, z]$, $\widetilde{b} = [\widetilde{w}, z']$ for a vertex $\widetilde{w} \in \widetilde{B}_i$, and $z \sim z'$ in the graph G.

Finally, we define the map $f_{i+1} : \widetilde{B}_{i+1} \to V(G)$ in the following way: if $\widetilde{a} \in \widetilde{B}_i$, then $f_{i+1}(\widetilde{a}) = f_i(\widetilde{a})$, otherwise, if $\widetilde{a} \in \widetilde{S}_{i+1}$ and $\widetilde{a} = [\widetilde{w}, z]$, then $f_{i+1}(\widetilde{a}) = z$. Notice that f_{i+1} is well-defined because all couples from the equivalence class representing $\widetilde{a}$ have one and the same vertex z in the second argument. In the sequel we follow our earlier convention for notations: all vertices of $\widetilde{B}_{i+1}$ will be denoted with the tilde and their images in G under f_{i+1} will be denoted without a tilde, e.g. if $\widetilde{w} \in \widetilde{B}_{i+1}$, then $w = f_{i+1}(\widetilde{w})$.

4.3. **Properties of $\widetilde{G}_{i+1}$ and f_{i+1}.** In this subsection we check our inductive assumptions, verifying the properties (P_{i+1}) through (U_{i+1}) for $\widetilde{G}_{i+1}$ and f_{i+1} defined above. In particular it allows us to define the corresponding complex $\widetilde{X}_{i+1}$.

Lemma 4.6 (Property (P_{i+1})). *The graph $\widetilde{G}_{i+1}$ satisfies the property (P_{i+1}), i.e., $B_j(v,\widetilde{G}_{i+1}) = \widetilde{B}_j$ for any $j \le i+1$.*

Proof. By the definition of edges of $\widetilde{G}_{i+1}$, any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is adjacent to at least one vertex of $\widetilde{B}_i$ and all such neighbors of $\widetilde{b}$ are vertices of the form $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, z]$ for a couple $(\widetilde{w}, z)$ of Z. By the definition of Z, $\widetilde{w} \in \widetilde{S}_i$, whence any vertex of $\widetilde{S}_{i+1}$ is adjacent only to vertices of $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$. Therefore, the distance between the basepoint $\widetilde{v}$ and any vertex $\widetilde{a} \in \widetilde{B}_i$ is the same in the graphs $\widetilde{G}_i$ and $\widetilde{G}_{i+1}$. On the other hand, the distance in $\widetilde{G}_{i+1}$ between $\widetilde{v}$ and any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is $i+1$. This shows that indeed $B_j(v,\widetilde{G}_{i+1}) = \widetilde{B}_j$ for any $j \le i+1$. □

Lemma 4.7 (Property (Q_{i+1})). *The graph $\widetilde{G}_{i+1}$ satisfies the property (Q_{i+1}), i.e. $\widetilde{G}_{i+1}$ satisfies the triangle and the square-pyramid conditions with respect to $\widetilde{v}$.*

Proof. First we show that $\widetilde{G}_{i+1}$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$. Pick two adjacent vertices $\widetilde{u}, \widetilde{w}$ having in $\widetilde{G}_{i+1}$ the same distance to $\widetilde{v}$. Since by Lemma 4.6, $\widetilde{G}_{i+1}$ satisfies the property (P_{i+1}) and, by (Q_i) the graph $\widetilde{G}_i$ satisfies the triangle condition with respect to $\widetilde{v}$, we can suppose that $\widetilde{u}, \widetilde{w} \in \widetilde{S}_{i+1}$. From the definition of the edges of $\widetilde{G}_{i+1}$, there exist two couples $(\widetilde{x}, z), (\widetilde{x}, z') \in Z$ such that $\widetilde{x} \in \widetilde{B}_i$, z is adjacent to z' in G, and $\widetilde{u} = [\widetilde{x}, z], \widetilde{w} = [\widetilde{x}, z']$. Since $\widetilde{x}$ is adjacent in $\widetilde{G}_{i+1}$ to both $\widetilde{u}$ and $\widetilde{w}$, the triangle condition $\mathrm{TC}(\widetilde{v})$ is established.

Now we establish the square-pyramid condition $\mathrm{SPC}(\widetilde{v})$. Again, by Lemma 4.6 and by (Q_i) for $\widetilde{G}_i$, it is enough to consider the situation when $\widetilde{u} \in \widetilde{S}_{i+1}$ and $\widetilde{u}$ is adjacent to mutually non-adjacent $\widetilde{w}, \widetilde{w}' \in \widetilde{S}_i$. By the definition of $\widetilde{S}_{i+1}$ we have that $\widetilde{u} = [\widetilde{w}, z] = [\widetilde{w}', z]$. Thus $(\widetilde{w}, z) \equiv (\widetilde{w}', z)$ and, by the definition of the equivalence relation $\equiv$, we have the following two cases.

Case 1: There exists a vertex $\widetilde{x} \in \widetilde{B}_{i-1}$ adjacent to $\widetilde{w}, \widetilde{w}'$. Then we obtain the square $\widetilde{u}\widetilde{w}\widetilde{x}\widetilde{w}'$ as required by the square-pyramid condition with respect to $\widetilde{v}$.

Case 2: There exists a square $\widetilde{w}\widetilde{x}\widetilde{w}'\widetilde{x}'$ in $\widetilde{S}_i$ such that vertices w, x, w', x', z induce a pyramid in G. Observe that by (R_i) and (T_i), we obtain $(\widetilde{x}, z), (\widetilde{x}', z) \in Z$, and by the definition of $\widetilde{S}_{i+1}$ we have $\widetilde{u} = [\widetilde{w}, z] = [\widetilde{x}, z] = [\widetilde{x}', z]$ and thus $\widetilde{u} \sim \widetilde{x}, \widetilde{x}'$. Moreover, by ($\mathrm{T}_i$) applied to $\widetilde{w}$, we obtain that $\widetilde{x} \nsim \widetilde{x}'$. Hence the square-pyramid condition is verified. □

Now we establish some properties of the map f_{i+1}. We first prove that the mapping f_{i+1} is a graph homomorphism (preserving edges) from $\widetilde{G}_{i+1}$ to G. In particular, this implies that two adjacent vertices of $\widetilde{G}_{i+1}$ are mapped in G to different vertices.

Lemma 4.8. *f_{i+1} is a graph homomorphism from $\widetilde{G}_{i+1}$ to G, i.e., for any edge $\widetilde{a}\widetilde{b}$ of $\widetilde{G}_{i+1}$, ab is an edge of G.*

Proof. Consider an edge $\widetilde{a}\widetilde{b}$ of $\widetilde{G}_{i+1}$. If $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$, the lemma holds by ($\mathrm{R}_i$) or ($\mathrm{T}_i$) applied to $\widetilde{a}$. Suppose that $\widetilde{a} \in \widetilde{S}_{i+1}$. If $\widetilde{b} \in \widetilde{B}_i$, then $\widetilde{a} = [\widetilde{b}, a]$, and ab is an edge of G. If $\widetilde{b} \in \widetilde{B}_{i+1}$, then the fact that $\widetilde{a}$ and $\widetilde{b}$ are adjacent implies that there exists a vertex $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{a} = [\widetilde{w}, a], \widetilde{b} = [\widetilde{w}, b]$, and such that $a \sim b$ in G. □

We now prove that f_{i+1} is locally surjective at any vertex in $\widetilde{B}_i$.

Lemma 4.9. *If $\widetilde{a} \in \widetilde{B}_i$ and if $b \sim a$ in G, then there exists a vertex $\widetilde{b}$ of $\widetilde{G}_{i+1}$ adjacent to $\widetilde{a}$ such that $f_{i+1}(\widetilde{b}) = b$.*

Proof. If $\widetilde{a} \in \widetilde{B}_{i-1}$, the lemma holds by (R_i). Suppose that $\widetilde{a} \in \widetilde{S}_i$ and consider $b \sim a$ in G. If $\widetilde{a}$ has a neighbor $\widetilde{b} \in \widetilde{B}_i$ mapped to b by f_i, we are done. Otherwise $(\widetilde{a}, b) \in Z$, $[\widetilde{a}, b] \sim \widetilde{a}$ in $\widetilde{G}_{i+1}$ and $[\widetilde{a}, b]$ is mapped to b by f_{i+1}. □

We now prove that f_{i+1} is locally injective.

Lemma 4.10. *If $\widetilde{a} \in \widetilde{B}_{i+1}$ and $\widetilde{b}, \widetilde{c}$ are distinct neighbors of $\widetilde{a}$ in $\widetilde{G}_{i+1}$, then $b \neq c$.*

Proof. If $\widetilde{a} \in \widetilde{B}_{i-1}$, then the assertion of the lemma follows directly from the condition (R_i) applied to $\widetilde{a}$. Further we proceed by contradiction, i.e., we assume that $b = c$.

Suppose now that $\widetilde{a} \in \widetilde{S}_i$. If $\widetilde{b}$ or $\widetilde{c}$ is in $\widetilde{B}_i$, then $(\widetilde{a}, b)$ is not in Z. Thus, if say $\widetilde{b} \in \widetilde{B}_i$, then $\widetilde{c}$ is not in $\widetilde{S}_{i+1}$, otherwise we would have $\widetilde{c} = [\widetilde{a}, b]$. If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, then we get a contradiction with (T_i) applied to $\widetilde{a}$. If $\widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, then, by the definition of vertices in $\widetilde{S}_{i+1}$, we have $\widetilde{b} = [\widetilde{a}, b] = \widetilde{c}$, contradicting the choice of $\widetilde{b}, \widetilde{c}$.

Thus further we assume that $\widetilde{a} \in \widetilde{S}_{i+1}$. If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, then $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$. By Lemma 4.8 we have that $\widetilde{b} \nsim \widetilde{c}$. Since $(\widetilde{b}, a) \equiv (\widetilde{c}, a)$, by the definition of the relation $\equiv$, there is a vertex $\widetilde{x} \in \widetilde{B}_i$ adjacent to $\widetilde{b}$ and $\widetilde{c}$. Then we get a contradiction by (R_i) or (T_i) applied to $\widetilde{x}$. Now we suppose that $\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$. By the definition of the edge $\widetilde{a}\widetilde{b}$ there is a vertex $\widetilde{d} \in \widetilde{S}_i$ adjacent to $\widetilde{a}$ and $\widetilde{b}$. Observe that $\widetilde{d} \nsim \widetilde{c}$ since otherwise we would get a contradiction with $(\widetilde{d}, b) \in Z$. Since $(\widetilde{d}, a) \equiv (\widetilde{c}, a)$, by the definition of the relation $\equiv$, we are either in the case (Z2) or in the case (Z3). Observe however that this is not possible since $f_{i+1}(\widetilde{d}) \sim f_{i+1}(\widetilde{c}) = b$. Consequently, we obtain that it is not possible that $\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$.

For the remaining part of the proof we thus suppose that $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$. By the definitions of edges $\widetilde{b}\widetilde{a}$ and $\widetilde{a}\widetilde{c}$ there exist vertices $\widetilde{w}, \widetilde{w}' \in \widetilde{S}_i$ with $\widetilde{w} \sim \widetilde{b}, \widetilde{a}$ and $\widetilde{w}' \sim \widetilde{a}, \widetilde{c}$. Observe that $\widetilde{w} \nsim \widetilde{w}'$ since otherwise we would have $\widetilde{b} = [\widetilde{w}, b] = [\widetilde{w}', b] = \widetilde{c}$. Since $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$, by the definition of $\equiv$, we are in the case (Z2) or (Z3). For (Z2) there is a vertex $\widetilde{x} \in \widetilde{B}_{i-1}$ such that $waw'x$ is a square in G. By (A_1), $b \neq x$ and then $wbw'x$ is also a square in G; consequently, by (Z2), $\widetilde{b} = [\widetilde{w}, b] = [\widetilde{w}', b] = \widetilde{c}$. Thus we are in the case (Z3), i.e., there exist vertices $\widetilde{x}, \widetilde{x}' \in \widetilde{S}_i$ adjacent to $\widetilde{w}, \widetilde{w}'$, mutually non-adjacent and such that w, w', a, x, x' induce a pyramid in G. Observe that, by (A_1), we have $b \notin \{x, x'\}$, and that b lies in the interval $I(w, w')$ in G. Consequently, $b \sim x, x'$, and by (Z3), $\widetilde{b} = [\widetilde{w}, b] = [\widetilde{w}', b] = \widetilde{c}$. □

Before proving the next result, we formulate two technical lemmas.

Lemma 4.11. *If a, b, c are distinct pairwise adjacent vertices of G such that $\widetilde{a} = [\widetilde{b}, a] \in \widetilde{S}_{i+1}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$ and $\widetilde{b} \sim \widetilde{c}$ in $\widetilde{G}_{i+1}$, then $(\widetilde{c}, a) \in Z$ and, in particular, $[\widetilde{c}, a] = \widetilde{a} \sim \widetilde{c}$.*

Proof. If $(\widetilde{c}, a) \notin Z$, then there exists $\widetilde{a}' \sim \widetilde{c}$ in $\widetilde{B}_i$ such that $f_i(\widetilde{a}') = a$. By (T_i) applied to $\widetilde{c}$ we conclude that $\widetilde{b} \sim \widetilde{a}'$, i.e., $(\widetilde{b}, a) \notin Z$, a contradiction. Thus $(\widetilde{c}, a) \in Z$; in particular, $\widetilde{a} = [\widetilde{c}, a]$ by (Z1), and by the definition of edges of $\widetilde{G}_{i+1}$, $\widetilde{a} \sim \widetilde{c}$. □

Lemma 4.12. *Let $\widetilde{a} \in \widetilde{S}_{i-1}$, $\widetilde{b} \in \widetilde{S}_i$ and $\widetilde{c} = [\widetilde{b}, c] \in \widetilde{S}_{i+1}$ be such that $\widetilde{b} \sim \widetilde{a}, \widetilde{c}$. Assume that there is a square abcd in G. Then there exists $\widetilde{d} \in \widetilde{S}_i$ such that $\widetilde{a} \sim \widetilde{d}$ and $(\widetilde{d}, c) \in Z$.*

Proof. By (R_i), there exists $\widetilde{d} \sim \widetilde{a}$ such that $f_i(\widetilde{d}) = d$. If $\widetilde{d} \in \widetilde{B}_{i-1}$, then by conditions ($\mathrm{P}_i$), ($\mathrm{R}_i$) and ($\mathrm{S}_i$), there exists $\widetilde{c}' \in \widetilde{B}_i$ such that $\widetilde{c}' \sim \widetilde{b}, \widetilde{d}$ and $f_i(\widetilde{c}') = c$; this is impossible since $(\widetilde{b}, c) \in Z$. Consequently, $\widetilde{d} \in \widetilde{S}_i$.

Suppose now that $(\widetilde{d}, c) \notin Z$. It means that there exists $\widetilde{c}' \in \widetilde{S}_{i-1} \cup \widetilde{S}_i$ such that $\widetilde{c}' \sim \widetilde{d}$ and $f_i(\widetilde{c}') = c$. We distinguish two cases depending on if $\widetilde{c}' \in \widetilde{S}_{i-1}$ or $\widetilde{c}' \in \widetilde{S}_i$.

Case 1: $\widetilde{c}' \in \widetilde{S}_{i-1}$.

We have $\widetilde{c}' \nsim \widetilde{a}$. By the triangle and square-pyramid conditions (Q_i), either there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{a}, \widetilde{c}'$, or there exist $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_{i-1}$ such that $\widetilde{y}, \widetilde{y}' \sim \widetilde{a}, \widetilde{c}', \widetilde{d}$ and $\widetilde{y} \nsim \widetilde{y}'$.

If there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{a}, \widetilde{c}'$, then $\widetilde{x} \nsim \widetilde{b}, \widetilde{d}$. By ($\mathrm{R}_i$), $x = f_i(\widetilde{x}) \sim a, c$ and $x \nsim b, d$. Since b, d, x belong to the interval $I(a, c)$ and are pairwise non-adjacent, we get a contradiction with the interval condition.

Assume that there exist $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_{i-1}$ such that $\widetilde{y}, \widetilde{y}' \sim \widetilde{a}, \widetilde{c}', \widetilde{d}$ and $\widetilde{y} \nsim \widetilde{y}'$. We have $y = f_i(\widetilde{y}) \sim a, c, d$, and by the interval condition, $y \sim b$. By (R_i) applied to $\widetilde{a}$ and then to $\widetilde{y}$, we obtain that $\widetilde{y} \sim \widetilde{b}$ and $\widetilde{b} \sim \widetilde{c}'$. Consequently, $(\widetilde{b}, c) \notin Z$, a contradiction.

Case 2: $\widetilde{c}' \in \widetilde{S}_i$.

By the triangle condition (Q_i), there exists $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{c}', \widetilde{d}$. By Lemma 4.3, we can assume that there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{a}$. Note that $\widetilde{x} \nsim \widetilde{b}, \widetilde{c}', \widetilde{d}$ and by (R_i), we have $x \nsim b, c, d$.

If $\widetilde{u} \sim \widetilde{a}$, then $u \sim a$ and by the interval condition applied to the interval $I(a, c)$, we have $u \sim b$. By (R_i) applied to $\widetilde{a}$ and to $\widetilde{u}$, we get $\widetilde{u} \sim \widetilde{b}$ and $\widetilde{b} \sim \widetilde{c}'$. Consequently, $(\widetilde{b}, c) \notin Z$, a contradiction.

Suppose now that $\widetilde{u} \nsim \widetilde{a}$. Note that if $\widetilde{u} \sim \widetilde{b}$, then $u \sim b$, $u \sim a$ by the interval condition, and $\widetilde{u} \sim \widetilde{a}$ by (R_i). Consequently, $\widetilde{u} \nsim \widetilde{a}, \widetilde{b}$ and $u \nsim a, b$. Therefore, the graph induced by the vertices a, b, c, d, u, x is an half open book since $x \nsim b, c, d$, $u \nsim a, b$, $a \nsim c$ and $b \nsim d$, a contradiction by Lemma 2.1. □

We now show that the subgraphs induced by $B_1(\widetilde{a}, \widetilde{G}_{i+1})$ and $f_{i+1}(B_1(\widetilde{a}, \widetilde{G}_{i+1}))$ are isomorphic.

Lemma 4.13. *Let $\widetilde{a}, \widetilde{b}, \widetilde{c}$ be three distinct vertices in $\widetilde{G}_{i+1}$ such that $\widetilde{a} \sim \widetilde{b}, \widetilde{c}$. Then $\widetilde{b} \sim \widetilde{c}$ if and only if $b \sim c$.*

Proof. By Lemma 4.10, $b \neq c$. If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, then the lemma holds by the condition (T_i) applied to $\widetilde{a}$. Thus further we assume that among $\widetilde{a}, \widetilde{b}, \widetilde{c}$ there is a vertex outside $\widetilde{B}_i$. Note that from Lemma 4.8, if $\widetilde{b} \sim \widetilde{c}$, then $b \sim c$, establishing one direction. Suppose now that $b \sim c$ in G; we will show that $\widetilde{b} \sim \widetilde{c}$ in $\widetilde{G}_{i+1}$.

Case 1: $\widetilde{a} \in \widetilde{B}_i$.

If $\widetilde{b},\widetilde{c} \in \widetilde{S}_{i+1}$, then $\widetilde{b} = [\widetilde{a}, b]$ and $\widetilde{c} = [\widetilde{a}, c]$. Since $b \sim c$, by construction, we have $\widetilde{b} \sim \widetilde{c}$ in $\widetilde{G}_{i+1}$. Suppose now that $\widetilde{b} = [\widetilde{a}, b] \in S_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$. Then $\widetilde{b} \sim \widetilde{c}$ by Lemma 4.11.

Case 2: $\widetilde{b},\widetilde{c} \in \widetilde{B}_i$ and $\widetilde{a} \in \widetilde{S}_{i+1}$.

We know that $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$. If $\widetilde{b} \nsim \widetilde{c}$, we are in one of the cases (Z2) or (Z3) from the definition of $\equiv$. Hence in G the vertices b and c are opposite vertices of a square, which is impossible because $b \sim c$.

Case 3: $\widetilde{a},\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$.

Since $\widetilde{c} \in \widetilde{B}_i$, there exists $\widetilde{b}' \in \widetilde{B}_{i+1}$ such that $\widetilde{b}' \sim \widetilde{c}$ and $f_{i+1}(\widetilde{b}') = b$. Applying Case 1 to the triplet $\widetilde{c},\widetilde{a},\widetilde{b}'$, we get that $\widetilde{b}' \sim \widetilde{a}$. By Lemma 4.10, we get that $\widetilde{b}' = \widetilde{b}$ and we are done.

Case 4: $\widetilde{a},\widetilde{b},\widetilde{c} \in \widetilde{S}_{i+1}$.

There exist $\widetilde{w},\widetilde{w}' \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$, $\widetilde{c} = [\widetilde{w}', c]$, and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{w}', a]$. If $\widetilde{w} \sim \widetilde{c}$ or $\widetilde{w}' \sim \widetilde{b}$, then $\widetilde{b} \sim \widetilde{c}$ because $b \sim c$. Suppose further that $\widetilde{w} \nsim \widetilde{c}$, $\widetilde{w}' \nsim \widetilde{b}$. From Case 3 applied to $\widetilde{a},\widetilde{b} \in \widetilde{S}_{i+1}$ (respectively, $\widetilde{a},\widetilde{c} \in \widetilde{S}_{i+1}$) and $\widetilde{w}' \in \widetilde{B}_i$ (respectively, $\widetilde{w} \in \widetilde{B}_i$), it follows that $w \nsim c$ and $w' \nsim b$. Since $[\widetilde{w}, a] = [\widetilde{w}', a]$, the vertices $\widetilde{w}$ and $\widetilde{w}'$ obey one of the conditions (Z1),(Z2),(Z3).

We show that we can assume that there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$. If $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$ by condition (Z2), we are done. If $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$ by condition (Z1), $\widetilde{w} \sim \widetilde{w}'$ and by the triangle condition, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$. Suppose now that $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$ by condition (Z3). Thus, there exist $\widetilde{y},\widetilde{y}' \in \widetilde{S}_i$ such that $\widetilde{y},\widetilde{y}' \sim \widetilde{a},\widetilde{w},\widetilde{w}'$ and $\widetilde{y} \nsim \widetilde{y}'$. Consider the triangles awy, awy' and awb, all three sharing the common edge aw. By Lemma 2.1, we get that b is adjacent to y or y', say $b \sim y$. By Case 3 for $\widetilde{b},\widetilde{a},\widetilde{y}$, we have $\widetilde{b} \sim \widetilde{y}$. Then, we replace $\widetilde{w}$ by $\widetilde{y}$, and $(\widetilde{y}, a) \equiv (\widetilde{w}', a)$ by condition (Z1): there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{y}, \widetilde{w}'$, by the triangle condition.

Note that $\widetilde{x} \nsim \widetilde{a},\widetilde{b},\widetilde{c}$ and that $x \sim w, w'$. By Lemma 4.11, $x \nsim a, b, c$. By the local triangle condition, there exists a vertex y adjacent to x, b, and c. By (R_i) there exists $\widetilde{y} \in \widetilde{B}_i$ such that $f_i(\widetilde{y}) = y$ and $\widetilde{y} \sim \widetilde{x}$. If $y \sim w$, first (R_i) implies that $\widetilde{y} \sim \widetilde{w}$ and then Lemma 4.11 shows that $\widetilde{y} \sim \widetilde{b}$. If $y \nsim w$, $xwby$ is a square of G and by Lemma 4.12, $\widetilde{y} \sim \widetilde{b}$ and thus $\widetilde{y} \in \widetilde{S}_i$. Using the same reasoning, one can show that $\widetilde{y} \sim \widetilde{c}$. Applying Case 1 to the triplet $\widetilde{y},\widetilde{b},\widetilde{c}$, we conclude that $\widetilde{b} \sim \widetilde{c}$. □

We can now prove that the image under f_{i+1} of a triangle or a square is a triangle or a square.

Lemma 4.14. *If $\widetilde{a}\widetilde{b}\widetilde{c}$ is a triangle in $\widetilde{G}_{i+1}$, then abc is a triangle in G. If $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ is a square in $\widetilde{G}_{i+1}$, then $abcd$ is a square in G. Moreover, $\widetilde{G}_{i+1}$ does not contain induced $K_{2,3}$ and W_4^-.*

Proof. For triangles, the assertion follows directly from Lemma 4.8. Consider now a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$. From Lemmas 4.8 and 4.10, the vertices a, b, c, and d are pairwise distinct and $a \sim b$, $b \sim c$, $c \sim d$, $d \sim a$. From Lemma 4.13, $a \nsim c$ and $b \nsim d$. Consequently, $abcd$ is a square in G.

Now, if $\widetilde{G}_{i+1}$ contains an induced $K_{2,3}$ or W_4^-, from the first assertion and Lemma 4.13 we conclude that the image under f_{i+1} of this subgraph will be an induced $K_{2,3}$ or W_4^- in the graph G, contrary to the interval condition. □

Lemma 4.14 implies that replacing all 3–cycles and all induced 4–cycles of $\widetilde{G}_{i+1}$ by triangle- and square-cells, we will obtain a triangle-square flag complex, which we denote by $\widetilde{X}_{i+1}$. Then, obviously, $\widetilde{G}_{i+1} = G(\widetilde{X}_{i+1})$. The first assertion of Lemma 4.14 and the flagness of X implies that f_{i+1} can be extended to a cellular map from $\widetilde{X}_{i+1}$ to X: f_{i+1} maps a triangle $\widetilde{a}\widetilde{b}\widetilde{c}$ to the triangle abc of X and a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ to the square $abcd$ of X.

Lemma 4.15 (Properties (R_{i+1}) and (T_{i+1}))**.** *The map f_{i+1} satisfies the conditions (R_{i+1}) and (T_{i+1}).*

Proof. From Lemmas 4.10 and 4.13, we know that for any $\widetilde{w} \in \widetilde{B}_{i+1}$, f_{i+1} induces an isomorphism between the subgraph of $\widetilde{G}_{i+1}$ induced by $B_1(\widetilde{w}, \widetilde{G}_{i+1})$ and the subgraph of G induced by $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1}))$. Consequently, the condition (T_{i+1}) holds. From Lemma 4.9, we know that for every $\widetilde{w} \in \widetilde{B}_i$, $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1})) = B_1(w, G)$ and consequently (R_{i+1}) holds as well. □

Lemma 4.16 (Property (S_{i+1}))**.** *For any $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that the vertices $w = f_{i+1}(\widetilde{w}), w' = f_{i+1}(\widetilde{w}')$ belong to a square $ww'u'u$ of X, there exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i+1}$ such that $f_{i+1}(\widetilde{u}) = u, f_{i+1}(\widetilde{u}') = u'$, and $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square of $\widetilde{X}_{i+1}$, i.e., $\widetilde{X}_{i+1}$ satisfies the property (S_{i+1}).*

Proof. Note that if $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_{i-1}$, the lemma holds by the condition (S_i). Let us assume further that $\widetilde{w} \in \widetilde{S}_i$. By the property ($\mathrm{R}_{i+1}$) (cf. Lemma 4.15) applied to $\widetilde{w}$ and $\widetilde{w}'$, we know that in $\widetilde{G}_{i+1}$ there exist $\widetilde{u}, \widetilde{u}'$ distinct from $\widetilde{w}, \widetilde{w}'$, such that $\widetilde{u} \sim \widetilde{w}$, $\widetilde{u}' \sim \widetilde{w}'$ and $f_{i+1}(\widetilde{u}) = u$, $f_{i+1}(\widetilde{u}') = u'$. Observe that, by ($\mathrm{R}_{i+1}$), we have $\widetilde{u} \nsim \widetilde{w}'$ and $\widetilde{u}' \nsim \widetilde{w}$.

Claim 4.17. *If there exists $y \notin \{u', w\}$ such that $y \sim u, w'$, then $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$. If there exists $\widetilde{y} \in \widetilde{B}_{i+1}$ such that $\widetilde{y} \notin \{\widetilde{u}', \widetilde{w}\}$ and $\widetilde{y} \sim \widetilde{u}, \widetilde{w}'$, then $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$.*

Proof. For the first statement: By the interval condition applied to the interval $I(u, w')$, we have $y \sim u, u', w, w'$. By (R_{i+1}) applied to $\widetilde{w}$ and $\widetilde{w}'$, there exists $\widetilde{y} \in \widetilde{B}_{i+1}$ such that $\widetilde{y} \sim \widetilde{u}, \widetilde{u}', \widetilde{w}$ and $f_{i+1}(\widetilde{y}) = y$. By ($\mathrm{R}_{i+1}$) or ($\mathrm{T}_{i+1}$) applied to $\widetilde{y}$, we have $\widetilde{u} \sim \widetilde{u}'$.

The second statement follows from the first one, and from the fact that, by (R_{i+1}), $y = f_{i+1}(\widetilde{y}) \notin \{u', w\}$ and $y \sim u, w'$. □

Thus, for the rest of the proof of the lemma, we assume the following (since otherwise the lemma follows from Claim 4.17):

- there does not exist $\widetilde{y} \in \widetilde{B}_{i+1}$ such that $\widetilde{y} \sim \widetilde{u}, \widetilde{w}'$ and $\widetilde{y} \neq \widetilde{u}', \widetilde{w}$, or such that $\widetilde{y} \sim \widetilde{u}', \widetilde{w}$ and $\widetilde{y} \neq \widetilde{u}, \widetilde{w}'$;
- there does not exist $y \in V(G)$ such that $y \sim u, w'$ and $y \neq u', w$, or such that $y \sim u', w$ and $y \neq u, w'$.

Case 1. $\widetilde{w} \in \widetilde{S}_i, \widetilde{w}' \in \widetilde{S}_{i-1}$.

If $\widetilde{u}' \in \widetilde{B}_{i-1}$ then, by (S_i) applied to $\widetilde{w}'$ and $\widetilde{u}'$, we conclude that $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$. Hence further we assume that $\widetilde{u}' \in \widetilde{S}_i$.

If $\widetilde{u} \in \widetilde{S}_{i-1}$ then, by (R$_{i+1}$) applied to $\widetilde{w}$, we conclude that $\widetilde{u}$ is not adjacent to $\widetilde{w}'$. By the square-pyramid condition (Q$_i$), there exists $\widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{y} \sim \widetilde{u}, \widetilde{w}'$, which contradicts our assumptions.

Suppose now that $\widetilde{u} \in \widetilde{S}_i$. By the triangle condition (Q$_i$), there exists $\widetilde{y} \in \widetilde{S}_{i-1}$ such that $\widetilde{y} \sim \widetilde{u}, \widetilde{w}$. By (R$_{i+1}$), we know that $y = f_i(\widetilde{y}) \notin \{u, u', w, w'\}$ and $y \sim u, w$. By our assumtions, we have $y \nsim u', w'$. By the local triangle condition, there exists $x \sim u', w', y$. By (R$_i$) applied to $\widetilde{w}'$ there exists $\widetilde{x} \in \widetilde{B}_i$ such that $\widetilde{x} \sim \widetilde{w}', \widetilde{u}'$ and $f_{i+1}(\widetilde{x}) = x$. Again, by our assumptions, we have $x \nsim u, w$, i.e., $wyxw'$ is a square of G. By the previous case applied to the square $wyxw'$ (i.e., with $\widetilde{y}$ and $\widetilde{x}$ playing respectively the roles of $\widetilde{u}$ and $\widetilde{u}'$), we get that $\widetilde{w}\widetilde{y}\widetilde{x}\widetilde{w}'$ is a square of $\widetilde{G}_{i+1}$. By the positioning condition (U$_i$) with respect to $\widetilde{v}$ for the square $\widetilde{w}\widetilde{y}\widetilde{x}\widetilde{w}'$, we get that $\widetilde{x} \in \widetilde{S}_{i-2}$. This is impossible since $\widetilde{x} \sim \widetilde{u}'$ and $\widetilde{u}' \in \widetilde{S}_i$.

Suppose now that $\widetilde{u} \in \widetilde{S}_{i+1}$, i.e., $\widetilde{u} = [\widetilde{w}, u]$. By Lemma 4.12, $(\widetilde{u}', u) \in Z$. Since $\widetilde{w}' \in \widetilde{S}_{i-1}$, by $(Z2)$, $\widetilde{u} = [\widetilde{u}', u] \sim \widetilde{u}'$ and we are done.

Case 2. $\widetilde{w}, \widetilde{w}' \in \widetilde{S}_i$.

By the triangle condition (Q$_i$), there exists $\widetilde{y} \in \widetilde{S}_{i-1}$ such that $\widetilde{y} \sim \widetilde{w}, \widetilde{w}'$. By (R$_{i+1}$), we have $y = f_{i+1}(\widetilde{y}) \sim w, w'$. By our assumptions we have that $y \nsim u, u'$. By the local triangle condition, there exists $x \sim y, u, u'$ and, again by our assumptions, we have $x \nsim w, w'$. By (R$_{i+1}$), there exists $\widetilde{x} \sim \widetilde{y}$ such that $f_{i+1}(\widetilde{x}) = x$. Applying Case 1 to the squares $wyxu$ and $w'yxu'$, we get that $\widetilde{x} \sim \widetilde{u}$ and $\widetilde{x} \sim \widetilde{u}'$. By (T$_{i+1}$) applied to $\widetilde{x}$, we conclude that $\widetilde{u} \sim \widetilde{u}'$. ☐

Lemma 4.18 (Property (U$_{i+1}$))**.** *The graph $\widetilde{G}_{i+1}$ satisfies the property (U$_{i+1}$), i.e., the squares of $\widetilde{G}_{i+1}$ satisfy the positioning condition PC($\widetilde{v}$).*

Proof. Suppose by way of contradiction that there exists a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ of $\widetilde{G}_{i+1}$ such that

$$(*) \qquad d(\widetilde{a}, \widetilde{v}) + d(\widetilde{c}, \widetilde{v}) < d(\widetilde{b}, \widetilde{v}) + d(\widetilde{d}, \widetilde{v}).$$

Let a, b, c, d be the respective images of $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ in G by f_{i+1}. By Lemma 4.15, vertices a, b, c, d induce a square in G. If $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \widetilde{B}_i$, then (U$_i$) leads to a contradiction. In view of (*), in the following we can assume that at least one of the vertices $\widetilde{b}, \widetilde{d}$ belongs to $\widetilde{S}_{i+1}$, say $\widetilde{d} \in \widetilde{S}_{i+1}$. Consequently, $\widetilde{a}, \widetilde{c} \in \widetilde{S}_i \cup \widetilde{S}_{i+1}$ and $\widetilde{b} \in \widetilde{S}_{i-1} \cup \widetilde{S}_i \cup \widetilde{S}_{i+1}$. Moreover, by (*), we may assume without loss of generality that $\widetilde{a} \in \widetilde{S}_i$.

Case 1. $\widetilde{c} \in \widetilde{S}_i$.

Note that the inequality (*) implies that $\widetilde{b} \in \widetilde{S}_i \cup \widetilde{S}_{i+1}$. Since $\widetilde{d} = [\widetilde{a}, d] = [\widetilde{c}, d]$, either there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{a}, \widetilde{c}$, or there exist $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_i$ such that $\widetilde{y}, \widetilde{y}' \sim \widetilde{a}, \widetilde{c}, \widetilde{d}$ and $\widetilde{y} \nsim \widetilde{y}'$.

If there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{a}, \widetilde{c}$, then $\widetilde{x} \neq \widetilde{b}$ since $\widetilde{b} \in \widetilde{S}_i \cup \widetilde{S}_{i+1}$. By (R$_i$), we have $x = f_{i+1}(\widetilde{x}) \sim a, c$ and $x \notin \{b, d\}$. By the interval condition applied to $I(a, c)$, we get that $x \sim b, d$. Consequently, by (R$_{i+1}$) (cf. Lemma 4.15), we get $\widetilde{x} \sim \widetilde{d}$. However, this is impossible since $\widetilde{x} \in \widetilde{S}_{i-1}$ and $\widetilde{d} \in \widetilde{S}_{i+1}$.

Suppose now that there exist $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_i$ such that $\widetilde{y}, \widetilde{y}' \sim \widetilde{a}, \widetilde{c}, \widetilde{d}$ and $\widetilde{y} \nsim \widetilde{y}'$. By (R$_{i+1}$), we have $y = f_{i+1}(\widetilde{y}) \sim a, c, d$ and $y' = f_{i+1}(\widetilde{y}') \sim a, c, d$. By the interval condition, $b \sim y, y'$, and by (R$_{i+1}$), we get $\widetilde{b} \sim \widetilde{y}, \widetilde{y}'$. By the triangle condition, there exists $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{a}, \widetilde{y}$.

Since $\widetilde{b} \in \widetilde{S}_i \cup \widetilde{S}_{i+1}$, we have $\widetilde{u} \neq \widetilde{b}$, and by ($\mathrm{R}_{i+1}$) we get $u = f_{i+1}(\widetilde{u}) \neq b$, $u \sim a, y$, and $u \nsim d$. If $u \sim c$, by the interval condition applied to $I(a,c)$, we have $u \sim d$. This is a contradiction and, therefore, $u \nsim c$. Consider the triangles, ayb, ayd, and ayu, all three sharing the common edge ay. By the no-propeller property (cf. Lemma 2.1), we get that $b \sim u$ and by (R_{i+1}), we have $\widetilde{b} \sim \widetilde{u}$. Consequently, $\widetilde{b} \in \widetilde{S}_i$.

By the triangle condition (Q_i), there exists $\widetilde{u}' \in \widetilde{S}_{i-1}$ such that $\widetilde{u}' \sim \widetilde{b}, \widetilde{c}$. By ($\mathrm{R}_{i+1}$), we have $u' = f_{i+1}(\widetilde{u}') \sim b, c$ and $u' \nsim d$. If $u' \sim a$, by the interval condition applied to $I(a,c)$, we get $u' \sim d$, a contradiction. Hence, $u' \nsim a$. By Lemma 4.3, we can assume that there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. By ($\mathrm{R}_{i+1}$), we have $x = f_{i+1}(\widetilde{x}) \sim u, u'$, and $x \nsim a, b, c$, i.e., $d(x,a) = d(x,b) = d(x,c) = 2$. By the local positioning condition applied to the square $abcd$ with respect to x, we get that $d(x,d) = 2$. By the local triangle condition, there exists $z \sim a, d, x$. By (R_{i+1}), there exists $\widetilde{z} \sim \widetilde{x}$ such that $f_{i+1}(\widetilde{z}) = z$. If $z \sim u$, by (R_{i+1}) applied to $\widetilde{x}$ and $\widetilde{u}$, we get that $\widetilde{z} \sim \widetilde{u}, \widetilde{a}$. If $z \nsim u$, by (S_i) applied to the square $xuaz$, we also get that $\widetilde{z} \sim \widetilde{a}$. By ($\mathrm{R}_{i+1}$) applied to $\widetilde{a}$, we get that $\widetilde{z} \sim \widetilde{d}$. However, this is impossible since $\widetilde{z}$ cannot be adjacent to $\widetilde{d} \in \widetilde{S}_{i+1}$ and $\widetilde{x} \in \widetilde{S}_{i-2}$.

Case 2. $\widetilde{c} \in \widetilde{S}_{i+1}$.

Since $d(\widetilde{a}, \widetilde{v}) + d(\widetilde{c}, \widetilde{v}) < d(\widetilde{b}, \widetilde{v}) + d(\widetilde{d}, \widetilde{v})$ by (*), we get $\widetilde{b} \in \widetilde{S}_{i+1}$. By ($\mathrm{Q}_{i+1}$) (cf. Lemma 4.7), there exists $\widetilde{u} \in \widetilde{S}_i$ such that $\widetilde{u} \sim \widetilde{b}, \widetilde{c}$. By ($\mathrm{T}_{i+1}$) (cf. Lemma 4.15), we have $u = f_{i+1}(\widetilde{u}) \sim b, c$ and $u \sim a$ iff $\widetilde{u} \sim \widetilde{a}$ (respectively, $u \sim d$ iff $\widetilde{u} \sim \widetilde{d}$). By the interval condition, $u \sim a$ iff $u \sim d$.

Suppose that $\widetilde{u} \sim \widetilde{a}$ (note that then $\widetilde{u} \sim \widetilde{d}$). Then, by the triangle condition, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{a}$. By ($\mathrm{R}_{i+1}$), we have $x = f_{i+1}(\widetilde{x}) \sim a, u$ and $x \nsim b, c, d$. Consider the triangles, aux, aub and aud, all three sharing the common edge au. By the no-propeller property (Lemma 2.1), we get a contradiction since x, b, d are pairwise non-adjacent.

Thus $\widetilde{u} \nsim \widetilde{a}$. By the square-pyramid condition (Q_{i+1}) (cf. Lemma 4.7), we obtain two cases (i) and (ii) below:

(i) There exists $\widetilde{x} \in \widetilde{S}_{i-1}$ with $\widetilde{x} \sim \widetilde{u}, \widetilde{a}$. Then the vertices $\widetilde{x}, \widetilde{a}, \widetilde{d}, \widetilde{u}, \widetilde{b}, \widetilde{c}$ induce a half open book. By properties (R_{i+1}) and (T_{i+1}) (cf. Lemma 4.15), we obtain that their images x, a, d, u, b, c induce a half open book in G, which contradicts Lemma 2.1.

(ii) There exist two non-adjacent vertices $\widetilde{y}, \widetilde{y}' \in \widetilde{S}_i$, both adjacent to $\widetilde{u}, \widetilde{a}, \widetilde{b}$. By ($\mathrm{R}_{i+1}$), we get then three triangles uby, uby', and ubc sharing the common edge ub. By the no-propeller property (Lemma 2.1), we conclude that $y \sim c$ or $y' \sim c$, say $y \sim c$. By (R_{i+1}), we get $\widetilde{y} \sim \widetilde{c}$, which reduces the case to the (impossible) situation when $\widetilde{u} \sim \widetilde{a}$ (obtained by replacing $\widetilde{u}$ with $\widetilde{y}$).

In all the cases we assumed the inequality (*) and reached a contradiction. This implies that $d(\widetilde{a}, \widetilde{v}) + d(\widetilde{c}, \widetilde{v}) = d(\widetilde{b}, \widetilde{v}) + d(\widetilde{d}, \widetilde{v})$ and establishes the positioning condition. □

4.4. **The universal cover $\widetilde{X}$.** Concluding our inductive construction, in this subsection we define the universal covering map $\widetilde{X} \to X$ and finish the proof of Theorem 5 by showing that $\widetilde{X}$ satisfies the interval and the positioning conditions.

Let $\widetilde{X}_v$ denote the triangle-square complex obtained as the directed union $\bigcup_{i\geq 0} \widetilde{X}_i$ with the vertex v of X as the basepoint. Denote by $\widetilde{G}_v$ the 1–skeleton of $\widetilde{X}_v$. Let $f = \bigcup_{i\geq 0} f_i$ be the cellular map from $\widetilde{X}_v$ to X.

Lemma 4.19. *For any $\widetilde{w} \in \widetilde{X}$, the restriction $f|_{St(\widetilde{w},\widetilde{X}_v)}$ of f is an isomorphism between the stars $St(\widetilde{w},\widetilde{X}_v)$ and $St(w,X)$. Consequently, the map $f\colon \widetilde{X}_v \to X$ is a covering map.*

Proof. Note that, since $\widetilde{X}_v$ is a flag complex, a vertex $\widetilde{x}$ of $\widetilde{X}_v$ belongs to $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ if and only if either $\widetilde{x} \in B_1(\widetilde{w},\widetilde{G}_v)$ or $\widetilde{x}$ has two non-adjacent neighbors in $B_1(\widetilde{w},\widetilde{G}_v)$.

Let $\widetilde{w} \in \widetilde{S}_i$, i.e., i is the distance between $\widetilde{v}$ and $\widetilde{w}$ in $\widetilde{G}_v$, and consider the set $B_{i+2}(\widetilde{v},\widetilde{G}_v)$. Then the vertex-set of $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ is included in $B_{i+2}(\widetilde{v},\widetilde{G}_v)$. From (R_{i+2}) we know that f is an isomorphism between the graphs induced by $B_1(\widetilde{w},\widetilde{G}_v)$ and $B_1(w,G)$.

For any vertex x in $\mathrm{St}(w,X)\smallsetminus B_1(w,G)$ there exists a square $wuxu'$ in G. By (R_{i+2}), there exist $\widetilde{u},\widetilde{u}'$ both adjacent to $\widetilde{w}$ in $\widetilde{G}_v$ and such that $\widetilde{u} \nsim \widetilde{u}'$, and $f(\widetilde{u}) = u, f(\widetilde{u}') = u'$. By (S_{i+2}) applied to $\widetilde{w},\widetilde{u}$ and since $\widetilde{w}$ has a unique neighbor $\widetilde{u}'$ mapped to u', there exists a vertex $\widetilde{x}$ in $\widetilde{G}_v$ such that $f(\widetilde{x}) = x$, $\widetilde{x} \sim \widetilde{u},\widetilde{u}'$ and $\widetilde{x} \nsim \widetilde{w}$. Consequently, $f|_{V(\mathrm{St}(\widetilde{w},\widetilde{X}_v))}$ is a surjection from $V(\mathrm{St}(\widetilde{w},\widetilde{X}_v))$ onto $V(\mathrm{St}(w,X))$.

Now we show that $f|_{V(\mathrm{St}(\widetilde{w},\widetilde{X}_v))}$ is injective. Suppose by way of contradiction that there exist two distinct vertices $\widetilde{u},\widetilde{u}'$ of $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ such that $f(\widetilde{u}) = f(\widetilde{u}') = u$. If $\widetilde{u},\widetilde{u}' \sim \widetilde{w}$, by condition (R_{i+1}) applied to $\widetilde{w}$, we get a contradiction. Suppose first that $\widetilde{u} \sim \widetilde{w}$ and $\widetilde{u}' \nsim \widetilde{w}$ and let $\widetilde{z} \sim \widetilde{w},\widetilde{u}'$. This implies that w,u,z are pairwise adjacent in G. Since f is an isomorphism between the graphs induced by $B_1(\widetilde{w},\widetilde{G}_v)$ and $B_1(w,G)$, we conclude that $\widetilde{z} \sim \widetilde{u}$. But then f is not locally injective around $\widetilde{z}$, contradicting the condition (R_{i+2}). Suppose now that $\widetilde{w} \nsim \widetilde{u},\widetilde{u}'$. Let $\widetilde{a} \nsim \widetilde{b}$, respectively $\widetilde{a}' \nsim \widetilde{b}'$, be vertices adjacent to both $\widetilde{u}$ and $\widetilde{w}$, and, respectively, $\widetilde{u}'$ and $\widetilde{w}'$. If $\widetilde{a}' = \widetilde{a}$ or $\widetilde{a}' = \widetilde{b}$, then applying (R_{i+2}) to $\widetilde{a}'$, we get that $f(\widetilde{u}) \neq f(\widetilde{u}')$. Hence further we suppose that $\widetilde{a}' \notin \{\widetilde{a},\widetilde{b}\}$. By (R_{i+1}) applied to $\widetilde{w}$ we have that $a' \neq a \neq b \neq a'$ and $a \nsim b$. In G, the vertices a,b,a',b' belong to the interval $I(w,u)$. Consequently, by the interval condition, $a' \sim a,b$. By (R_{i+2}) applied to $\widetilde{w}$ and $\widetilde{a}$, $\widetilde{a}' \sim \widetilde{a}$ and $\widetilde{a}' \sim \widetilde{u}$. Thus, by (R_{i+2}) applied to $\widetilde{a}'$, $\widetilde{u} = \widetilde{u}'$, contradicting our choice of $\widetilde{u},\widetilde{u}'$. In all cases, we get a contradiction, thus $\widetilde{u}$ and $\widetilde{u}'$ as above do not exist.

Hence $f|_{V(\mathrm{St}(\widetilde{w},\widetilde{X}_v))}$ is a bijection between the vertex-sets of $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ and $\mathrm{St}(w,X)$. We show now that $\widetilde{u} \sim \widetilde{u}'$ in $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ if and only if $u \sim u'$ in $\mathrm{St}(w,X)$. If $\widetilde{u} \sim \widetilde{u}'$ then $u \sim u'$ by (R_{i+2}). Assume now that $\widetilde{u} \nsim \widetilde{u}'$ and $u \sim u'$. By (R_{i+2}), there exists $\widetilde{u}'' \sim \widetilde{u}$ with $f(\widetilde{u}'') = u'$. This leads however to a contradiction by the local injectivity of f.

By (R_{i+2}) applied to w and since X and $\widetilde{X}_v$ are flag complexes, $\widetilde{a}\widetilde{b}\widetilde{w}$ is a triangle in $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ if and only if abw is a triangle in $\mathrm{St}(w,X)$. By (R_{i+2}) and since X is a flag complex, if $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{w}$ is a square in $\mathrm{St}(\widetilde{w},\widetilde{X})$, then $abcw$ is a square in $\mathrm{St}(w,X)$. Conversely, by the conditions (R_{i+2}) and (S_{i+2}) and the flagness of $\widetilde{X}_v$, we conclude that if $abcw$ is a square in $\mathrm{St}(w,X)$, then $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{w}$ is a square in $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$. Consequently, for any $\widetilde{w} \in \widetilde{X}_v$, the map $f|_{\mathrm{St}(\widetilde{w},\widetilde{X}_v)}$ is an isomorphism between $\mathrm{St}(\widetilde{w},\widetilde{X}_v)$ and $\mathrm{St}(w,X)$, and thus f is a covering map. □

Lemma 4.20. *The graph $\widetilde{G} = G(\widetilde{X}_v)$ satisfies the interval condition and the positioning condition with respect to $\widetilde{v}$.*

Proof. For every $\widetilde{w}, \widetilde{w}' \in V(\widetilde{G})$ such that $d(\widetilde{w}, \widetilde{w}') = 2$, by Lemma 4.19, $d(w, w') = 2$, and by the interval condition in G, there exists a square $uwu'w'$ in $\mathrm{St}(w, X)$. By Lemma 4.19, $\widetilde{w}' \in V(\mathrm{St}(\widetilde{w}, \widetilde{X}))$, and the interval $I(\widetilde{w}, \widetilde{w}')$ is contained in $\mathrm{St}(\widetilde{w}, \widetilde{X})$. Since the map $f|_{\mathrm{St}(\widetilde{w}, \widetilde{X}_v)}$ is an isomorphism onto its image (cf. Lemma 4.19), the interval condition for $I(\widetilde{w}, \widetilde{w}')$ is satisfied.

The positioning condition with respect to $\widetilde{v}$, i.e. $\mathrm{PC}(\widetilde{v})$, is a consequence of (U_i) for sufficiently large i. □

Lemma 4.21. *The complex $\widetilde{X}_v$ is simply-connected for any basepoint $v \in V(X)$. For any two vertices $\widetilde{v}$ and $\widetilde{v}'$ the corresponding complexes $\widetilde{X}_v$ and $\widetilde{X}_{v'}$ are isomorphic.*

Proof. Simple connectedness follows from Lemma 2.2 and from the fact that $\widetilde{X}_v$ satisfies the condition (Q_i) for every i. It follows then, by Lemma 4.19, that $\widetilde{X}_v$ is the universal cover of X, and the second statement is a consequence of the uniqueness of the universal cover (cf. e.g. [Hat02, page 67]). □

Thus, for any choice of the basepoint we obtain the same universal cover $\widetilde{X}$ of X. By Lemma 4.20, its 1–skeleton satisfies the interval and the positioning conditions. This finishes the proof of Theorem 5.

5. Examples and extensions

5.1. **Examples.** Here, we provide examples of graphs satisfying our local conditions and not being the basis graphs of matroids. Of course, in view of Theorem 5 such examples arise as quotients of basis graphs of matroids under free actions of groups — for basics on relations between group actions and covering spaces see e.g. [Hat02, Chapter 1.3]. Note, that the quotient should be a graph (without multiple loops etc.) so that the replacement function for the group action should be large enough. For example, there is no such nontrivial action on C_4.

In fact our examples are the same as examples given in [DHT77, Theorem 2.3] for slightly different purposes (see comments below). We follow the notations of [DHT77]. Let $B_{n,n}$ be the basis graph of the complete matroid $M_{n,n}$, i.e., the one formed by the family of all n–element subsets of a set of cardinality $2n$. Define a $\mathbb{Z}_2$–action on $B_{n,n}$ in the way that each vertex v of $B_{n,n}$ is mapped by the generator of $\mathbb{Z}_2$ to the antipodal vertex v^*, i.e., the unique vertex at distance n from v (this is in fact the vertex corresponding to the complement of the n–element set v). It is easy to observe that this defines an action by graph automorphisms and that, for $n \geq 2$, this action is free. It can be observed, that a combinatorial ball of radius $\lfloor n/2 \rfloor - 1$ in $H_n := B_{n,n}/\mathbb{Z}_2$ is isomorphic to a ball of the same radius in $B_{n,n}$. Thus, for $n \geq 8$, balls of radii up to 3 look as corresponding balls in $B_{n,n}$, i.e., H_n satisfies our local conditions. Moreover, for such n, the quotient map $B_{n,n} \to H_n$ induces a map of the corresponding triangle-square complexes $X(B_{n,n}) \to X(H_n)$, being a covering map. It follows that $\pi_1(X(H_n)) = \mathbb{Z}_2$, and hence H_n is not the basis graph of a matroid.

Remark 5.1. It is stated in [DHT77] (cf. discussion after Theorem 2.6 there) that "the graphs H_n offer counterexamples to any number of futile conjectures(...), including Conjectures 2 and 3 of Maurer's thesis [Mau73]". As shown by our result a general form of Maurer's Conjecture 3 — saying that the triangle-square complexes of basis graphs of matroids may be characterized as simply connected complexes satisfying some local conditions — is true. In fact, as shown above, the existence of graphs H_n is consistent with the picture, since the corresponding complexes are not simply connected for large n.

Remark 5.2. Note that the counterexamples to the original Maurer's Conjecture 3 [Mau73] provided in [DHT77] do not satisfy our local conditions. The second example, cf. [DHT77, Fig. 1], does not satisfy the local positioning condition, while the first example, cf. [DHT77, Fig. 3], does not even satisfy the local triangle condition.

5.2. **Extension to even Δ–matroids.** Now, we will show that our Theorem 2 can be extended to even Δ–matroids. A Δ*–matroid* is a collection $\mathcal{B}$ of subsets of a finite set I, called *bases* (not necessarily equicardinal) satisfying the symmetric exchange property: for any $A, B \in \mathcal{B}$ and $a \in A\Delta B$, there exists $b \in A\Delta B$ such that $A\Delta\{a,b\} \in \mathcal{B}$. A Δ–matroid whose bases all have the same cardinality modulo 2 is called an *even* Δ*–matroid*. The *basis graph* $G = G(\mathcal{B})$ of an even Δ–matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by a single exchange, i.e., $|A\Delta B| = 2$. Extending Maurer's characterization of basis graphs of matroids, it was shown in [Che07] that a graph G is the basis graph of an even Δ–matroid if and only if G satisfies the positioning condition, the *generalized link condition* (the neighborhood of each vertex is the line graph of a finite graph) and the *generalized interval condition* (IC4) (each 2–interval of G contains a square and is an induced subgraph of the 4–dimensional octahedron). It was also noted in [Che07] that the generalized link condition is necessary, i.e., the interval condition (IC4) and the positioning condition solely do not characterize basis graphs of even Δ–matroids. Wenzel [Wen95] showed that the triangle-square complexes defined by basis graphs of even Δ-matroids are simply connected.

Let G be a (not necessarily finite) graph satisfying the local positioning, the generalized link and the generalized interval conditions. Inspecting the proofs of Lemma 2.1 and Theorem 5 (namely, noting that each use of the interval condition either employs a square or a pyramid in a 2–interval), analogously one can conclude that the 1–skeleton of the universal cover $\widetilde{X} = \widetilde{X(G)}$ of the triangle-square complex $X(G)$ of G satisfies the positioning condition and the generalized interval condition (IC4). Now, for any choice of the basepoint v, the triangle-square complex $\widetilde{X}_v$ is isomorphic to $\widetilde{X}$. Since the neighborhood of $\tilde{v}$ in the 1–skeleton of $\widetilde{X}_v$ coincides with $N(v)$ and thus is a line graph by the generalized link condition, we conclude that the 1–skeleton $G(\widetilde{X})$ of $\widetilde{X}$ satisfies the generalized link condition. From the result of [Che07] it follows that $G(\widetilde{X})$ is the basis graph of an even Δ–matroid, thus establishing the following result:

Theorem 6. *For a graph G the following conditions are equivalent:*

(i) G is the basis graph of an even Δ–matroid;

(ii) the triangle-square complex $X(G)$ is simply connected, every ball of radius 3 in G is isomorphic to a ball of radius 3 in the basis graph of an even Δ–matroid;

(iii) the triangle-square complex $X(G)$ is simply connected, G satisfies the generalized interval condition (IC4), the generalized link condition, and the local positioning condition.

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