Weak Convergence of Nonlinear High-Gain Tracking Differentiator
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Abstract—In this technical note, the weak convergence of a nonlinear high-gain tracking differentiator based on finite-time stable system is presented under some easy checkable conditions. An example is constructed by using homogeneity. Numerical simulation shows that this tracking differentiator takes advantages over the existing ones. This result relaxes the strict conditions required in existing literature that the Lyapunov function satisfies the global Lipschitz condition and the setting-time function is continuous at zero, both of them seem very restrictive in applications.

Index Terms—Finite-time stability, homogeneity, tracking differentiator.

I. INTRODUCTION
The differential tracking for a given signal is a well known yet challenging problem in control theory and practice. The numerous researches have been contributed to differential trackers like high-gain observer based differentiator [6], the super-twisting second-order sliding-mode algorithm [8], linear time-derivative tracker [16], robust exact differentiation [17], name just a few. For a nice comparison with different differential trackers, we refer to [21]. In this technical note, we study the following tracking differentiator first proposed in [12]:
\[ \dot{x}_H(t) = A_n x_H(t) + B_n R^n f \left( x_1 H(t) - v(t), \frac{x_2 H(t)}{R}, \ldots, \frac{x_n H(t)}{R^{n-1}} \right) \] (1.1)
where
\[ x_H = (x_1 H, x_2 H, \ldots, x_n H)^T \in \mathbb{R}^n, \quad A_n = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \]
\[ B_n = (0, \ldots, 0, 1)^T \]

\( I_{n-1} \) is an \((n-1) \times (n-1)\) identity matrix. It is claimed without proof in [12] that if the free system (i.e., \( v \equiv 0 \) and \( R = 1 \) in (1.1)) is globally attractive, that is, every solution of the free system tends to zero as time goes to infinity, then for any given measurable signal \( v \) and any given time \( T > 0 \), the solution of (1.1) satisfies
\[ \lim_{H \to \infty} \int_0^T |x_{1H}(t) - v(t)| dt = 0. \] (1.2)

In applications, the signal \( v \) may be only locally integrable like the piecewise continuous signal or bounded measurable \( v \), and hence its classical (pointwise) derivative may not exist but its \( i - 1 \)th generalized derivative, still denoted by \( v^{(i-1)} \), always exist in the sense of distribution which is defined as a functional of \( C_{0}^{\infty}(0, T) \) for any \( T > 0 \) as
\[ v^{(i-1)}(\varphi) = (-1)^{i-1} \int_0^T v(t) \varphi^{(i-1)}(t) dt \] (1.3)
where \( \varphi \in C_{0}^{\infty}(0, T), i > 1 \). The above is the standard definition of the generalized derivative ([1, Eq.(15), page 21]). From this definition, we see that any order of the generalized derivative \( v^{(i)} \) always exist provided that \( v \) is bounded measurable. Suppose that (1.2) holds true. Then considering \( x_{1H} \) as a function of \( C_{0}^{\infty}(0, T) \) ([1, Eq.(13), p.20])
\[ \lim_{H \to \infty} x_{1H} = v^{(i-1)} \] (1.4)
Comparing the right-hand sides of (1.3) and (1.4), we see that
\[ \lim_{R \to \infty} x_H = v \] in the sense of distribution ([1, pages 20–21]). So \( x_H \) can be regarded as an approximation of the \((i - 1)\)th generalized derivative \( v^{(i-1)} \) of \( v \) in \( 0, T \). For more details, we refer to [14].

Definition 1: For any given initial value, the tracking differentiator (1.1) is said to be weak convergent if \( \lim_{H \to \infty} x_{1H}(t) = v(t) \) for almost all \( t > 0 \) (and hence (1.2) is valid for any \( T > 0 \)). Although the first weak convergence is given in [13] and re-appeared later in [19], it is indicated in [11] that the proofs of [13], [19] are incorrect. A rigorous proof for weak convergence of the linear tracking differentiator is given in [6], [10], and recently the weak convergence for nonlinear one in [11]. A convergence result with the more accurate error estimation than [13], [11] is given in [20], which is based on a strict condition that there is a Lyapunov function \( V: \mathbb{R}^n \to \mathbb{R}^+ \) satisfying

I. \( \sum_{i=1}^{n-1} \left( \frac{\partial V}{\partial x_i} x_{i+1} + \frac{\partial V}{\partial x_n} f(x) \right) \leq -c V(x)^\theta \) for some \( c > 0, \theta \in (0, 1) \).

II. \( V \) is globally Lipschitz continuous or the gradient \( \nabla x V \) is bounded in \( \mathbb{R}^n \).

The condition I above is to guarantee that the free system of (1.1) (i.e., \( R = 1, v = 0 \)) is globally finite-time stable. The differentiator based on such a system has its independent significance due to its fast convergence. The second condition is quite strong and is actually not necessary as we shall see later in present technical note. A differentiator based on finite-time stable system given in [20] is as follows:
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -s t e^t \left\{ \text{sign} (\phi_s (x_1 (t) - v(t), e^t x_2 (t))) \right\} + \phi_s (x_1 (t) - v(t), e^t x_2 (t)) \| x_2(t) \|^{2s-2} \\
\phi_s (x_1 (t) - v(t), e^t x_2 (t)) &= -s t \text{sat} \{ \text{sign} (x_2 (t)) \} \| x_2(t) \|^2
\end{align*}
\] (1.5)
with \( \phi_n(x, y) = x + (\sigma_n(y))^{2-n}/(2 - \alpha) \),\( \phi_n(x) = x \) for \( |x| \leq \varepsilon_n \), \( \sigma_n(x) = x \) for \( |x| \geq \varepsilon_n \). (1.5) is a special tracking differentiator (1.1) with \( \varepsilon = 1, \gamma = 2 \). The conclusion of [20] is as follows: For the given signal \( v \), there exists a \( \gamma > 0 \) with \( \rho \gamma > 2 \) and \( \rho = \alpha/(2 - \alpha) \) such that

\[
x_i(t) - v_i^{(i-1)}(t) = O(e^{\gamma t}), 
\]

for all \( t > T \) for some positive number \( T \).

There is no direct verification of the conditions I, II (finite-time stability and global Lipschitz continuous of Lyapunov function) in [20] for system (1.5) instead, is refereed to [5]. Although the finite-time stability of the free system of (1.5) (i.e. \( \varepsilon = 1, \gamma = 2 \)) is studied in [5], the Lyapunov function is not available in [5] so it is not clear for us how to verify the global Lipschitz condition for Lyapunov function required in [20] for the system (1.5), which, to our knowledge, is far from simple. For instance, a very simple Lyapunov function like \( V(x_1, x_2) = x_1^2 + x_2^2 \) does not satisfy the global Lipschitz condition.

Moreover, the choice of the parameter \( \gamma \) is also a big deal. According to the proof in [20], \( \gamma = (1 - \theta)/\theta \), where \( \theta \) is the power exponent in its assumption I. However since the Lyapunov function is not given explicitly in both [20] and [5], we are not clear why the required parameter condition \( \gamma > 2 \) is satisfied. Notice that the finite-time stability of the free system of (1.5) is concluded from the following system:

\[
\begin{aligned}
\dot{x}_i(t) &= -x_i(t), \\
\dot{x}_i(t) &= -\dot{x}_i(t) \\
\phi_n(x(t)) &= \phi_n(x(t)) \\
\end{aligned}
\]

which is equal to (1.5) in some neighborhood of zero in \( \mathbb{R}^2 \). For system (1.6), the Lyapunov function satisfying assumption I is given in [5] with \( \theta = (2/(3 - \alpha)) \), \( \gamma = (1 - \theta)/(\theta \alpha) \). By a simple computation, we get \( \varepsilon = (0, 1/2) \) and \( \rho = (\alpha/(2 - \alpha)) \) \( \in (0, 1) \). So it seems impossible to choose \( \gamma > 2 \) even for system (1.6).

In this technical note, we first generalize, in Section II, the stability result for the perturbed finite-time stable systems studied in [3], and then apply the result to the proof of weak convergence of the finite-time stable based tracking differentiator without assuming the global Lipschitz continuous for Lyapunov function. In Section III, a first order tracking differentiator based on finite-time stable system is constructed using homogeneity, which is simpler than (1.5). All required conditions are verified. In Section IV, numerical experiments are performed to illustrate the advantage of the proposed differentiator over the existing ones. Finally, in Section V, we apply the tracking differentiator presented in Section III to stabilize a one-dimensional wave equation under the boundary control for which the derivative of the output signal is no more than locally square integrable. Numerical simulations are illustrated the effectiveness of the differentiator.

II. WEAK CONVERGENCE OF TRACKING DIFFERENTIATOR

The main purpose of this section is to prove the weak convergence of (1.1) by removing the global Lipschitz continuity on Lyapunov function required in [20]. Before going on, we give the definition and some preliminary results on finite-time stability. For notational simplicity, we drop the subscript \( \phi \) in all state variables by abuse of notation.

Definition 2: The following system:

\[
\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n
\]

is said to be globally finite-time stable, if a) it is Lyapunov stable; b) for any \( x_0 \in \mathbb{R}^n \), there exists a \( T(x_0) > 0 \) such that the solution of (2.7) satisfies \( \lim_{t \to \infty} x(t) = 0 \), and \( x(t) \neq 0 \) for all \( t \in [T(x_0), \infty) \).

The function \( T_f : \mathbb{R}^n \to \mathbb{R} \) in Definition 2 is called the settling-time function.

Remark 1: In [20], another additional restrictive assumption is imposed on setting-time function that \( T_f \) is continuous at zero, which is also almost impossible to check. This assumption is removed in present technical note.

The following Lemma 2.1 follows from (iii) of proposition 2.4 and theorem 4.2 of [3].

Lemma 2.1: Suppose that there exists a continuous, positive definite function \( V : \mathbb{R}^n \to \mathbb{R} \), constants \( c, \alpha \in (0, 1) \) such that

\[
L_f V(x) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x) \leq -c V(x)^\alpha, \quad x \neq 0
\]

where \( f \) denotes the \( i \)th component of \( f \). Then (2.7) is globally finite-time stable. Furthermore, there exists a \( \sigma > 0 \) such that for any \( \|x_0\| \leq \sigma \), it has

\[
\|x(t)\| \leq \frac{1}{c(1 - \alpha)} (V(x_0))^{1-\alpha}, \quad t \geq 0
\]

We remark that in theorem 4.2 of [3], it is shown that \( T_f(x_0) \leq (1/c)(1-\alpha)/(V(x_0))^{1-\alpha} \) for all \( x_0 \) in a neighborhood of origin, and in proposition 2.4 of [3], \( T_f(x_0) > \|x_0\| \) for all \( x_0 \) in another neighborhood of origin. These two facts together lead to (2.9).

The following Lemma 2.2 is a generalization of theorem 5.2 of [3] for the case of attract basin being the whole space \( \mathbb{R}^n \) by removing the global Lipschitz continuity for Lyapunov function \( V \).

Lemma 2.2: Consider the following perturbed system of (2.7):

\[
\dot{y}(t) = f(y(t)) + g(t, y(t)), \quad y(0) = y_0, \quad \|y_0\| \leq H
\]

where \( H > 0 \) is a constant. If there exist a continuous, positive definite and radially unbounded (i.e., \( \lim_{|y| \to \infty} V(x) = \infty \)) function \( V : \mathbb{R}^n \to \mathbb{R} \) with all continuous partial derivatives in its variables, and constants \( c, \alpha \in (0, 1) \), such that (2.8) holds, then for any \( H > 0 \), there exists a \( H \)-dependent constant \( \varepsilon_{H_H} > 0 \) such that for any continuous function \( g : \mathbb{R}^{n+1} \to \mathbb{R}^n \) with

\[
\dot{y}(t) \leq L \frac{\|y(t)\|}{\|x_0\|}, \quad \forall t \in [T_c, \infty)
\]

the solution of (2.10) is bounded and

\[
\|y(t)\| \leq L \frac{\|y_0\|}{\|x_0\|}, \quad \forall t \in [T_c, \infty)
\]

where the constants \( L, \varepsilon > 0, T_c > 0 \) depend on the initial value \( y_0 \).

Proof: We split the proof into two steps.

Step 1. There exists a \( \varepsilon_{H_H} > 0 \) such that for any \( \varepsilon < \varepsilon_{H_H} \), where \( \delta \) is defined in (2.11), the solution of (2.10) is bounded.

Let

\[
b_H = \max \{1, \sup_{y \neq 0} V(y)\}
\]

be a \( H \)-dependent constant, and

\[
\varepsilon_{H_H} = \frac{b_H}{M_H}, \quad M_H = \max \{1, \sup_{x \neq 0} V(x)\}
\]

the radial unbounded positive definite function \( V \), by lemma 4.3 of [15] on page 145, there are strictly increase functions \( \kappa_1, \kappa_2 : [0, \infty) \to [0, \infty) \) such that \( \lim_{x \to \infty} \kappa_1(x) = \kappa_2(0) = 0 \), and \( \kappa_1(\|y\|) \leq V(x) \leq \kappa_2(\|y\|) \). Since \( x \in [x : V(x) \leq b_H + 1] \), \( \kappa_2(\|y\|) \leq b_H + 1, \|y\| \leq \kappa_1(\|y\|) \), it is known that the set \( x \in [x : V(x) \leq b_H + 1] \) is bounded. This together with the continuity of \( \nabla V \) concludes that \( M_H < \infty \). Hence \( \delta_{H_H} \) is a positive number (even for \( H = 0 \)). We assume that the claim in Step 1 is not true and obtain a contradiction. From the definition of \( b_H, V(y(0)) \leq b_H \). This together with the continuity of \( y \) guarantees that for \( \delta < \delta_{H_H} \), there exist \( t_1, t_2 : 0 < t_1 < t_2 \) such that the solution of (2.10) satisfies

\[
V(y(t_1)) - b_H, V(y(t_2)) > b_H,
\]

\[
y(t) \subseteq \{x : b_H < V(x) < b_H + 1\}, \forall t \in [t_1, t_2].
\]

(2.13)
Finding the derivative of $V(y(t))$ in $[t_1, t_2]$ gives

$$
\dot{V}(y(t)) = \nabla V(y(t)) + \left(\nabla^2 V(y(t)) + \varepsilon(t)\right),
$$

with $\nabla V(x) = -c(V(x))^\alpha$ and $h$ is the vector field: $h(x) = (x_2, x_3, \ldots, x_{n-1}, f(x))^T$.

Then for any initial value of (1.1) and constant $\alpha > 0$, there exists a $R_c > 0$ depending on the initial value of (1.1), input $\varepsilon$ and its derivatives, such that

$$
x(t, \varepsilon(t)) \leq (1/R)^{\gamma t}, \forall t > o, R > R_0
$$

where $I$ is a positive constant depending on the initial value of (1.1), input $\varepsilon$ and its derivatives, $\gamma = \{(1 - \alpha)/\alpha\}$, $\theta = \min\{\theta_1, \theta_2, \ldots, \theta_n\}$, $\xi_i(1 \leq i \leq n)$ are the solutions of (1.1).
there exists a $R_c > 0$, depending on initial value, and $v$, such that for all $R > R_0$ and $t \in (t_j + a, t_{j+1})$ or $t > t_{m} + a$, it has
\[
|x_1(t) - v(t)| \leq L (1/R)^{\gamma_1}, \forall i = 1, 2, \ldots, n \quad \text{(2.29)}
\]
where $v_-$ denotes the left derivative and $v_+$ the right one, $L$ is some positive constant depending on the initial value of (1.1), input $v$ and its derivatives, $\gamma = (1 - \alpha)/\alpha$, $\theta = m + n \{\theta_2, \theta_3, \ldots, \theta_n\}$, $x_1(1 \leq i \leq n)$ are the solutions of (1.1).

III. A FIRST ORDER TRACKING DIFFERENTIATOR

In this section, we construct a first order differentiator based on finite-time stable system by the help of homogeneity.

Definition 3: A function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is called homogeneous of degree $d$ with weights $r_i$, if
\[
V(\lambda x_1, \lambda^{r_2} x_2, \ldots, \lambda^{r_n} x_n) = \lambda^d V(x_1, \ldots, x_n). \quad \text{(3.1)}
\]
For any $\lambda > 0$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$,
\[
g_i(\lambda x_1, \lambda^{r_2} x_2, \ldots, \lambda^{r_n} x_n) = \lambda^{d_i} g_i(x_1, x_2, \ldots, x_n) \quad \text{(3.2)}
\]
where $g_i$ is the $i$th component of $g$.

Consider the second order system:
\[
\dot{x} = f(x) = (f_1(x), f_2(x))^T, x = (x_1, x_2)^T \quad \text{(3.3)}
\]
where
\[
\begin{cases}
f_1(x_1, x_2) = x_2, \\
f_2(x_1, x_2) = -k_1 |x_1|^{\alpha} - k_2 |x_2|^\beta,
\end{cases}
\]
with $|x|^\alpha = \text{sign}(x)|x|^\alpha$, and
\[
\alpha, \beta, a, b, d, k_1, k_2 > 0, a \alpha = b \beta = b - d, a = b + d. \quad \text{(3.5)}
\]
It is seen that for any $\lambda > 0$
\[
\begin{align*}
f_1(\lambda x_1, \lambda^{\beta} x_2) &= \lambda^\beta x_2 = \lambda^{-d-a} f_1(x_1, x_2), \\
f_2(\lambda x_1, \lambda^{\beta} x_2) &= -k_1 \lambda^{\alpha} x_1^\alpha - k_2 \lambda^{\beta} |x_2|^{\beta} \\
&= \lambda^{-d-a} f_2(x_1, x_2).
\end{align*}
\]
Therefore, the vector field $f$ is homogeneous of degree $-d$ with weights $a, b$.

Let $W: \mathbb{R}^d \rightarrow \mathbb{R}$ be given by
\[
W(x_1, x_2) = \frac{1}{2k_1} x_2^2 + \frac{|x_1|^{1+\alpha}}{1+\alpha}. 
\]
A direct computation shows that
\[
L_f W(x) = -\frac{k_2}{k_1} |x_2|^{\beta+1} \leq 0. \quad \text{(3.8)}
\]
By LaSalle’s invariance principle, system (3.3) is globally asymptotically stable.

From theorem 6.2 of [4], there is a continuous, positive definite Lyapunov function $V: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\nabla_x V$ is continuous on $\mathbb{R}^n$. Moreover, $V$ is homogeneous of degree $d > \max \{d, a, b\}$, and $L_f V$ is homogeneous of degree $i - d$, both with the same weights $(a, b)$. We also know from theorem 2 of [18] that $V$ is radially unbounded. By virtue of lemma 4.2 of [4], there exists a $c > 0$ such that
\[
L_f V(x) \leq -c (V(x))^{1+\alpha}. \quad \text{(3.9)}
\]
The finite-time stability of (3.3) is also studied in [2] on page 191. But here we are more interested in inequality (3.9) since it means that our condition (2.20) in Theorem 1 is valid.

Now, we show that $f_2$ satisfies condition (2.19) in Theorem 1. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\phi(x) = x^a - \alpha - (x - a)^{\alpha}$ for some $a > 0$, $\theta \in (0, 1)$. $\phi$ is decreasing since $\phi'(x) = \theta x^{a-1} - (x - a)^{a-1} < 0$ for all $x > a$. It follows that
\[
x^a - \frac{y^a}{a} \leq (x - y)^{\alpha}, \forall x > y > 0. \quad \text{(3.10)}
\]
Furthermore, since $\phi''(x) = \theta (\theta - 1) x^{a-2} < 0$ for all $x > 0$, $\phi$ is convex on $(0, \infty)$. By Jessen’s inequality, we have
\[
x^a + y^a \leq (x + y)^{\alpha}, \forall x, y > 0. \quad \text{(3.11)}
\]
Combining inequalities (3.10), (3.11), we get
\[
|f_2(x_1, x_2) - f_2(y_1, y_2)| \leq k_2 |x_1 - y_1|^{\alpha} + k_2 |x_2 - y_2|^\beta. \quad \text{(3.12)}
\]
That is, the condition (2.19) of Theorem 1 is satisfied for $f_2$. Based on finite-time stable system (3.3), we can construct the following tracking differentiator:
\[
\begin{align*}
\dot{x}_2(t) &= x_2(t), \\
\dot{x}_3(t) &= R^2 \left( -k_1 (x_1(t) - v(t))^\alpha - k_2 \frac{|x_2(t)|^\beta}{R^2} \right). 
\end{align*}
\]
By Theorem 1, (3.13) is weak convergent.

For differentiator (3.13), using the notation (2.22) for $n = 2$, the error equation becomes
\[
\begin{align*}
\dot{e}_1(t) &= e_2(t), \\
\dot{e}_2(t) &= -k_1 |e_1(t)|^{\alpha} - k_2 |e_2(t)|^\beta + \Delta(t).
\end{align*}
\]
where $|\Delta(t)| \leq \kappa |e(t)|^a$ for some positive constant $\kappa$. Since $|x_2|$ is homogeneous of degree $b$ with weights $a, b$, it follows from lemma 4.2 of [4] that $|x_2| \leq C_1 (V(x_1, x_2))^{1/a}$.

Let the Lyapunov function $V$ be defined in (3.9). Finding its derivative along the solution of (3.14) gives
\[
\frac{dV}{dt}_{a \left| a \right. (3.14)} \leq -cV(t)^{1-a/b} + C_1 V(t)^{(1-a)/(b-a)}. \quad \text{(3.14)}
\]
It is easy to verify that if $V > \left( (V_1(t)/V^*)^{-b/a} \right)^{1/a}$, then $dV/dt < 0$ along the solution of (3.14). Hence there exist an initial value dependent constant $T > 0$ such that $V(t) \leq C_1 V(t)^{(1-a)/(b-a)}$ for all $t < T$.

Since the functions $|x_1|, |x_2|$ are homogeneous of degree $a$ and $b$ with weights $a$ and $b$ respectively, using lemma 4.2 of [4] again, we get that
\[
|e_1(t)| \leq C_1 e_1(t)^{a/b} \leq M_1 (V(t))^{b/a}, \\
|e_2(t)| \leq C_1 e_2(t)^{b/a} \leq M_2 (V(t))^{b/b} \quad \forall t > T.
\]
We have thus proved the following Theorem 2.

Theorem 2: If the signal $e$ satisfies $e \in \mathcal{C}_0 \cap \mathcal{C}_1 \ni e_i(t) < \infty$ for $i = 1, 2$, then the first order high-gain finite-time stable based differentiator (3.13) is convergent in the sense that for any initial value of (3.13) and $T_1 > 0$, there exists a $R_c > 0$ depending on initial value and $e$ such that $\forall R > R_0, t > T_1$
\[
|x_1(t) - v(t)| \leq M_1 (V(t))^{b/a} \quad \forall t > T_1 
\]
where $M_1, M_2$ are constants depending on initial value and $e$, and $b - d = \theta > 0$ by (3.5).
IV. NUMERICAL SIMULATIONS

In this section, we give some numerical simulations to compare the following three differentiators.

**D1.** Robust exact differentiator using sliding mode technique from [17]:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) - 1.5C \frac{1}{2} |x_1(t) - v(t)|^{\frac{1}{2}}, \\
\dot{x}_2(t) &= -1.1C \text{sign} (x_1(t) - v(t)).
\end{align*}
\]  

**DII.** Linear tracking differentiator in [10], which is equivalent to the high-gain observer presented in [6] under the coordinate transformation:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= R^2 (|x_1(t) - v(t)| - x_2(t)/R).
\end{align*}
\]  

**DIII.** High-gain finite-time stable system based tracking differentiator (3.13) (taking \( a = 4, \beta = 3, k_1 = k_2 = 1, \alpha = 1/2, \beta = 2/3 \) in (3.13)):

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= R^2 \left( -|x_1(t) - v(t)|^{\frac{1}{2}} - \frac{x_2(t)}{R} \right)^{\alpha/2}.
\end{align*}
\]  

The Matlab program of Euler method is adopted in investigation. We choose the same zero initial value and \( v(t) = \sin t \) in all simulations.

The results by differentiator D1 are plotted in Fig. 1 where in Fig. 1(a), \( C = 2 \) and integral step \( h = 0.01 \). Fig. 1(b) is the magnification of Fig. 1(a). The results by differentiator DII are plotted in Fig. 2 where in Fig. 2(a), \( R = 100 \) and integral step \( h = 0.0001 \), while in Fig. 2(b) \( R = 10 \) and integral step \( h = 0.0001 \). The results by differentiator DIII are plotted in Fig. 3 where in Fig. 3(a), \( R = 100 \) and integral step \( h = 0.0001 \), while in Fig. 3(b) \( R = 10 \) and integral step \( h = 0.0001 \). Fig. 3(c) is the magnification of Fig. 3(a). Fig. 3(d) plots the results of differentiator DIII with delayed signal \( v \) of delay 0.05, \( R = 100 \) and integral step \( h = 0.0001 \). In Fig. 4(a), we take \( R = 10 \), integral step \( h = 0.0001 \), while in Fig. 4(b), \( R = 20 \), \( h = 0.0001 \). From Figs. 1, 2 and 3, we see that our tracking differentiator based on finite-time stable system DIII is smoother than differentiator D1 in which the discontinuous of function produces problems like chattering. And differentiator DII tracks faster than linear differentiator DII. Moreover, it seems that the our finite-time stable system based tracking differentiator DIII is more accurate than linear DII with the same high-gain parameters. Finally, the finite-time stable system based tracking differentiator is tolerant to small time delay and noise. From Fig. 4 we see that the tuning parameter \( R \) in differentiator DIII plays a significant role in convergence and noise tolerance: the larger the \( R \) is, the more accurate the tracking effect would be, but more sensitive to the noise. This suggests that the choice of parameter \( R \) in DIII is a tradeoff between tracking accuracy and noise tolerance in practice.

V. APPLICATION TO STABILIZATION OF A STRING EQUATION

Consider the following one-dimensional wave equation

\[
\begin{align*}
\frac{\partial^2 w(x, t)}{\partial x^2} - \frac{\partial w(x, t)}{\partial t} &= 0, \\
0 < x < 1, \\
t > 0.
\end{align*}
\]  

which describes the vibration of string, where \( w \) is the amplitude, \( u_1 \) is the velocity, \( u_x \) is the vertical force, \( |u_1, u_x| \in H^1_0 \times L^2_0 \),...
\[ H^1_1(0,1) = \{ f \in H^1_1(0,1) | f(0) = 0 \} \]
is the initial value, and \( u \) is the boundary control input. It is well-known that with the boundary feedback \( u(t) = -\omega w_1(1, t), \quad \alpha > 0, \) the system (5.1) is exponentially stable in the sense that

\[ E(t) \leq M e^{-\omega t} E(0), \quad E(t) = \frac{1}{2} \int_0^1 \left[ w_2^2(x, t) + w_1^2(x, t) \right] dx \]

for some positive constants \( M, \omega > 0, \) where \( E(t) \) is the energy of the vibrating string. The feedback controller \( u(t) = -\alpha w_1(1, t) \) requires the velocity \( w_1(1, t) \) which is hard to measure ([7]). Instead, the difference is obtained from \( u(t) = -\omega w_1(1, t), \) and the result is plotted in Fig. 5(b). It is seen that both of them are convergent satisfactorily. We use numerical method to study the stability of the system (5.2).

VI. Conclusion

In this technical note, we have proved the weak convergence for the high-gain nonlinear tracking differentiator based on finite-time stable systems. An unnecessary restrictive condition of the global Lipschitz continuity for Lyapunov function in literature is removed, by which the conditions become simple and practically checkable. All the conditions are implemented in the construction of a first order finite-time stable system based tracking differentiator. By this example, numerical experiment is carried out to show the fast tracking, accuracy, smooth, anti-chattering, small time delay and small noise tolerance of this differentiator compared with the existing ones. We also apply this nonlinear differentiator to the boundary stabilization of a one-dimensional wave equation. The numerical simulation shows that it is very effective and noise tolerance even for finite-dimensional systems. However, the other issues with a differentiator design like evaluating its accuracy in the presence of a measurement noise (robustness) and the settling-time function estimation are needed for further investigations theoretically, although it is believed that our differentiator is robust to noise as proved in linear differentiator of the same kind in [6], [10] and confirmed by numerical simulations in this technical note. A recent effort on this aspect can be found in [9].

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REFERENCES

Low-Frequency Learning and Fast Adaptation in Model Reference Adaptive Control

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Abstract—While adaptive control has been used in numerous applications to achieve system performance without excessive reliance on dynamical system models, the necessity of high-gain learning rates to achieve fast adaptation can be a serious limitation of adaptive controllers. This is due to the fact that fast adaptation using high-gain learning rates can cause high-frequency oscillations in the control response resulting in system instability. In this note, we present a new adaptive control architecture for nonlinear uncertain dynamical systems to address the problem of achieving fast adaptation using high-gain learning rates. The proposed framework involves a new and novel controller architecture involving a modification term in the update law. Specifically, this modification term filters out the high-frequency content contained in the update law while preserving asymptotic stability of the system error dynamics. This key feature of our framework allows for robust, fast adaptation in the face of high-gain learning rates. Furthermore, we show that transient and steady-state system performance is guaranteed with the proposed architecture. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

Index Terms—Adaptive control, command following, fast adaptation, high-gain learning rate, low-frequency learning, nonlinear uncertain dynamical systems, stabilization, transient and steady state performance.

I. INTRODUCTION

While adaptive control has been used in numerous applications to achieve system performance without excessive reliance on system models, the necessity of high-gain learning rates for achieving fast adaptation can be a serious limitation of adaptive controllers [1]. Specifically, in certain applications fast adaptation is required to achieve stringent tracking performance specifications in the face of large system uncertainties and abrupt changes in system dynamics.

This, for example, is the case for high performance aircraft systems that are subjected to system faults or structural damage which can result in major changes in aerodynamic system parameters. In such situations, adaptive control with high-gain learning rates is necessary in order to rapidly reduce and maintain system tracking errors. However, fast adaptation using high-gain learning rates can cause high-frequency oscillations in the control response resulting in system instability [2]–[4]. Hence, there exists a critical trade-off between system stability and adaptation learning rate (i.e., adaptation gain).

In this note, we present a new adaptive control architecture for nonlinear uncertain dynamical systems to address the problem of achieving fast adaptation using high-gain learning rates. The proposed framework involves a new and novel controller architecture involving a modification term in the update law. Specifically, this modification term filters out the high-frequency content contained in the update law while preserving asymptotic stability of the system error dynamics. This key feature of our framework allows for robust, fast adaptation in the face of high-gain learning rates. We further show that transient and steady-state system performance is guaranteed with the proposed architecture. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

The notation used in this technical note is fairly standard. Specifically, $R$ denotes the set of real numbers, $R^n$ denotes the set of $n \times 1$ real column vectors, $R^{n \times m}$ denotes the set of $n \times m$ real matrices, $\cdot^T$ denotes transpose, $\cdot^{-1}$ denotes inverse, $\cdot \cdot \cdot_2$ denotes the Euclidian norm, and $\cdot \cdot \cdot_F$ denotes the Frobenius matrix norm. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $M$ and $\text{tr}(\cdot)$ for the trace operator.

II. MODEL REFERENCE ADAPTIVE CONTROL

We begin by presenting a brief review of the model reference adaptive control problem. Specifically, consider the nonlinear uncertain dynamical system given by

$$\dot{x}(t) = Ax(t) + B\Delta(x(t)) + Bu(t), \quad x(0) = x_0, \quad t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n, t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m, t \geq 0$, is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices such that the pair $(A, B)$ is controllable, and $\Delta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a matched system uncertainty. We assume that the full state is available for feedback and the control input $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m, t \geq 0$. In addition, we consider the reference system given by

$$\dot{x}_m(t) = A_mx_m(t) + B_mr(t), \quad x_m(0) = x_0, \quad t \geq 0 \tag{2}$$

where $x_m(t) \in \mathbb{R}^n, t \geq 0$, is the reference state vector, $r(t) \in \mathbb{R}^r, t \geq 0$, is a bounded piecewise continuous reference input, $A_m \in \mathbb{R}^{n \times n}$ is Hurwitz, and $B_m \in \mathbb{R}^{n \times r}$.

Assumption 2.1: The matched uncertainty in (1) is linearly parameterized as

$$\Delta(x) = W^T \beta(x), \quad x \in \mathbb{R}^n \tag{3}$$

where $W \in \mathbb{R}^{n \times m}$ is an unknown constant weighting matrix and $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^r$ is a basis function of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_r(x)]$.

Here, the aim is to construct a feedback control law $u(t), t \geq 0$, such that the state of the nonlinear uncertain dynamical system given by (1) asymptotically tracks the state of the reference model given by (2) in the presence of matched uncertainty satisfying (3).