# Families of Wildly Ramified Covers of Curves

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**Abstract**

In this paper, I investigate wildly ramified $G$-Galois covers of curves $\phi : Y \to \mathbb{P}^1_k$ branched at exactly one point over an algebraically closed field $k$ of characteristic $p$. I answer a question of M. Raynaud by showing that proper families of such covers of a twisted projective line are isotrivial. The method is to construct an affine moduli space for covers whose inertia group is of the form $I = \mathbb{Z}/p \rtimes \mu_m$. There are two other applications of this space in the case that $I = \mathbb{Z}/p \rtimes \mu_m$. The first uses formal patching to compute the dimension of the space of non-isotrivial deformations of $\phi$ in terms of the lower jump of the filtration of higher inertia groups. The second gives necessary criteria for good reduction of families of such covers. These results will be used in a future paper to prove the existence of such covers $\phi$ with specified ramification data.

## 1 Introduction

Let $R$ be an equal characteristic complete discrete valuation ring with fraction field $K$ and algebraically closed residue field $k$ of characteristic $p$. Let $G$ be a finite group.

### 1.1 Questions and Results

The results of this paper concern families of smooth connected $G$-Galois covers of the projective line which are branched at exactly one point. (See Section 1.2 for precise definitions.) In particular, one can ask:

**Question 1.1.1.** Do there exist complete non-isotrivial families of smooth connected $G$-Galois covers of the projective line branched at exactly one point?

Here a family of covers is isotrivial if the fibres of the family are isomorphic as curves. Question 1.1.1 relates to the following:

**Question 1.1.2.** Given a $G$-Galois cover $\phi : Y_k \to \mathbb{P}^1_k$ of smooth connected curves branched at exactly one point, under what conditions does there exist a deformation of the cover $\phi$ so that the corresponding deformation of the curve $Y$ is non-isotrivial?

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**Question 1.1.3.** Given a $G$-Galois cover $\phi : Y_K \to \mathbb{P}^1_K$ of smooth connected relative curves branched at exactly one $K$-point, under what conditions does $\phi$ have good reduction?

In Section 3, I show that the answer to Question 1.1.1 is no:

**Theorem 3.3.4.** *Let $\Omega$ be a proper irreducible $k$-scheme. Let $P_\Omega$ be a ruled scheme over $\Omega$. Let $\phi_\Omega : Y_\Omega \to P_\Omega$ be a family of $G$-Galois covers of smooth connected curves branched exactly above one section. Let $g$ be the genus of the fibres of $Y_\Omega$. Then $\phi_\Omega$ is isotrivial. In other words, the morphism $\Omega \to M_g$, taking $\omega$ to the isomorphism class of the fibre $Y_\omega$, is constant.*

Theorem 3.3.2 gives another result that proper families of covers of a certain type must be constant. These results are of interest since the moduli space $M_g$ does contain projective subvarieties [Dia87].

The proof of Theorem 3.3.4 uses induction to reduce to the case that the inertia group is of the form $I = \mathbb{Z}/p \rtimes \mu_m$. Under this assumption, the filtration of higher ramification groups is determined by one integer $j$ for which $\gcd(j, p) = 1$, namely by the (lower) jump (also called the conductor).

Suppose that $\phi : Y \to \mathbb{P}^1_k$ is a $G$-Galois cover branched at exactly one point with inertia $I = \mathbb{Z}/p \rtimes \mu_m$ and jump $j$. If $G \neq \mathbb{Z}/p$, then there are several necessary conditions on $j$: there is a congruence condition on $j \mod m$; if $j \equiv a \mod m$ then there exists a lower bound $j_{\min}(I, a)$ for $j$. As a result there is a small set of values $j_{\min}(I)$, depending only on $I$ and not on $G$, consisting of the minimal possible jump for each congruence value of $j \mod m$ (Section 1.4).

With the assumption that $I = \mathbb{Z}/p \rtimes \mu_m$, Section 2 describes a moduli space for $I$-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$ under a certain equivalence. This space is affine and is of positive dimension if and only if $j \notin j_{\min}(I)$. Analyzing the moduli space allows one to answer Question 1.1.2 in this case:

**Theorem 3.1.10.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$. Let $\phi : Y_k \to \mathbb{P}^1_k$ be a $G$-Galois cover of smooth connected curves branched exactly at $\infty$ with inertia $I$ and jump $j$. Suppose $g = \mathrm{genus}(Y_k) \geq 2$. Then there exists a $G$-Galois deformation $\phi_R : Y_R \to \mathbb{P}^1_R$ of $\phi$ over $\mathrm{Spec}(R)$ which is branched only at $\infty_R$ and which induces a non-constant morphism $\mathrm{Spec}(R) \to M_g$ if and only if $j \notin j_{\min}(I)$.*

Furthermore, analyzing the moduli space gives necessary conditions for good reduction for Question 1.1.3 in the case that $I = \mathbb{Z}/p \rtimes \mu_m$. See Proposition 3.2.4 for this result.

These theorems are used in [Pri] to realize the existence of $G$-Galois covers $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at exactly one point for some inertia group $I = \mathbb{Z}/p \rtimes \mu_m \subset G$ and small jump $j \in j_{\min}(I)$. The method is to consider such a cover $\phi$ with sufficiently large jump. Using Theorem 3.1.10 one can construct a non-isotrivial deformation of $\phi$. This deformation yields a proper variety of $\overline{M_g}$ and Theorem 3.3.4 allows one to say that it intersects the boundary. This produces a cover of this type with smaller jump. I would like to thank D. Harbater for help with my thesis, in which I developed a large part of this material.

## 1.2 Notation and Background

Let $k$ be an algebraically closed field of characteristic $p$. Let $R$ be an equal characteristic complete discrete valuation ring with residue field $k$ and fraction field $K$. Let $G$ be a finite group and let $\mathrm{Syl}_p$ be a chosen $p$-Sylow subgroup of $G$.

If $X$ is a scheme over $R$, we assume that the morphism $f : X \to \mathrm{Spec}(R)$ is separated, flat and of finite type. An *R-curve* is a scheme $X$ over $\mathrm{Spec}(R)$ such that each irreducible component of $X$ is of relative dimension 1 over $R$. The $R$-curve $X$ is *proper* if $f$ is proper. Let $X_K = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ and $X_k = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ be the generic and special fibres of $X$ respectively. If $X$ is a $k$-curve and $S$ is a $k$-scheme, let $X_S = X \times_k S$.

Suppose a scheme $X$ is reduced and connected, but not necessarily irreducible. A morphism $\phi : Y \to X$ of schemes is a (possibly branched) *cover* if $\phi$ is finite and generically separable. Note that if $\phi$ is a cover of $R$-curves, then $\phi_k$ may not be generically separable although $\phi_K$ will be. A *G-Galois cover* is a cover $\phi : Y \to X$ along with a choice of homomorphism $G \to \mathrm{Aut}_X(Y)$ by which $G$ acts simply transitively on each generic geometric fibre of $\phi$ (again allowing branching).

Suppose that $\xi$ is a point of a scheme $X$. The *germ* $\hat{X}_\xi$ of $X$ at $\xi$ is defined to be the spectrum of the complete local ring of functions of $X$ at $\xi$. Note that if $\xi$ is a closed point of a curve $X$ then $\hat{X}_\xi$ has dimension 1 if $X$ is a $k$-curve and has dimension 2 if $X$ is an $R$-curve. Suppose $\phi : Y \to X$ is a cover. Let $\eta \in Y$ be a closed point and let $\xi = \phi(\eta)$. The cover $\hat{\phi}_\eta : \hat{Y}_\eta \to \hat{X}_\xi$ will be called the *germ* of $\phi$ at $\eta$. If $\phi$ is Galois and $D$ is the decomposition group of $\phi$ at $\eta$ then $\hat{\phi}_\eta$ is Galois with group $D$.

For all $m \in \mathbb{N}$ with $\gcd(m, p) = 1$, choose an $m$th root of unity $\zeta_m \in k$ such that $\zeta_{m_2} = \zeta_{m_2 m_1}^{m_1}$. For $g \in \mathbb{N}$, let $M_g$ denote the coarse moduli space of proper smooth connected curves of genus $g$, and let $\overline{M}_g$ denote the compactification of $M_g$ as in [DM69]. If $Y$ is a curve of genus $g$, let $[Y]$ denote its isomorphism class. For a $k$-scheme $S$, let $U_S = \underline{\mathrm{Spec}}(O_S[[u^{-1}]])$; if $S = \mathrm{Spec}(A)$ then $U_S = \mathrm{Spec}(A[[u^{-1}]])$.

For completeness, here is some information about inertia groups of wildly ramified covers from [Ser68, Chapter IV]. Consider a (wildly ramified) $G$-Galois cover of curves $\phi : Y \to X$ with branch locus $B$. For a closed point $\xi \in B$. let $\pi_\eta$ be a uniformizer of $Y$ at $\eta \in \phi^{-1}(\xi)$.

**Definition 1.2.1.** a) The *decomposition group D* of $\phi$ at $\eta$ is the subgroup $D = \{g \in G : g(\eta) = \eta\})$. The *inertia group I* of $\phi$ at $\eta$ is the normal subgroup $I \subset D$ which acts by the identity on $\hat{O}_{Y,\eta}/\pi_\eta$. (If $\xi$ is a $k$-point, then $I = D$.)

b) The *filtration of higher ramification groups* $\{I_i : i \in \mathbb{N}^+\}$ of $I$ at $\eta$ is defined as follows: If $g \in I$ then $g \in I_i$ if and only if $g(\pi_\eta) \equiv \pi_\eta \mod \pi_\eta^{i+1}$.

c) The *(lower) jumps* for $\phi$ at $\eta$ are the integers $j > 0$ such that $I_j \neq I_{j+1}$; in other words, $j$ is a lower jump for $\phi$ if and only if $j + 1 = \mathrm{val}(g(\pi_\eta) - \pi_\eta)$ for some $g \in I$.

Note that, up to isomorphism, these objects do not depend on the choice of $\eta$ above $\xi$. If $\eta$ is not a closed point of $X$, the decomposition group, inertia group, filtration of higher ramification groups, and jumps for $\phi$ at $\eta$ are the corresponding objects over the generic point of $\eta$. If $\eta$ is a point of height one then the inertia group $I$ of $\phi$ at $\eta$ is of the form $I = P \rtimes \mu_m$ where $P$ is a $p$-group and $\gcd(m, p) = 1$.

Let $X$ be a smooth connected proper curve of genus $g_X$ with $r_X$ marked points $\{x_i\}$ over an algebraically closed field $k$ of characteristic $p$. Let $\phi : Y \to X$ be a $G$-Galois cover of smooth connected curves branched only at $\{x_i\}$. An important problem is to classify all groups $G$ for which such a cover exists, as well as to classify all the possibilities for the inertia groups above the branch points, along with their ramification filtrations.

Abhyankar's Conjecture [Abh57, Conjecture 1] is the main result in this area.

**Theorem 1.2.2 (Abhyankar's Conjecture).** *(Raynaud, Harbater)*
*Let $G$ be a finite group. Let $p(G) \subset G$ be the subgroup generated by all the Sylow-$p$ subgroups of $G$. Let $X$ be a proper smooth connected curve of genus $g_X$ over an algebraically closed field of characteristic $p$ and let $\{x_i\}$ be a finite set of $r_X$ points of $X$ with $r_X \geq 1$. There exists a $G$-Galois cover $\phi : Y \to X$ of smooth connected curves branched only at $\{x_i\}$ if and only if the quotient $G/p(G)$ can be generated by at most $2g_X + r_X - 1$ generators.*

*Proof.* [Ray94, Corollary 6.5.3]; [Har94, Theorem 6.2]. □

**Definition 1.2.3.** A finite group $G$ is *quasi-$p$* if $G$ is generated by its Sylow-$p$ subgroups.

Furthermore, the inertia groups of $\phi$ in Theorem 1.2.2 can be chosen to be the $p$-Sylow subgroups of $G$ [Har93]. In the case that $X \simeq \mathbb{P}^1_k$ and $r_X = 1$, then the group theoretic condition in Theorem 1.2.2 is that $G = p(G)$ or equivalently that $G$ is quasi-$p$.

**Corollary 1.2.4.** *Let $G$ be a finite group. There exists a $G$-Galois cover $\phi : Y \to \mathbb{P}^1_k$ of smooth connected curves branched at only one point and having inertia group $I = \mathrm{Syl}_p$ if and only if $G$ is quasi-$p$.*

*Proof.* [Ray94, Corollary 6.5.3]. □

Since there are many unanswered questions about inertia groups for $G$-Galois covers $\phi : Y \to X$ of smooth curves over $k$, this paper restricts to the case where $X \simeq \mathbb{P}^1_k$ and where $\phi$ is branched at exactly one point. In this case $G$ must be a quasi-$p$ group.

## 1.3 Inertia Group Invariants

Let $I = \mathbb{Z}/p \rtimes \mu_m$ with $\gcd(m, p) = 1$. Let $c$ denote the chosen generator $\zeta_m$ of $\mu_m$ from Section 1.2 and choose a generator $q$ of $\mathbb{Z}/p$. The invariants $\alpha(I)$ and $a(I)$ defined below give information about the structure of $I$ by describing the action of $\mu_m$ on $\mathbb{Z}/p$.

**Definition 1.3.1.** Let $I = \mathbb{Z}/p \rtimes \mu_m$ with generators $c$ of $\mu_m$ and $q$ of $\mathbb{Z}/p$ as above. Consider the homomorphism $g : \mu_m \to \mathrm{Aut}(\mathbb{Z}/p)$ given by $g(c) : q \mapsto cqc^{-1}$. Define $\alpha(I) \in (\mathbb{Z}/p\mathbb{Z})^*$ to be the number such that $cqc^{-1} = q^{\alpha(I)}$. Let $n' = \#\mathrm{Ker}(g)$ and let $n$ be such that $m = nn'$.

The invariant $\alpha(I)$ does not depend on the choice of generator $q$ (Lemma 1.3.3). Note that the order of the prime-to-$p$ part of the center of $I$ is $n'$. The subgroup $\mathrm{Ker}(g) = \langle c^n \rangle$ is precisely the subgroup which acts trivially on $\mathbb{Z}/p$. The number $n$ satisfies $n|(p-1)$.

**Definition 1.3.2.** Let $I = \mathbb{Z}/p \rtimes \mu_m$, with generators $c$ of $\mu_m$ and $q$ of $\mathbb{Z}/p$ as above. Let $g : \mu_m \to \mathrm{Aut}(\mathbb{Z}/p)$, $\alpha(I) \in (\mathbb{Z}/p\mathbb{Z})^*$, and $n|(p-1)$ be as in Definition 1.3.1. Write $p-1 = n_1 n$. Let $\zeta_{p-1} \in k$ be the chosen root of unity from Section 1.2. Let $a'(I) \in \{1, \dots p-1\}$ be such that $\alpha(I) = \zeta_{p-1}^{a'(I)}$. Define the *invariant* of $I$ to be $a(I) = a'(I)/n_1$.

**Lemma 1.3.3.** *i) The invariant $a(I) \in \mathbb{Z}$, $1 \le a(I) \le n$.*

*ii) The invariant $a(I)$ satisfies* $\gcd(a(I), n) = 1$.

*iii) The invariant $a(I)$ does not depend on the choice of $q$.*

*iv) The invariant $a(I)$ depends on the choice of $c$ as follows: if $c_\circ = c^\beta$ for some $\beta$ such that* $\gcd(\beta, m) = 1$ *then $a_\circ(I) \equiv a(I)\beta \mod n$.*

*Proof.* i) For the first statement, it is sufficient to show that $n_1 | a'(I)$. By Definition 1.3.1, $c^n$ acts trivially on $q$. So $q = c^n q c^{-n} = q^{\alpha(I)^n}$. So $1 = \zeta_{p-1}^{a'(I)n} = \zeta_{n_1}^{a'(I)}$. The second statement is automatic from the bounds on $a'(I)$.

ii) Consider $g : \langle c \rangle \to \mathrm{Aut}(\mathbb{Z}/p)$ where $c : q \to q^{\alpha(I)}$. By definition $n = m/n'$ is the order of $\alpha(I) = \zeta_{p-1}^{a'(I)} \in (\mathbb{Z}/p\mathbb{Z})^*$. Thus $n = (p-1)/\gcd(p-1, a'(I))$ and so $n_1 = \gcd(p-1, a'(I))$. Thus $\gcd(n, a(I)) = 1$.

iii) If $q' = q^\beta$ then $g(c) : q' \mapsto c q' c^{-1}$; but $c q' c^{-1} = c q^\beta c^{-1} = c q c^{-1} \cdots c q c^{-1} = (q^{\alpha(I)})^\beta = (q')^{\alpha(I)}$; since this change of $q$ does not affect $g(c)$, it does not affect $\alpha(I)$ or $a(I)$.

iv) This follows from the fact that $g$ is a homomorphism; in other words, if $c_1 = c^\beta$ (with $\gcd(\beta, m) = 1$) then $g(c_1) : q \mapsto q^{\alpha(I)^\beta}$; thus $\alpha(I) \equiv \alpha(I)^\beta \mod p$; thus $\zeta_{p-1}^{a_1'(I)} = \zeta_{p-1}^{a'(I)\beta}$ and the claim follows by dividing each exponent by $n_1$.

□

## 1.4 Covers of Complete Local Rings

Let $U = \mathrm{Spec}(k[[u^{-1}]])$ and let $\xi$ be the closed point of $U$. (The choice of $u^{-1}$ rather than $u$ simplifies later notation.) Any connected normal Galois cover of $U$ has Galois group $I = P \rtimes \mu_m$ where $I$ is a $p$-group and $\gcd(m, p) = 1$. Assume $I = \mathbb{Z}/p \rtimes \mu_m$ and describe $I$ as in Definition 1.3.1 and Definition 1.3.2. This section describes the structure of $I$-Galois covers $\phi : Y \to U$ of germs of curves with lower jump $j$ in the filtration of higher ramification groups above $\xi$.

**Lemma 1.4.1.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$. Consider generators $c$ and $q$ of $\mu_m$ and $\mathbb{Z}/p$ and $\alpha(I) \in (\mathbb{Z}/p\mathbb{Z})^*$ as in Definition 1.3.1. If $\phi : Y \to U$ is an $I$-Galois cover of germs of smooth connected curves with jump $j$ as in Definition 1.2.1 then the following are true:*

*i) The jump satisfies $\gcd(j, p) = 1$; there exists $f(x) \in k[x]$ with degree $j$ so that the equations for $\phi$ generically can be chosen to be:*

$$x^m = u, y^p - y = f(x).$$

*ii) The Galois action on the generic fibre is given by the following equations for some $\gamma$ with $\gcd(\gamma, m) = 1$ (after possibly changing the choice of $q$)*

$$c(x) = \zeta_m^{\gamma} x, c(y) = y\alpha(I)^{-1}, q(x) = x, q(y) = y + 1.$$

*iii) There is a condition on the exponents of $f(x)$ modulo $m$, namely $f(x) = x^j g_u$ where $g_u \in k[u^{-1}]$. Also, $f(x)$ and $\alpha(I)$ satisfy:*

$$f(x) = \alpha(I) f(\zeta_m^{\gamma} x), \alpha(I) = \zeta_m^{-\gamma j}.$$

*iv) For $n'$ as in Definition 1.3.1, the cover satisfies $n' = \gcd(m, j)$.*

*v) The invariant $a(I)$ satisfies $a(I)n' \equiv -\gamma j \mod m$.*

*Proof.* i) The proof (using Kummer theory, Artin-Schreier theory and [Ser68, Chapter IV]) is classical.

ii) By Kummer theory, (with no change in choice of generator $c$), $c(x) = \zeta_m^{\gamma} x$ for some $\gamma$ such that $\gcd(\gamma, m) = 1$. By Artin-Schreier theory, there is a choice of $q$ with the desired action on $y$. The action of $c$ extends to $y$ and is of the form $c(y) = y\alpha'$ for some $\alpha' \in \mu_m$. Since $cqc^{-1} = q^{\alpha(I)}$ notice that $cq(y) = y\alpha' + 1$ must equal $q^{\alpha(I)} c(y) = y\alpha' + \alpha(I)\alpha'$. Thus $\alpha' = \alpha(I)^{-1}$.

iii) Applying the automorphism $c$ to the equation $y^p - y = f(x)$ and noting that $\alpha(I)^p = \alpha(I)$ we see that $\alpha(I)^{-1}(y^p - y) = f(\zeta_m^{\gamma} x)$. Thus $f(x) = \alpha(I) f(\zeta_m^{\gamma} x)$ which implies that all the exponents of $f(x)$ must be congruent modulo $m$. Since $j$ is the degree of $f(x)$, it follows that $f(x) = x^j g_u$ where $g_u \in k[x^{-m}] = k[u^{-1}]$ and $\alpha(I) = \zeta_m^{-\gamma j}$.

iv) By Definition 1.3.1, $n' = m/n$ where $n$ is the smallest integer such that $g(c^n) = id$. Note that $c^n(q) = q^{\alpha(I)^n}$ and by item iii) $\alpha(I) = \zeta_m^{-\gamma j}$. Thus $n$ is the smallest value among all $n_0$ such that $1 = \zeta_{nn'}^{n_0 \gamma j}$. Since $\gcd(\gamma, m) = 1$ this is only possible if $n' = \gcd(nn', j) = \gcd(m, j)$.

v) This follows immediately from $\zeta_m^{a(I)n'} = \zeta_n^{a(I)} = \zeta_{p-1}^{a'(I)} = \alpha(I) = \zeta_m^{-\gamma j}$.

□

There are some necessary conditions on the invariants associated to a Galois cover $\phi : Y \to \mathbb{P}^1_k$ which is branched at exactly one point. In particular, if $\mathrm{genus}(Y) \geq 2$ then there is a lower bound on $j$ which can be easily computed in the case that $I = \mathbb{Z}/p \rtimes \mu_m$.

**Definition 1.4.2.** Suppose $I \simeq \mathbb{Z}/p \rtimes \mu_m$ with $\gcd(m,p) = 1$. Let $n'$ be the order of the prime-to-$p$ part of the center of $I$. Let $n$ be such that $m = nn'$. Let $a$ be such that $1 \leq a \leq n$ and $\gcd(a,n) = 1$. Define $j_{\min}(I,a) = 2m + n'$ if $a = 1$ and $n = p-1$ and define $j_{\min}(I,a) = m + an'$ otherwise. Define $j_{\min}(I)$ to be the set of all $j_{\min}(I,a)$ for $1 \leq a \leq n$ and $\gcd(a,n) = 1$.

Note that $j \leq m(2 + 1/(p-1))$. Also, if $j \in j_{\min}(I)$ then $\gcd(j,p) = 1$. To see this when $n = p-1$ and $a = 1$, note that $\gcd(n',p) = 1$ and $j_{\min}(I,1) = 2m + n' = n'(2n+1) = n'(2p-1)$ is prime to $p$. If either $n \neq p-1$ or $a \neq 1$, then $j_{\min}(I,a) = n'(n+a)$. Consider the inequalities $2 \leq n + a \leq 2n \leq 2p-2$. The only possibility for $p$ to divide $n + a(I)$ is if $n = p-1$ and $a(I) = 1$ which is the earlier case.

**Lemma 1.4.3.** *Suppose $\phi : Y \to \mathbb{P}^1_k$ is a $G$-Galois cover of smooth connected curves, branched at exactly one point with inertia group $I = \mathbb{Z}/p \rtimes \mu_m$, jump $j$, and $\mathrm{genus}(Y) = g$. Then the values of $m$, $j$, and $g$ satisfy the following conditions:*

*i) The numbers $j$ and $m$ satisfy $j \in \mathbb{N}^+$, $m \in \mathbb{N}^+$, $\gcd(j,p) = 1$, $\gcd(m,p) = 1$; and $j \equiv an' \mod m$ for some $a$ such that $1 \leq a \leq n$ and $\gcd(a,n) = 1$.*

*ii) If $d = \#G$ and $r = d/mp$ is the number of ramification points then the genus satisfies*

$$g = 1 + r(j(p-1) - mp - 1)/2;$$

*iii) If $g = 0$ then $G \simeq \mathbb{Z}/p$, $m = 1$, and $j = 1$. If $g = 1$ then $m = 1$, $p = 3$ and $j = 2$ or $m = 1$, $p = 2$ and $j = 3$. (These cases occur; see Example 1.4.4.)*

*iv) Suppose $g \geq 2$ and $j \equiv an' \mod m$ with $a$ as in i). Then $j \geq j_{\min}(I,a)$.*

*Proof.* i) Let $\xi$ be the branch point of $\phi$ and let $\eta \in f^{-1}(\xi)$. Let $I \subset G$ be the inertia group of $\phi$ at $\eta$. Apply Lemma 1.4.1 to the $I$-Galois cover $\hat{\phi}$ of the germs of the curves at the points $\xi$ and $\eta$.

ii) This follows by simplifying the Riemann-Hurwitz formula, $2g - 2 = -2d + r((mp-1) + j(p-1))$, [Ser68, Proposition 4.4] and [Har77, Corollary 4.2.4].

iii) If $g = 0$ then simplifying (ii) yields $2 = r(mp + 1 - j(p-1))$. Since $r$ divides 2, the subgroup $I$ is normal in $G$. Thus the quotient of $Y$ by $I$ is an unramified cover of $\mathbb{P}^1_k$ and is thus trivial. This implies that $G = I$. Then the Sylow-$p$ subgroup $\mathbb{Z}/p$ is normal in a quasi-$p$ group $G$ and thus $G = \mathbb{Z}/p$. In particular, this implies that $m = 1$. Then simplifying again gives $2 = p + 1 - j(p-1)$ and so $j = 1$.

If $g = 1$ then simplifying (ii) yields $j = m + (m+1)/(p-1)$. Since $n$ divides $p-1$ (Definition 1.3.1) and $j \in \mathbb{Z}$, $n$ must divide $m+1$. Since $n$ also divides $m$, it must be true that $n=1$. Likewise since $n'$ divides both $j$ and $m$, it must divide $m+1$. So $n'=1$ as well. Thus $j=1+2/(p-1)$ is an integer. So $p=2$ or $p=3$ and $j=1+2/(p-1)$ so $j=3$ or $j=2$ respectively.

iv) If $g \geq 2$, the first claim is that $j > m+(m+1)/(p-1)$. By ii), if $g \geq 2$ then $r(j(p-1)-mp-1)/2 \geq 1$. Thus $j(p-1) \geq mp+1+2/r$ and $j \geq m+(m+1)/(p-1)+2/r(p-1)$. If $2/r(p-1) < 1$ then $j > m+(m+1)/(p-1)$ since $j \in \mathbb{Z}$. If $2/r(p-1) \in \{1,2\}$ then the claim can be verified directly.

Now suppose that $a$ is such that $1 \leq a \leq n$ and $\gcd(a,n)=1$ and $j \equiv an' \mod m$. If $n=p-1$ and $a=1$ then $j > m+(m+1)/n = m+n'+1/n$. Thus $j \geq 2m+n' = j_{\min}(I,1)$. If either $n \neq p-1$ or $a \neq 1$, then $an' > (m+1)/(p-1)$ so $j \geq m+an' = j_{\min}(I,a)$.

□

**Example 1.4.4.** Let $G \simeq \mathbb{Z}/p$. There exists a $G$-Galois cover $\phi : Y \to \mathbb{P}^1_k$ branched at one point with inertia $\mathbb{Z}/p$, jump $j$, and $\text{genus}(Y)=g$ if and only if $j \in \mathbb{N}^+$ with $\gcd(j,p)=1$ and $g=(j-1)(p-1)/2$.

*Proof.* In this case $m=r=1$, $d=p$. By Lemma 1.4.3, the conditions are necessary. If $j \in \mathbb{N}^+$ with $\gcd(j,p)=1$ then the equation $y^p - y = x^j$ will determine a cover $\phi$ with jump $j$ and genus $g=(j-1)(p-1)/2$. □

# 2 Moduli Space

## 2.1 Families of Covers

Let $S$ be an irreducible $k$-scheme. A *family of curves* over $S$ is a flat morphism $X \to S$ of schemes of relative dimension 1. A *family of (G-Galois) covers of curves* is a flat $S$-morphism $\phi_S : Y_S \to X_S$ where $Y_S$ and $X_S$ are families of curves over $S$ and the fibres of $\phi_S$ are ($G$-Galois) covers. There are several types of isomorphism between families of Galois covers of curves over $S$.

**Definition 2.1.1.** Let $\phi_S : Y_S \to X_S$ and $\phi'_S : Y'_S \to X_S$ be two families of $G$-Galois covers of smooth connected curves over an irreducible scheme $S$. Let $g$ be the genus of the fibres of $Y_S$.

a) The covers $\phi_S$ and $\phi'_S$ are *isomorphic* if there exists an $S$-isomorphism $f : Y'_S \to Y_S$ such that $f$ is $G$-equivariant and $\phi'_S = \phi_S f$.

b) The covers $\phi_S$ and $\phi'_S$ are *weakly isomorphic* if there exist $S$-isomorphisms $f : Y'_S \to Y_S$ and $f_1 : X_S \to X_S$ such that $f$ is $G$-equivariant and $f_1\phi'_S = \phi_S f$.

c) The cover $\phi_S$ is *constant* if there exists a $G$-Galois cover $\phi_k : Y_k \to X_k$ such that $\phi_S$ is isomorphic to $\phi_k \times_k S : Y_k \times_k S \to X_k \times_k S$.

d) The cover $\phi_S$ is *weakly constant* if there exists a $G$-Galois cover $\phi_k : Y_k \to X_k$ such that $\phi_S$ is weakly isomorphic to $\phi_k \times_k S : Y_k \times_k S \to X_k \times_k S$.

e) The relative curve $Y_S$ is *isotrivial* if the corresponding morphism $\tau : S \to M_g$ taking $S \mapsto [Y_S]$ is constant.

Clearly, (taking the isomorphism $f_1$ to be the identity) if $\phi_S$ and $\phi'_S$ are isomorphic then they are weakly isomorphic, and if $\phi_S$ is constant then it is weakly constant. If $\phi_S$ is weakly constant, then $\phi_S$ and $Y_S$ are isotrivial because the image of $\tau$ in $M_g$ is exactly the unique point of $M_g$ corresponding to $Y_k$. The converse is false since $M_g$ is not a fine moduli space. However, one can say the following:

**Lemma 2.1.2.** *Let $\phi_S : Y_S \to X_S$ be a family of $G$-Galois covers of smooth connected curves over an irreducible $k$-scheme $S$. Suppose that the relative curve $Y_S$ is isotrivial. Suppose the genus of the fibres of $Y_S$ satisfies $g \geq 2$. Then there exists an étale cover $i : S_0 \to S$ with $S_0$ irreducible such that the pullback $\phi_{S_0} = i^*\phi_S$ is weakly constant.*

*Proof.* Choose $l \geq 3$ such that $\gcd(l,p) = 1$. Let $S_2$ be the closed subset of the Jacobian $J = \mathrm{Pic}_S(Y)$ consisting of the kernel of the finite morphism $[l]$ of multiplication by $l$. Then $S_2$ is étale over $S$ with degree $l^{2g}$. Let $S_1$ be an irreducible component of the fibre product over $S$ of the irreducible components of $S_2$. Note that $S_1$ is irreducible and $i : S_1 \to S$ is étale. Consider the pullbacks $J_{S_1}$ and $Y_{S_1}$. The morphism $J_{S_1} \to S_1$ is now equipped with $l^{2g}$ disjoint sections. Thus one can choose a Jacobi structure of level $l$ on $Y_{S_1}$, namely an isomorphism between the $l$-torsion of $J_{S_1}$ and $(\mathbb{Z}/l\mathbb{Z})^{2g}_{S_1}$.

The data of $Y_{S_1}$ and its Jacobi structure of level $l$ gives a morphism $\tau : S_1 \to M_{g,l}$, the fine moduli space of curves of genus $g$ with Jacobi structure of level $l$ [Gro61, Theorem 3.1], [DM69, 5.14]. Furthermore, since $Y_{S_1}$ is isotrivial and $S_1$ is connected, the image of $\tau$ consists of exactly one point which corresponds to an isomorphism class of curves $Y_k$ with a Jacobi structure of level $l$. Thus there is an $S_1$-isomorphism $f_{S_1} : Y_{S_1} \simeq Y_k \times_k S_1$.

The action of $G$ on $Y_{S_1}$ and the $S_1$-isomorphism $f_{S_1}$ induce an $S_1$-homomorphism from $G$ to $\underline{\mathrm{Aut}}_{S_1}(Y_k \times_k S_1)$. Since $g \geq 2$, $\underline{\mathrm{Aut}}_{S_1}(Y_k \times_k S_1)$ is finite and unramified over $S_1$ [DM69, Theorem 1.11]. Thus there is a finite and étale map $S_0 \to S_1$ such that the pullback of $\underline{\mathrm{Hom}}_{S_1}(G, \underline{\mathrm{Aut}}_{S_1}(Y_k \times_k S_1))$ to $S_0$ is trivial. Let $X_k$ be the quotient of $Y_k$ by $G$ and let $\phi'_{S_0} : Y_k \times_k S_0 \to X_k \times_k S_0$ be the constant cover. Replace $S_0$ with an irreducible component of the fibre product over $S$ of the irreducible components of $S_0$. The finite étale cover $i : S_0 \to S$ trivializes the $G$-action induced from $f_{S_0}$ on $Y_k \times S_0$. Define $f_1 : X_{S_0} \to X_k \times_k S_0$ by $f_1 = \phi_{S_0}^{-1} f_{S_0} \phi'_{S_0}$. The cover $i^*\phi_{S_0}$ is weakly isomorphic to $\phi'_{S_0}$ and thus weakly constant by Definition 2.1.1. □

In certain circumstances, one can prove that a family of covers is constant if it is constant near the branch points.

**Proposition 2.1.3.** *Let $X_k$ be a smooth connected curve and let $B_k = \{\xi_i\} \subset X_k$ be a finite set of points. Let $S$ be a connected $k$-scheme. Let $\phi_S : Y_S \to X_k \times_k S$ be a $G$-Galois cover of* *smooth connected curves branched only at $\xi_{i,S} = \xi_i \times_k S$ for $\xi_i \in B_k$. Let $\hat{\phi}_{i,S}$ be the germ of the cover at a point above $\xi_{i,S}$. Then $\phi_S$ is constant if and only if $\hat{\phi}_{i,S}$ is constant for each $\xi_i \in B_k$.*

*Proof.* The forward implication is immediate. For the converse, let $\phi'_S = \phi_k \times_k S : Y_k \times_k S \to X_k \times_k S$. Pull back the cover $\phi_S : Y_S \to X_k \times_k S$ by the constant family $\phi'_S$. By hypothesis, $\hat{\phi}_S \simeq \hat{\phi}'_S$ for each $\xi_i \in B_k$. Thus after normalization, the cover $(\phi'_S)^*\phi_S$ is étale. This unramified cover of the constant family $Y_k \times_k S$ yields a family of étale covers of the proper curve $Y_k$. Such a family of étale covers of a proper curve over an algebraically closed field lifts to characteristic 0 and so must be constant [Gro71, Exposé X, Corollary 3.9 and 2.12]. But the fibre over the chosen $k$-point of $S$ is trivial. So the family is trivial. Thus $\phi'_S \simeq \phi_S$ which implies that $\phi_S$ is constant. □

Since the inertia group and decomposition group of a generic ramification point might not be equal, the following lemma will be needed in Section 3.

**Lemma 2.1.4.** *Let $S$ be an irreducible $k$-scheme. Let $\phi_S : Y_S \to X_S$ be a $G$-Galois cover of smooth connected curves over $S$ branched at a finite set of $S$-sections with disjoint support. Then there exists an irreducible scheme $S'$ and a finite surjective morphism $i : S' \to S$ so that the pullback $\phi_{S'} = i^*\phi_S$ has the following property: for any generic point $\eta$ of the ramification locus of $\phi_{S'}$ the decomposition group $D_\eta$ equals the inertia group $I_\eta$.*

*Proof.* Let $\xi$ be the generic point corresponding to a branch point $\xi_S$. For any $\eta \in \phi_S^{-1}(\xi)$ there exists a $D_\eta$-Galois extension $\hat{\phi}_\eta : \hat{Y}_\eta \to \hat{X}_\xi$. Since $I_\eta$ is normal in $D_\eta$ there is a $D_\eta/I_\eta$-Galois cover of $\hat{X}_\xi$. Since $\xi$ is the generic geometric point corresponding to the $S$-point $\xi_S$, this yields a finite cover $S'_\xi \to S$. Let $S'$ be an irreducible component of the fibre product over $S$ of $S'_\xi$ for all branch points $\xi$ and let $i : S' \to S$. Consider the pullback $\phi_{S'} = i^*\phi_S$. This pullback trivializes the residue field extension at every generic ramification point $\eta$ and thus $D_\eta = I_\eta$. □

Section 2.2 describes a moduli space for certain $I$-Galois covers up to isomorphism for $I = \mathbb{Z}/p \rtimes \mu_m$. A difficulty in parametrizing these covers and counting the dimension of these spaces is that it is possible to change the equations of an $I$-Galois cover without changing its isomorphism class.

**Lemma 2.1.5.** *Let $S$ be an affine $k$-scheme. Suppose $f(x) \in O(S)((x^{-1}))$ and $f_1(x_1) \in O(S)((x_1^{-1}))$. Suppose the degree of $f(x)$ is $j$. Consider an $I$-Galois cover $\phi_S$ (respectively $\phi'_S$) of $U_S = \mathrm{Spec}(O_S[[u^{-1}]])$ given by the following equation: $x^m = u$ and $y^p - y = f(x)$ (respectively $x_1^m = u$ and $y_1^p - y_1 = f_1(x_1)$). There is an $I$-Galois isomorphism between the covers $\phi_S$ and $\phi'_S$ if and only if $f(x_1) = z f_1(x_1) + \delta^p - \delta$ for some $z \in \mu_{p-1}$ and some $\delta \in x^j O(S)((u^{-1}))$.*

*Proof.* Let $c$ and $q$ (respectively $c_1$ and $q_1$) be generators for the $I$-Galois action as in Lemma 1.4.1. If there is an isomorphism $\tau$ between the two covers then $\tau(x) = \zeta_m^a x_1$ for some $a$ since $x^m = u = x_1^m$. Since $\tau$ must commute with the $\mu_m$-Galois action, it follows that $\tau(c) = c_1$. Also since $\tau$ is invertible, $\tau(y) = z_1 y_1 + \delta_1$ for some $z_1, \delta \in O(S)((x^{-1}))$ with $z_1$ a unit. Thus

$$z_1^p y_1^p - z_1 y_1 + \delta_1^p - \delta_1 = \tau(y^p - y) = \tau(f(x)) = \zeta_m^{aj} f(x_1).$$

Substituting $y_1^p = y_1 + f_1(x_1)$ implies that $(z_1^p - z_1)y_1 \in O(S)((x_1^{-1}))$ and thus that $z_1 \in \mu_{p-1}$. Thus $f(x_1) = (z_1 f_1(x_1) + \delta_1^p - \delta_1)/\zeta_m^{aj}$. Note that $\zeta_m^{aj} \in \mu_{p-1}$. The proof follows by setting $z = z_1/\zeta_m^{aj}$ and $\delta = \delta_1/\zeta_m^{aj}$.

Furthermore, since $\tau$ must commute with the $\mathbb{Z}/p$-Galois action, it follows that $\tau(q) = q_1^{a_q}$ where $a_q \in \mathbb{Z}/p^* = \mathbb{F}_p^*$ corresponds to the element $1/z\zeta_m^{aj} \in \mu_{p-1}$. The compatibility $c_1(\tau(y)) = \tau(c(y))$ implies that $c_1(\delta) = \zeta_m^{\gamma j}\delta$ and thus that $\delta \in x^j O(S)((u^{-1}))$.

The converse is straight-forward: if $f(x_1) = z f_1(x_1) + \delta^p - \delta$, the isomorphism $\tau$ is defined by $\tau(x) = x_1$ and $\tau(y) = zy_1 + \delta$. □

The following example of a family of covers with group $I = \mathbb{Z}/p$ illustrates another subtlety that needs to be addressed.

**Example 2.1.6.** Let $S = \mathrm{Spec}(k((t)))$ and let $i : S' \to S$ be the finite radicial morphism given by $(t')^p = t$. Consider the cover $\phi$ of $\mathrm{Spec}(O(S)[[x^{-1}]])$ given by $y^p - y = tx^p$ and the cover $\phi'$ of $\mathrm{Spec}(O(S')[[x^{-1}]])$ given by $y_1^p - y_1 = t'x$. Then there is an isomorphism between $\phi'$ and the pullback $i^*\phi$ taking $y \mapsto y_1 + t'x$.

The rest of Section 2 investigates $I$-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$ in the case that $I = \mathbb{Z}/p \rtimes \mu_m$. The main conclusion used in Section 3 is Proposition 2.2.6 which states that there is an affine space parametrizing these covers whose dimension can be computed explicitly (Notation 2.2.4).

## 2.2 Configuration Space

This section constructs an explicit parameter space $C(I, j)$ associated with a functor of covers of $k[[u^{-1}]]$. Due to the presence of purely inseparable morphisms (Example 2.1.6) the scheme $C(I, j)$ does not have all the properties of a moduli space for the natural functor of covers. As a result, the results will first be phrased in terms of a weaker concept of a *configuration space*. The slightly more complicated situation in which $C(I, j)$ is a fine moduli space is introduced at the end of Section 2.2.

**Definition 2.2.1.** Let $F$ be a contravariant functor from irreducible $k$-schemes $S$ to sets. A $k$-scheme $C$ is a *configuration space* for the functor $F$ if the following conditions hold:

i) There is a morphism $T : \mathrm{Hom}(\circ, C) \to F(\circ)$;

ii) The morphism $T$ induces a bijection between the $k$-points of the configuration space $C$ and $F(\mathrm{Spec}(k))$;

iii) If $\phi_S \in F(S)$ then there exists a finite radicial morphism $i : S' \to S$ and a unique morphism $f : S' \to C$ such that $T(f) = i^*\phi_S$ in $F(S')$.

**Remark 2.2.2.** The morphism $f : S' \to C$ in Definition 2.2.1 (iii) may not descend to $f : S \to C$. As a result, $C$ will not be a moduli space for the functor $F$ unless one considers a category in which finite purely inseparable morphisms between schemes are invertible; see Definition 2.2.9.

This configuration space will parametrize $I$-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$.

**Notation 2.2.3.** Let $I = \mathbb{Z}/p \rtimes \mu_m$ where $m \in \mathbb{N}^+$ with $\gcd(m, p) = 1$. Recall the definition of $n'$ from Section 1.3. Let $j \in \mathbb{N}^+$ with $\gcd(j, p) = 1$ and $n' = \gcd(j, m)$. Let $S$ be an irreducible $k$-scheme. Let $U_S = \underline{\mathrm{Spec}}(O_S[[u^{-1}]])$ and let $\infty_S$ be the $S$-point defined by $u^{-1} = 0$.

Let $F_{I,j}$ be the contravariant functor from irreducible $k$-schemes $S$ to sets defined as follows: let $F_{I,j}(S)$ be the set of equivalence classes of $I$-Galois covers $\phi_S : Y_S \to U_S$ of smooth connected germs of curves which are branched only above $\infty_S$ and whose fibres have constant inertia $I$ and jump $j$. Two such covers $\phi_S$ and $\phi'_S$ will be said to be *equivalent* if they are isomorphic after pullback by a finite radicial morphism $S' \to S$.

Lemma 2.1.5 and Example 2.1.6 illustrate why the configuration space for $F_{I,j}$ (Definition 2.2.5 and Proposition 2.2.6) will be complicated. In particular, the ambiguity of modifying by $\delta^p - \delta$ in Lemma 2.1.5 is addressed in Notation 2.2.4; the ambiguity of $\mu_{p-1}$ in Lemma 2.1.5 explains why the configuration space will be a quotient of an affine space by an action of the cyclic group of $(p-1)$-roots of unity. Finally Example 2.1.6 shows that finite purely inseparable morphisms of the base will be necessary.

**Notation 2.2.4.** Let $I$ and $j$ be as in Notation 2.2.3. Define sets:

$$E_0(I, j) = \{e \in \mathbb{Z} : 1 \le e \le j; e \equiv j \mod m\};$$

$$E(I, j) = \{e \in E_0(I, j) : \forall \nu \in \mathbb{N}^+, p^\nu e \notin E_0(I, j)\}.$$

Define the numbers $r_0 = r_0(I, j)$ and $r = r(I, j)$ to be the cardinality of $E_0(I, j)$ and $E(I, j)$ respectively. In other words, $E_0(I, j)$ consists of the integers $e_i = j - im$ for $0 \le i < j/m$ and $E(I, j)$ consists of the subset of all $e_i$ such that $p^\nu e_i \neq e_k$ for any $\nu \in \mathbb{N}^+$.

The number $r(I, j)$ will be the dimension of the configuration space of $\mathbb{Z}/p \rtimes \mu_m$-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$. The proof of Theorem 3.1.10 shows that $j \notin j_{\min}(I)$ if and only if $r(I, j) \ge 3$.

**Definition 2.2.5.** Consider $I$ and $j$ as in Notation 2.2.3. Let $\tilde{C}(I, j)$ be the affine scheme $\mathbb{G}_m \times \mathbb{G}_a^{r(I,j)-1}$. Let $a_{j-im}$ for $j - im \in E(I, j)$ be the $r(I, j)$ coordinates of $\vec{a} \in \tilde{C}(I, j)$. Consider the action of $\mu_{p-1}$ on $\tilde{C}(I, j)$ which takes $\vec{a}$ to $\zeta_{p-1}\vec{a}$. The *configuration space* $C(I, j)$ is the quotient of $\tilde{C}(I, j)$ by this (possibly non-faithful) action of $\mu_{p-1}$.

**Proposition 2.2.6.** *With $I = \mathbb{Z}/p \rtimes \mu_m$, $j$, $C(I,j)$ and $F_{I,j}$ as in 2.2.3, 2.2.4 and 2.2.5, the scheme $C(I,j)$ is a configuration space for the functor $F_{I,j}$.*

*Proof.* The proof is to verify conditions i) - iii) of Definition 2.2.1.

i) Let $S$ be an irreducible $k$-scheme. Given a morphism $f : S \to C(I,j)$, choose a lifting to a morphism $S \to \tilde{C}(I,j)$. This yields coordinates $a_{j-im} \in O(S)$ for $j - im \in E(I,j)$ with $a_j \in O(S)^*$. Consider the equation given by

$$x^m = u, y^p - y = f(x) = a_j x^j + \dots a_{j-im} x^{j-im} + \dots a_{j-(r_\circ-1)m} x^{j-(r_\circ-1)m}$$

where $a_{j-im} = 0$ if $j - im \notin E(I,j)$. This equation defines an $I$-Galois cover $\phi_S : Y_S \to U_S$ of smooth connected germs of curves which is branched only above $\infty_S$ and has jump $j$. This gives an element $\phi_S \in F_{I,j}(S)$.

The cover $\phi_S$ is well-defined. Changing the lifting multiplies all the coordinates $a_{j-im}$ by a $(p-1)$th root of unity. Let $\phi'_S$ be the cover determined by the equations $x^m = u$ and $(y')^p - y' = \zeta_{p-1} f(x)$ (where the coefficients of $\zeta_{p-1} f(x)$ are the coordinates $\zeta_{p-1} a_{j-im}$ of the new lifting). Then $\phi_S$ and $\phi'_S$ are isomorphic by Lemma 2.1.5 (where the isomorphism takes $y' \mapsto \zeta_{p-1} y$). Thus $\phi_S$ is well-defined. Hence the map $T(S) : \mathrm{Hom}(S, C(I,j)) \to F_{I,j}(S)$ given by $f \mapsto \phi_S$ is well-defined. The map $T(S)$ depends functorially on $S$ and so defines a morphism $T$.

ii) Take $S = \mathrm{Spec}(k)$. By part (i), the morphism $T$ yields a map from the $k$-points of the configuration space $C(I,j)$ to $F_{I,j}(\mathrm{Spec}(k))$. By the statement of part (iii) (proven below) if $\phi_S \in F_{I,j}(S)$ then there exists a finite radicial morphism $i : S' \to S$ and a unique morphism $f : S' \to C(I,j)$ such that $T(f) = i^* \phi_S$ in $F_{I,j}(S')$. However, if $S = \mathrm{Spec}(k)$ then the cover $i : S' \to S$ must be trivial. Thus given $\phi_k \in F_{I,j}(\mathrm{Spec}(k))$ there exists a unique $k$-point $f : \mathrm{Spec}(k) \to C(I,j)$ such that $T(f) = \phi_k$. The existence of $f$ shows that the map $T$ is surjective on $k$-points, and the uniqueness shows that $T$ is injective on $k$-points.

iii) The goal is to show that if $\phi_S \in F(S)$ then there exists a finite radicial morphism $i : S' \to S$ and a unique morphism $f : S' \to C(I,j)$. If $\phi_S \in F_{I,j}(S)$ then $\phi_S$ corresponds to an equivalence class of $I$-Galois covers $\phi_S : Y_S \to U_S$ of smooth connected germs of curves which is branched only above $\infty_S$ and has jump $j$.

It suffices to show the statement for $S$ affine for sheaf theoretic reasons. In particular, if $S$ is not affine, then consider a collection of affine subvarieties which cover $S$. Then the equations for the cover $\phi_S$ are determined by the equations over this affine covering of $S$. The modifications made to these equations are also determined by the modifications over the affine covering.

By Kummer Theory and Artin-Schreier Theory, the cover $\phi_S$ corresponds to an extension of $O(S)[[u^{-1}]]$ given by the equations:

$$x^m = a_0 u, y^p - y = f_S(x)$$

with $a_0 \in O(S)^*$ and $f_S(x) \in O(S)((x^{-1}))$. By Lemma 2.1.5 it is possible to assume that $f_S(x) \in xO(S)[x]$ since any element of $O(S)[[x^{-1}]]$ is of the form $\delta^p - \delta$ for some $\delta \in O(S)[[x^{-1}]]$.

Furthermore, one can choose $f_S(x)$ so that the largest exponent of $x$ which is not divisible by $p$ is $j$ (as in Lemma 1.4.1). If the leading term of $f_S(x)$ is of the form $a_J x^{pJ}$ for some $a_J \in O(S)$, then consider the finite radicial morphism $i : S' \to S$ given by the equation $a^p = a_J$. Then adding $ax^J - (ax^J)^p$ to $f_S(x)$ does not change the isomorphism class of the pullback $\phi_{S'}$ of $\phi_S$. Thus one can remove the term $a_J x^{pJ}$ from the equation. Thus after taking a uniquely determined finite radicial morphism $i : S' \to S$, one can assume that $j$ is the degree of $f_S(x)$. Furthermore, the fact that $\phi_S$ is $I$-Galois forces all the exponents of $f_S(x)$ to be congruent to $j$ mod $m$.

Thus the cover $\phi_{S'}$ corresponds to an extension of $O(S')[[u^{-1}]]$ given by the equations:

$$x^m = a_0 u, y^p - y = f_2(x) = a''_j x^j + \dots a''_{j-im} x^{j-im} + \dots a''_{j-(r_0-1)m} x^{j-(r_0-1)m}$$

where $a''_{j-im} \in O(S')$ for $1 \leq i \leq (r_0 - 1)$, and $a''_j \in O(S')^*$.

Furthermore, if $p^\nu(j - im) = j - i'm$ for some $\nu \in \mathbb{N}^+$ and $1 \leq i, i' \leq (r_0 - 1)$ then one can modify $f_2(x)$ by $(a''_{j-im})^{p^\nu} x^{j-i'm} - a''_{j-im} x^{j-im}$ without changing the isomorphism class of the cover. Thus $\phi_{S'}$ is isomorphic to a cover which is given by the equations

$$x^m = a_0 u, y^p - y = f_1(x) = a'_j x^j + \dots a'_{j-im} x^{j-im} + \dots a'_{j-(r_0-1)m} x^{j-(r_0-1)m}$$

where $a'_{j-im} = 0$ if $p^\nu(j - im) = j - i'm$. In other words $a'_{j-im} = 0$ if $j - im \notin E(I, j)$. Note that it is still the case that $a'_{j-im} \in O(S')$. Also the coefficient $a'_j = a''_j$ since $\gcd(j, p) = 1$ and thus $a'_j \in O(S')^*$.

It is possible to absorb the coefficient $a_0$ into the second equation by taking an étale extension $S'' \to S'$ given by the equation $(a'_0)^m = a_0$ and replacing $x$ by $xa'_0$ in both equations. This changes the coefficients $a'_{j-im}$ to new coefficients $a_{j-im} \in O(S'')$. Note that $a_j \in O(S'')^*$ since $a_0 \in O(S')^*$. The new equations are the following (with $a'_{j-im} = 0$ if $j - im \notin E(I, j)$):

$$x^m = u, y^p - y = f(x) = a_j x^j + \dots a_{j-im} x^{j-im} + \dots a_{j-(r_0-1)m} x^{j-(r_0-1)m}.$$

Consider the map from $S''$ to $\tilde{C}(I, j)$ which is given by:

$$S'' \mapsto (a_j, \dots, a_{j-(r_0-1)m})$$

eliminating the coordinates $a_{j-im}$ for which $j - im \notin E(I, j)$. Note that there are exactly $r(I, j)$ terms by definition.

Furthermore, the choice of $a_0'$ is defined up to multiplication by $\zeta_m$. Multiplying $a_0'$ by $\zeta_m$ multiplies the coefficients $a_{j-im}$ by $\zeta_m^{j-im} = \zeta_m^j = \zeta_n^{j'}$. (Recall that $j = j'n'$ where $n' = \gcd(m, j)$). Thus the map $S'' \to \tilde{C}(I, j)$ is defined up to multiplication of each coordinate by $\zeta_n^{j'} = \zeta_{p-1}^{j'n_1}$. Furthermore, $f(x)$ is determined uniquely up to multiplication by $\zeta_{p-1}$ for all representatives of the isomorphism class of $\phi_S$. Recall that $C(I, j)$ is the quotient of $\tilde{C}(I, j)$ by this action of $\mu_{p-1}$. Thus the map $S'' \to \tilde{C}(I, j)$ descends to a map $f : S' \to C(I, j)$.

Thus $\phi_S$ determines a unique finite radicial morphism $i : S' \to S$ and a well-defined, unique map $f : S' \to C(I, j)(S')$. Note that $T(f) \simeq i^*\phi_S$ in $F_{I,j}(S')$ by construction.

□

The space $C(I, j)$ also parametrizes Katz-Gabber $I$-Galois covers of $\mathbb{P}^1_k$ which have jump $j$. Let $\mathbb{P}^1_S = \mathbb{P}^1_k \times_k S$ and let $0_S = 0 \times_k S$ and $\infty_S = \infty \times_k S$.

**Definition 2.2.7.** Let $S$ be an irreducible $k$-scheme. A cover $\psi : W \to \mathbb{P}^1_S$ is an $I$-Galois *Katz-Gabber cover* if $\psi$ is an $I$-Galois cover of smooth connected curves; the branch locus of $\psi$ is $B = \{\infty_S, 0_S\}$; the cover $\psi$ is totally ramified above $\infty_S$, and is tamely ramified above $0_S$ with inertia group $\mu_m$. Let $F_{KG}(I, j)$ be the Katz-Gabber functor from irreducible $k$-schemes $S$ to sets defined as follows: let $F_{I,j}(S)$ be the set of equivalence classes of $I$-Galois Katz-Gabber covers having jump $j$ above $\infty_S$.

**Lemma 2.2.8.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$. The space $C(I, j)$ is a configuration space for the functor $F_{KG}(I, j)$ parametrizing Katz-Gabber I-Galois covers of $\mathbb{P}^1_k$ with jump $j$.*

*Proof.* The bijections between $k$-points of $C(I, j)$, $I$-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$ with jump $j$, and $I$-Galois Katz-Gabber covers of $\mathbb{P}^1_k$ with jump $j$ follow from Proposition 2.2.6 and the theorem of Katz-Gabber [Kat86, Theorem 1.4.1]. The proofs of (i) and (iii) are the same as for the corresponding statements in Proposition 2.2.6. □

One can show that $C(I, j)$ is a fine moduli space in a slightly different context.

**Definition 2.2.9.** Let $(Sch)'$ be the category whose objects are the objects of the category $(Sch)$ of schemes, and whose morphisms consist of all morphisms of schemes plus formal inverses to finite radicial morphisms between schemes.

Note that any such finite inseparable morphism is a composition of Frobenius morphisms [Har77, IV.2.5]. Thus the category $(Sch)'$ can be obtained by localizing the category $(Sch)$ by the multiplicative system of morphisms which are powers of Frobenius [Har66, Proposition 3.1].

**Theorem 2.2.10.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$, $j$, $C(I, j)$, and $F_{I,j}$ be as in 2.2.3, 2.2.4, and 2.2.5. In the category $(Sch)'$ (where finite radicial morphisms are invertible) the scheme $C(I, j)$ is a fine moduli space for the functor $F_{I,j}$ of equivalence classes of I-Galois covers of $\mathrm{Spec}(k[[u^{-1}]])$ with jump $j$.*

*Proof.* By Proposition 2.2.6, the scheme $C(I,j)$ is a configuration space for $F_{I,j}$ as in Definition 2.2.1. So if $\phi_S \in F_{I,j}(S)$ then there exists a finite radicial morphism $i : S' \to S$ and a unique morphism $f : S' \to C(I,j)$. Also there is a morphism $T : \mathrm{Hom}(\circ, C(I,j)) \to F_{I,j}(\circ)$ for which $T(f) = i^*\phi_S$. Thus in the category $(Sch)'$ where finite radicial morphisms are invertible there is a natural transformation $\tau$ from $F_{I,j}$ to the functor of points of the scheme $C(I,j)$.

This transformation $\tau$ is surjective by Proposition 2.2.6 (i). Suppose that $\phi_S \in F_{I,j}(S)$ is an equivalence class of covers and assume that the corresponding morphism $f : S' \to C(i,j)$ is constant. Then $T(f)$ is constant by Proposition 2.2.6 (i). Recall that $\phi_S$ is isomorphic to $T(f)$ after pullback by a finite radicial morphism. Thus the transformation is injective in this category. Since $\tau$ is an isomorphism, $C(I,j)$ is a fine moduli space for the functor $F_{I,j}$ (in the category $(Sch)'$ where finite radicial morphisms are invertible). $\square$

There are several reasons, however, why analyzing the moduli space $C(I,j)$ does not fully answer the questions in Section 1.1. First, we need to determine when a deformation of $\phi$ corresponding to an $R$-point of $C(I,j)$ is isotrivial. Secondly, the *boundary* of $C(I,j)$ is not well-behaved in the following sense: there are examples of families of covers with good reduction for which the closure of the corresponding locus of $C(I,j)$ is not contained in $C(I,j)$. To resolve these problems it is necessary to consider the action of affine linear transformations on $C(I,j)$.

## 2.3 Affine Linear Transformations

Let $\Omega$ be an irreducible $k$-scheme. Given a $G$-Galois cover $\phi' : Y' \to \mathbb{P}^1_k$ branched at $\infty_k$, let $\phi'_\Omega : Y'_\Omega \to \mathbb{P}^1_\Omega$ be the constant cover corresponding to $\phi'$. Namely $Y'_\Omega = Y' \times_k \Omega$, $\phi'_\Omega = \phi' \times_k \Omega$, and $\phi'_\Omega$ is branched exactly at $\infty_\Omega$.

**Notation 2.3.1.** The group of affine linear transformations $\mathbf{A}_\Omega$ of $\mathbb{P}^1_\Omega$ is the subgroup of $\Omega$-automorphisms of $\mathbb{P}^1_\Omega$ which fix $\infty_\Omega$. If $A \in \mathbf{A}_\Omega$ then $A$ is an $\Omega$-automorphism of $O(\Omega)[u]$ and $A$ can be written as $A(u) = au + b$ for some $a \in O(\Omega)^*$, $b \in O(\Omega)$. Any affine linear transformation can be factored as $A = A_2A_1$ where $A_1(u) = au$ and $A_2(u) = u + d$ for $d = b/a$. If $A \in \mathbf{A}_\Omega$ then $A$ acts upon a cover $\phi_\Omega$ in the following way: $\phi^A = A^{-1}\phi : Y_\Omega \to \mathbb{P}^1_\Omega$. Note that for any cover $\phi$, $\phi^{AB} = (AB)^{-1}\phi = B^{-1}A^{-1}\phi = B^{-1}(\phi^A) = (\phi^A)^B$. The equations of $\phi_\Omega$ are modified under the action of $A$ by replacing $u$ with $A(u)$.

Let $\phi_\Omega : Y_\Omega \to \mathbb{P}^1_\Omega$ be a $G$-Galois cover of smooth connected curves which is branched only above $\infty_\Omega$. Note by Definition 2.1.1 that $\phi_\Omega$ is weakly constant if and only if there exists an affine linear transformation $A \in \mathbf{A}_\Omega$ such that $(\phi_\Omega)^A$ is constant. In fact the condition that $\phi_S$ is weakly constant depends only on the equivalence class of $\hat{\phi}_S$ modulo the action of affine linear transformations.

**Proposition 2.3.2.** *Let $\Omega$ be a connected $k$-scheme. Let $\phi_\Omega : Y_\Omega \to \mathbb{P}^1_\Omega$ be a $G$-Galois cover which is branched only above $\infty_\Omega$. Let $\hat{\phi}_\Omega$ be the germ of the cover at a point above $\infty_\Omega$. Then $\phi_\Omega$ is weakly constant if and only if there exists an affine linear transformation $A \in \mathbf{A}_\Omega$ such that the cover $(\hat{\phi}_\Omega)^A$ of germs of curves is constant.*

*Proof.* Suppose that $\phi_\Omega$ is weakly constant. There exists an affine linear transformation $A \in \mathbf{A}_\Omega$ such that $(\phi_\Omega)^A$ is constant. Thus the germ $(\hat{\phi}_\Omega)^A$ of the cover is constant. For the converse, consider $(\phi_\Omega)^A : Y'_\Omega \to \mathbb{P}^1_\Omega$. By hypothesis, $(\hat{\phi}_\Omega)^A$ is constant. By Proposition 2.1.3, this implies that $(\phi_\Omega)^A$ is constant. Thus $\phi_\Omega$ is weakly constant. □

Thus to investigate isotriviality (and also good reduction), it is natural to consider the orbit of covers of $U_S = \underline{\mathrm{Spec}}(O_S[[u^{-1}]])$ under the action of affine linear transformations. If $S$ is a $k$-scheme and $\overline{I = \mathbb{Z}/p} \rtimes \mu_m$ there is a natural action of $\mathbf{A}_S$ on the $S$ points of $C(I,j)$. In particular, suppose $A \in \mathbf{A}_S$ and $\hat{\phi}_S \in F_{I,j}(S)$ corresponds to an $S$-point of $C(I,j)$. Then the action of $A$ on $\phi_S$ is given by $\phi_S^A = A^{-1}\phi$ which yields another $S$-point of $C(I,j)$.

The following lemma will be used to find a particular representative of the orbit of an equivalence class of covers under affine linear transformations.

**Lemma 2.3.3.** *Given an $S$-point $\tilde{\gamma}$ of $\tilde{C}(I,j)$ there is an étale extension $S'' \to S$ and an affine linear transformation $A \in \mathbf{A}_{S''}$ such that the $S''$-point $\tilde{\gamma}^A$ is contained in the locus of $(1,0,c_{j-2m},\dots)$ of $\tilde{C}(I,j)$. There are finitely many such $A$ with this property.*

*Proof.* Consider the following equations corresponding to the $S$-point $\tilde{\gamma}$ of $\tilde{C}(I,j)$:

$$x^m = u, y^p - y = f(x) = a_j x^j + \dots a_{j-im} x^{j-im} + \dots a_{j-(r_0-1)m} x^{j-(r_0-1)m}$$

for $j - im \in E(I,j)$ with $a_{j-im} \in O(S)$ and $a_j \in O(S)^*$.

Consider the cover $i' : S' \to S$ given by the equation $a_\circ^j = a_j$ which is étale since $a_j \in O(S)^*$ and $\gcd(j,p) = 1$. Let $A = A_2 A_1 \in \mathbf{A}_{S'}$ where $A_1(u) = u a_\circ^{-m}$ and $A_2(u) = u + d$ for some (not yet determined) $d$. (There are finitely many such $A_1$ once $d$ is uniquely determined). First notice that $A_1(x) = x a_j^{-1/j}$ and $A_2(x) = (u+d)^{1/m} = x(1+d/u)^{1/m}$. The goal is to show that after another étale cover $i'' : S'' \to S'$, there exists $d \in O(S'')$ so that $\tilde{\gamma}^A$ is in the locus of $(1,0,c_{j-2m},\dots)$ in $\tilde{C}(I,j)(S'')$.

Let $b_{j-im} = a_{j-im} a_\circ^{-(j-im)} \in O(S')$. The cover $\phi^{A_1}$ is given by:

$$x^m = u, y^p - y = f_1(x) = x^j + \dots + b_{j-im} x^{j-im} + \dots + b_{j-(r_0-1)m} x^{j-(r_0-1)m}.$$

This is because

$$f_1(x) = f(A_1(x)) = \sum_{i=0}^{r_0-1} a_{j-im} x^{j-im} a_j^{-(j-im)/j}.$$

The coefficients satisfy $b_{j-im} = a_{j-im} a_j^{-(j-im)/j}$ by definition.

The cover $\phi^A$ is given by:

$$x^m = u, y^p - y = f_2(x) = x^j + \dots + c^\circ_{j-im} x^{j-im} + \dots + c^\circ_{j-(r_0-1)m} x^{j-(r_0-1)m}$$

where

$$c^\circ_{j-im} = \sum_{h=0}^{i} \binom{\frac{j}{m} - h}{i-h} b_{j-hm} d^{i-h}.$$

To see this compute $f_2(x) = f_1(A_2(x))$.

$$f_2(x) = \sum_{i=0}^{r_0-1} b_{j-im} x^{j-im}(1+d/u)^{(j-im)/m}.$$

Expand the binomial series. Namely,

$$(1+d/u)^{j/m-i} = 1 + (j/m-i)d/u + \binom{j/m-i}{2} d^2/u^2 + \dots.$$

Thus

$$x^{j-im}(1+d/u)^{j/m-i} = x^{j-im} + x^{j-(i+1)m}(j/m-i)d + x^{j-(i+2)m}\binom{j/m-i}{2} d^2 + \dots.$$

The expansion $(1+d/u)^{j/m-i}$ is a Taylor series in $u^{-1}$. However, there exists $f_3(x) \in xO(S')[x]$ (a polynomial with no constant term) with the following properties: $f_2(x) - f_3(x) \in O(S')[[x^{-1}]]$ (the coefficients of the terms of $f_2(x)$ and $f_3(x)$ are the same); and a representative of the isomorphism class of the cover $\phi^A$ is given by the equations $x^m = u$ and $y^p - y = f_3(x)$. The reason is that any element of $O(S')[[x^{-1}]]$ is of the form $\delta^p - \delta$ for some $\delta \in O(S')[[x^{-1}]]$. Thus the covers given by these two equations are isomorphic by Lemma 2.1.5.

Collect terms of $f_3(x)$ of the same degree into coefficients $c^\circ_{j-im}$. For example, $c^\circ_j = 1$, $c^\circ_{j-m} = dj/m + b_{j-m}$ and $c^\circ_{j-2m} = \binom{j/m}{2} d^2 + b_{j-m}(j/m-1)d + b_{j-2m}$.

It is possible to modify the equation $f_3(x)$ by another element of the form $\delta^p - \delta$ without changing the isomorphism class of the cover. As a result one can write

$$f_3(x) = x^j + \dots + c_{j-im}x^{j-im} + \dots + c_{j-(r_0-1)m}x^{j-(r_0-1)m}$$

so that $c_{j-im} = 0$ if $j - im \notin E(I, j)$.

After adjusting the terms by the appropriate element $\delta^p - \delta$, the second coefficient satisfies $c_{j-m} = dj/m + b_{j-m} + \delta_0^p$ for some $\delta_0 \in O(S')[b_1, \dots, b_{j-(r_0-1)m}][d]$. Consider the generically étale finite cover $i'' : S'' \to S'$ given by $0 = h(d) := dj/m + b_{j-m} + \delta_0^p$. For any choice of $d \in O(S'')$ which is a root of $h(d)$, the coefficient $c_{j-im} = 0$.

Thus there exists an étale cover $i'' : S'' \to S'$ and (finitely many) roots $d \in O(S'')$ of $h(d)$ so that $c_{j-m} = 0$ and thus so that $\tilde{\gamma}^A$ lies in the locus $(1, 0, c_{j-2m}, \dots, c_{j-(r_0-1)m})$ in $\tilde{C}(I, j)(S'')$. □

**Remark 2.3.4.** Given $I$ and $j$, the polynomial $h(d)$ defining the generically étale finite cover in the above proof can be explicately described. Let $E_1(I, j) = \{e \in E_0(I, j) | j - m = p^{\nu_e} e \text{ for some } \nu_e \in \mathbb{N}^+\}$. Then $h(d) = b_{j-m} + dj/m + \sum_{e \in E_1(I,j)} (c^\circ_e)^{p^{\nu_e}}$. In particular when $E_1(I, j)$ is empty (for example when $\gcd(j-m, p) = 1$) then $h(d)$ has degree 1 and $c_{j-m} = b_{j-m} + dj/m$. Then the cover $i'' : S'' \to S'$ is trivial and there is a unique choice $d = -b_{j-m}m/j$ in $O(S'')$ so that $c_{j-m} = 0$.

It is possible to define a contravariant functor $F'_{I,j}$ parametrizing equivalence classes of $I$-Galois covers $\hat{\phi}_S : Y_S \to U_S$ branched only at $\infty_S$ with jump $j$ under the action of affine linear transformations. One might ask whether there exists a configuration space $C'(I,j)$ for this functor. By Lemma 2.3.3, there is an affine scheme $C'(I,j)$ of dimension $r(I,j)-2$ (a quotient of $\mathbb{G}_a^{r(I,j)-2}$ by $\mu_{j/n'}$) with the following property: elements of $F'_{I,j}(S)$ are in bijection with finite sets of $S'$-points of $C'(I,j)$ (where $S' \to S$ is a finite radicial morphism). Furthermore, this finite set of $S'$-points is characterized precisely by the polynomial $h(d)$. In the case that $h(d)$ has degree 1, one can prove that the scheme $C'(I,j)$ is a configuration space for the functor $F'_{I,j}$. The details are omitted since they do not have direct applications in Section 3.

# 3 Applications

## 3.1 Deformation

Let $\phi : Y \to \mathbb{P}^1_k$ be a $G$-Galois cover of smooth connected curves branched exactly at $\infty$ over which it has inertia group $I = \mathbb{Z}/p \rtimes \mu_m \subset G$ and jump $j$. Note that $G$ is a quasi-$p$ group. Let $\hat{\phi} : \hat{Y} \to \mathrm{Spec}(k[[u^{-1}]])$ be the $I$-Galois cover of germs of curves determined by $\phi$ at a ramification point above $\infty$. Let $g$ be the genus of $Y$. Recall that $R$ is a complete discrete valuation ring of equal characteristic $p$.

**Definition 3.1.1.** Let $\phi : Y \to \mathbb{P}^1_k$ be a $G$-Galois cover as above. A *deformation* of $\phi$ over $\mathrm{Spec}(R)$ is a $G$-Galois cover $\phi_R : Y_R \to \mathbb{P}^1_R$ with the following properties: the closed fibre of $\phi_R$ is isomorphic to $\phi$; $\phi_R$ is branched at a unique $R$-point $\xi_R$ which specializes to $\infty$ on the closed fibre; and $I$ is still an inertia group of $\phi_R$ over the generic point of $\xi_R$.

**Definition 3.1.2.** A deformation $\phi_R : Y_R \to \mathbb{P}^1_R$ of $\phi$ is *smooth* if each fibre of $Y_R$ corresponds to a point in $M_g$ for some fixed $g$. A smooth deformation is *isotrivial* if the corresponding morphism $\tau : \mathrm{Spec}(R) \to M_g$ taking $\mathrm{Spec}(R) \mapsto [Y_R]$ is constant.

With this definition in mind, Question 1.1.2 can be rephrased as:

**Question 1.1.2** Given a $G$-Galois cover $\phi : Y_k \to \mathbb{P}^1_k$ of smooth connected curves branched at exactly one point, under what conditions does there exist a smooth non-isotrivial deformation $\phi_R : Y_R \to \mathbb{P}^1_R$ of $\phi$?

To answer this question in the case that $I = \mathbb{Z}/p \rtimes \mu_m$, the following technique of Harbater and Stevenson [HS99] will be used. This technique can also be deduced from results of Ferrand and Raynaud, [FR70]; see [Pri00].

Let $X$ be a projective smooth connected reduced $k$-curve. Let $B$ be a finite closed subset of $X$. Let $\pi$ be the maximal ideal of $R$.

**Definition 3.1.3.** A *(relative) thickening problem of covers* for $(X,B)$ consists of the following data:

a) A cover $f : Y \to X$ of geometrically connected reduced projective $k$-curves;

b) For each $b \in B$, a Noetherian normal complete local domain $R_b$ containing $R$ such that $\pi$ is contained in the maximal ideal of $R_b$, along with a finite generically separable $R_b$-algebra $A_b$;

c) For each $b \in B$, a pair of $k$-algebra isomorphisms $F_b : R_b/(\pi) \to \hat{O}_{X,b}$ and $E_b : A_b/(\pi) \to \hat{O}_{Y,b}$ which are compatible with the inclusion morphisms.

d) (for the relative case only) A *thickening* of $X$ (namely a projective normal $R$-curve $X^*$ which is a trivial deformation of $X$ away from $B$) satisfying that $X^*_k \simeq X$ and that the pullback of $X^*$ to the complete local ring at a point $b \in B$ is isomorphic to $R_b$.

**Definition 3.1.4.** A thickening problem is *G-Galois* if $f$ and $R_b \subset A_b$ are $G$-Galois and the isomorphisms $F_b$ are compatible with the $G$-Galois action (for all $b \in B$).

**Definition 3.1.5.** A *solution* to a (relative) thickening problem of covers is a cover $f^* : Y^* \to X^*$ of projective normal $R$-curves, whose closed fibre is isomorphic to $f$, whose pullback to the formal completion of $X^*$ along $X' = X - B$ is a trivial deformation of the restriction of $f$ over $X'$, and whose pullback to the complete local ring at a point $b \in B$ is isomorphic to $R_b \subset A_b$ (and such that $f^*$ is compatible with the isomorphisms above).

**Theorem 3.1.6.** *(Harbater, Stevenson.) Every thickening problem for covers has a solution. The solution is unique if the thickening problem is relative. The solution is G-Galois if the thickening problem is G-Galois.*

*Proof.* [HS99, Theorem 4]. □

A description of a few steps of the proof in the relative $G$-Galois case will be helpful. The data of the thickening problem consists of a $G$-Galois cover $f : Y \to X$, a thickening $X^*$ of $X$ and inclusions $R_b \to A_b$ for $b \in B$. Consider the trivial deformation of $f$ over $R$ away from $B$. In other words, let $X'_{tr}$ be the formal completion of $X^*$ along $X' = X - B$. Let $f_{tr} : Y_{tr} \xrightarrow{G} X'_{tr}$ be the trivial deformation of the restriction of $f$ to $X'$. For each $b \in B$, let $\hat{X}^*_b = \mathrm{Spec}(\hat{O}_{X^*,b})$ and let $\eta \in f^{-1}(b)$. Consider the (possibly disconnected) covers $\hat{f}_b : \hat{Y}_\eta \to \hat{X}^*_b$ determined by the inclusions $R_b \to A_b$.

For each $b \in B$ let $K_b = \mathrm{Spec}(\mathrm{Frac}(\hat{O}_{X,b}))$. The covers $f_{tr}$ and $\hat{f}_b$ are étale over $K_b$ and are isomorphic over $K_b$ by Definition 3.1.3 (c). Let $\tilde{K}_b$ be the formal completion of $\hat{X}^*_b$ along $K_b$. The formal deformations of an étale cover are all trivial [Gro71, I, Corollary 6.2] and hence isomorphic. Thus there exists a unique isomorphism between the pullbacks of $f_{tr}$ and $\hat{f}_b$ to $\tilde{K}_b$. This isomorphism extends the identity on the closed fibre. The covers $f_{tr}$ and $\hat{f}_b$ for $b \in B$ and the isomorphisms over $\tilde{K}_b$ constitute a patching problem as in [HS99]. The (unique) solution [HS99, Theorem 1] to this patching problem yields the $G$-Galois cover $f^* : Y^* \to X^*$. The pullbacks of $f^*$ and $f_{tr}$ to $X'_{tr}$ are isomorphic and the pullbacks of $f^*$ and $\hat{f}_b$ to $\hat{X}^*_b$ are isomorphic.

The technique of [HS99] and the space $C(I,j)$ can be used to find deformations of $G$-Galois covers. Although it is possible to generalize the following lemma to the case where genus$(X)$ and the number of branch points are arbitrary, the statement in this case parallels Theorem 3.1.10.

**Lemma 3.1.7.** *Let $\phi : Y_k \to \mathbb{P}^1_k$ be a $G$-Galois cover of smooth connected curves branched at exactly one point $\infty$ above which it has inertia $I = \mathbb{Z}/p \rtimes \mu_m$ and jump $j$. Let $\vec{a}$ be the point of $C(I,j)$ corresponding to the $I$-Galois cover $\hat{\phi} : \hat{Y}_k \to \mathrm{Spec}(k[[u^{-1}]])$ at a ramification point above $\infty$. Let $S = \mathrm{Spec}(R)$. Smooth deformations of $\phi$ over $S$ are parametrized by the $S$-points of the space $C(I,j)$ whose closed point is $\vec{a}$. In particular, there exists a smooth deformation of $\phi$ over $S$ since $r(I,j) \geq 1$.*

*Proof.* By Proposition 2.2.6, any $S$-point of the space $C(I,j)$ whose closed point is $\vec{a}$ determines an $I$-Galois cover $\hat{\phi}_S$ of $\mathrm{Spec}(O(S)[[u^{-1}]])$ with jump $j$ whose closed fibre is isomorphic to $\hat{\phi}$. Let $X^* = \mathbb{P}^1_S$. The covers $\hat{\phi}_S$ and $\phi$, and the isomorphism between them on the closed fibre, determine a relative $G$-Galois thickening problem as in Definition 3.1.3. The unique solution from Theorem 3.1.6 to this thickening problem by [HS99] yields the deformation $\phi_S$. The cover $\phi_S$ is isomorphic to $\hat{\phi}_S$ over $\hat{X}_S$ and is a trivial deformation of $\phi_k$ away from $\infty$. Thus the deformation $\phi_S$ is smooth since there are no singularities over $\hat{X}_S$ or $\mathbb{A}^1_k$.

For the reverse correspondence, consider any smooth deformation $\phi_S : Y_S \to \mathbb{P}^1_S$ of $\phi$. It is possible to choose a coordinate on $\mathbb{P}^1_S$ so that $\phi_S$ is branched at $\infty_S$. By Lemma 2.1.4 there exists an irreducible scheme $S'$ and a finite cover $i' : S' \to S$ so that the pullback $\phi_{S'}$ has the following property: for any generic ramification point $\eta$ of $\phi_{S'}$ the decomposition group $D_\eta$ equals the inertia group $I_\eta$. Consider the cover $\hat{\phi}_{S'}$ of $U_{S'}$ near $\infty_{S'}$. By Proposition 2.2.6, there exists a finite radicial morphism $i' : S'' \to S'$ and a unique morphism $f : S'' \to C(I,j)$ such that $\vec{a}$ is the image of the closed point under $f$.

In particular, since $r(I,j) \geq 1$ for all $j$, there exist non-constant $S$-points of $C(I,j)$. □

**Proposition 3.1.8.** *In the situation of Lemma 3.1.7, suppose* genus$(Y_k) \geq 2$. *A smooth deformation $\phi_S$ of $\phi_k$ is isotrivial if and only if there exists an étale cover $S' \to S$ so that the pullback $\phi_{S'}$ is in the orbit of $\phi$ under $\mathbf{A}_{S'}$. Furthermore, the smooth deformation $\phi_S$ is isotrivial if and only if there exists some étale cover $S' \to S$ and some $A \in \mathbf{A}_{S'}$ such that $(\hat{\phi}_{S'})^A$ is constant.*

*Proof.* The statements are automatic from Lemma 2.1.2 and Proposition 2.3.2. □

**Lemma 3.1.9.** *In the situation of Lemma 3.1.7, suppose* genus$(Y_k) \geq 2$. *Then smooth non-isotrivial deformations of $\phi$ over $S$ are parametrized by the $S$-points of the space $C(I,j)$ whose closed point is $\vec{a}$ and which are not contained in the orbit of $\vec{a}$ under the action of $\mathbf{A}_{S'}$ for any finite étale cover $S' \to S$. In particular, there exists a smooth non-isotrivial deformation of $\phi$ over $S$ if and only if $r(I,j) \geq 3$.*

*Proof.* By Lemma 3.1.7, smooth deformations of $\phi$ over $S$ are parametrized by $S$-points of the space $C(I,j)$ for which the base point maps to $\vec{a}$. By Proposition 3.1.8, the condition that the deformation is isotrivial is equivalent to whether there exists some étale cover $S' \to S$ and some $A \in \mathbf{A}_{S'}$ such that $(\hat{\phi}_{S'})^A$ is constant. There is a two dimensional orbit of $\vec{a}$ under the action of $\mathbf{A}$. Thus there exist $S$-points with base point $\vec{a}$ which are not contained in such an orbit if and only if $r(I,j) \geq 3$. □

The following theorem characterizes the values of $j$ for which there is a non-isotrivial smooth deformation.

**Theorem 3.1.10.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$. Let $\phi : Y_k \to \mathbb{P}^1_k$ be a G-Galois cover of smooth connected curves branched only at $\infty$ with inertia $I$ and jump $j$. Suppose* $\text{genus}(Y_k) \geq 2$. *Then there exists a non-isotrivial smooth deformation of $\phi$ over $R$ if and only if $j \notin j_{\min}(I)$.*

*Proof.* The configuration space $C(I,j)$ gives an $r = r(I,j)$ dimensional space of (possibly isotrivial) smooth deformations. By Lemma 3.1.9, there exists a smooth non-isotrivial deformation of $\phi$ over $S = \text{Spec}(R)$ if and only if $r \geq 3$. The claim is that $r \geq 3$ if and only if $j \notin j_{\min}(I)$. By Lemma 1.4.3 and Definition 1.4.2, there is some $a$ such that $1 \leq a \leq n$, $\gcd(a,n) = 1$, $j \equiv an' \mod m$ and $j \geq j_{\min}(I,a)$. Thus it is sufficient to show $r \geq 3$ if and only if $j \geq j_{\min}(I,a) + m$.

For the proof in either direction, it is possible to assume that $j \geq 2m + an'$. If $r \geq 3$ then $r_0 \geq 3$ and hence $j \geq 2m + an'$. If $j \geq j_{\min}(I,a) + m$ then $j \geq 2m + an'$.

Let $j_0 = j$, $j_1 = j - m$, and $j_2 = j - 2m$ and note that $j_0, j_1, j_2 \in E_0(I,j)$.

Then $r < 3$ if and only if $\{j_0, j_1, j_2\} \not\subset E(I,j)$. But this is the case if and only if $j_i = p^\nu j_j$ for some $\nu \in \mathbb{N}^+$. Since $\gcd(j,p) = 1$, $r < 3$ if and only if $j_1 = p^\nu j_2$. By Lemma 1.4.1, if $n'$ is the order of the tame part of the center of $I$, then $j = n'j'$ and $m = nn'$. Thus $r < 3$ if and only if $j' - n = p^\nu(j' - 2n)$ which is if and only if $j' = n(2p^\nu - 1)(p^\nu - 1)^{-1} = 2n + n(p^\nu - 1)^{-1}$. Now $n|(p-1)$ so $r < 3$ if and only if $\nu = 1$, $n = p - 1$, and $j' = 2n + 1$.

For the first case, suppose that if $n = p - 1$ then $a \neq 1$. Then $r \geq 3$ implies $j \geq 2m + an' = j_{\min}(I,a) + m$. Also, the last condition of the previous paragraph is not satisfied, so $j \geq 2m + an' = j_{\min}(I,a) + m$ implies that $r \geq 3$.

For the second case, suppose that $n = p - 1$ and $a = 1$. Then $r \geq 3$ implies that $j \geq 2m + an'$ and $j' \neq 2n + 1$. Thus $r \geq 3$ implies $j \neq 2m + n'$ and so $j \geq 3m + an' = j_{\min}(I,a) + m$. Also, if $j \geq j_{\min}(I,a) + m$ then $j \neq 2m + n'$ so $j' \neq 2n + 1$. Thus the last condition of the second to last paragraph is not satisfied and so $r \geq 3$. □

In conclusion, Theorem 3.1.10 answers Question 1.1.2 in the case that $p$ strictly divides the order of the inertia group by considering the configuration space $C(I,j)$ of covers of germs of curves under the action of affine linear transformations.

## 3.2 Good Reduction

**Notation 3.2.1.** Let $R$ be an equal characteristic complete discrete valuation ring with fraction field $K$ and residue field $k$. Let $G$ be a quasi-$p$ group. Let $I = \mathbb{Z}/p \rtimes \mu_m \subset G$.

Let $\phi_K : Y_K \to \mathbb{P}^1_K$ be a family of $G$-Galois covers ramified only over $\infty_K$ over which it has inertia $I$ and jump $j$. By Lemma 2.1.4, after a finite extension of $K$, one can assume that the germ $\hat{\phi}_K : \hat{Y}_K \to \mathrm{Spec}(K[[u^{-1}]])$ of the cover near the branch point is Galois with group $I$.

By Proposition 2.2.6, after a finite extension of $K$, one can associate the following data to $\hat{\phi}_K$: a unique $K$-point $\gamma : \mathrm{Spec}(K) \to C(I, j)$ in the first configuration space; and an Artin-Schreier equation $f_K(x) \in K[x]$ of degree $j$ describing this $K$-point of $C(I, j)$ (which is unique up to multiplication by $\zeta_{p-1}$).

Suppose $A$ is an affine linear transformation over $K$. Let $\gamma^A$ be the $K$-point in the configuration space $C(I, j)$ corresponding to $A^{-1}\phi_K : Y_K \to \mathbb{P}^1_K$ which is described by $f_K(A(x))$. Let $A_0$ be one of the finitely many affine linear transformations such that the $K$-point $\gamma_0 = \gamma^{A_0}$ corresponding to $\phi^{A_0}$ is in the image of the locus of $(1, 0, c_{j-2m}, \ldots)$ in $\tilde{C}(I, j)$. Note that $A_0 \in \mathbf{A}_{K'}$ for some field extension $K \to K'$. Let $f_0(x)$ be an Artin-Schreier equation corresponding to the cover $\phi^{A_0}$.

**Definition 3.2.2.** The cover $\phi_K$ has *good reduction* if (after a finite extension of $K$) there exists a normalization $\phi'_R : Y'_R \to \mathbb{P}^1_R$ of $\phi_K$ such that

a) the covers $\phi'_K$ and $\phi_K$ are weakly isomorphic;

b) the curve $Y'_k$ is smooth;

c) the cover $\phi'_k$ is generically étale (automatic from (b) if $\mathrm{genus}(Y_K) \geq 2$).

The weak isomorphism condition occurs in Definition 3.2.2 for the following reason. The cover $\phi_K$ will have good reduction if and only if there exists a choice of an integral model $\mathbb{P}^1_R$ for $\mathbb{P}^1_K$ such that the normalization $Y_R$ of $\mathbb{P}^1_R$ in $Y_K$ is smooth and the cover is generically étale over the special fibre. The choice of an integral model $\mathbb{P}^1_R$ depends on the choice of a parameter for $\mathbb{P}^1_K$. Thus the choice of an integral model is given up to an automorphism of $\mathbb{P}^1_K$, in other words up to weak isomorphism. Without loss of generality, the wild branch point of $\phi'_R$ can be chosen to be $\infty_R$. Thus the affine linear transformations $\mathbf{A}_K$ are the only automorphisms of $\mathbb{P}^1_K$ which are necessary to consider.

There are examples of families of covers of curves over $\mathrm{Spec}(K)$ which have good reduction over $R$, but for which the corresponding $K$-point $\gamma$ of $C(I, j)$ does not extend to an $R$-point of $C(I, j)$.

**Example 3.2.3.** The equation $y^p - y = tx^2 + x$ yields an example of a cover $\phi_K$ which has good reduction but for which $\gamma$ does not extend to a morphism $\bar{\gamma} : \mathrm{Spec}(R) \to C(I, j)$: The problem is that there exists an affine linear transformation $A$ taking this family to one which is isomorphic to the constant family given by $y^p - y = x^2$. In particular, $A = A_2A_1$ where $A_1(x) = x/\sqrt{t}$ and $A_2(x) = x - (2\sqrt{t})^{-1}$. This constant family has good reduction.

This example shows that if a $K$-point of $C(I, j)$ does not extend to an $R$-point of $C(I, j)$ then the corresponding family of covers may still have good reduction. For this reason, Proposition 3.2.4 investigates good reduction in terms of orbits of points of $C(I, j)$ under the action of affine linear transformations on $I$-Galois covers of germs of curves.

**Proposition 3.2.4.** *Let $I = \mathbb{Z}/p \rtimes \mu_m$. Suppose $\phi_K : Y_K \to \mathbb{P}^1_K$ is a $G$-Galois cover of smooth connected curves over* $\mathrm{Spec}(K)$ *which is branched exactly at $\infty_K$ over which it has inertia $I$ and jump $j$. Let $\gamma_K$ be the* $\mathrm{Spec}(K)$*-point of $C(I, j)$ corresponding to $\hat{\phi}_K$. If the family $\phi_K : Y_K \to \mathbb{P}^1_K$ has good reduction then (after a finite extension of $K$) there exists an affine linear transformation $A \in \mathbf{A}_K$ such that $\gamma^A$ extends to a morphism $\bar{\gamma}^A$ :* $\mathrm{Spec}(R) \to C(I, j)$.

*Proof.* If $\phi_K$ has good reduction then by Definition 3.2.2 (after possibly extending $K$ and $R$) there exists a normalization $\phi'_R : Y'_R \to \mathbb{P}^1_R$ of $\phi_K$ such that the covers $\phi'_K$ and $\phi_K$ are weakly isomorphic, the curve $Y'_k$ is smooth, and the cover $\phi'_k$ is generically étale. Thus there exists $A \in \mathbf{A}_K$ such that $\phi_K^A \simeq \phi'_K$. By Proposition 2.2.6, after a finite extension of $K$ there is a $K$-point $\gamma^A$ of $C(I, j)$ corresponding to the cover $\hat{\phi}_K^A \simeq \hat{\phi}'_K$. The cover $\hat{\phi}'_R$ yields an $R$-point of $C(I, j)$ by the conditions on $\phi'_k$. Thus $\gamma^A$ extends to a morphism $\bar{\gamma}^A : \mathrm{Spec}(R) \to C(I, j)$. □

The converse of Proposition 3.2.4 is only clear for the trivial case $G = \mathbb{Z}/p\mathbb{Z}$.

**Lemma 3.2.5.** *With notation as in 3.2.1, suppose that $G = \mathbb{Z}/p\mathbb{Z}$. Suppose $\phi_K : Y_K \to \mathbb{P}^1_K$ is a family of $G$-Galois covers corresponding to a $K$-point $\gamma : \mathrm{Spec}(K) \to C(I, j)$ in the configuration space. Then the family $\phi_K$ has good reduction if and only if (after a finite extension of $K$) there exists an affine linear transformation $A \in \mathbf{A}_K$ such that $\gamma^A$ extends to a morphism $\bar{\gamma}^A : \mathrm{Spec}(R) \to C(I, j)$.*

*Proof.* The forward direction follows from Proposition 3.2.4. For the converse, Lemma 2.2.8 states that the $R$-point $\bar{\gamma}^A$ of $C(I, j)$ gives a family $\phi_R^A : Y_R \to \mathbb{P}^1_R$ corresponding to the equation $f_R(A(x))$. The cover $\phi_R^A$ is a generically étale $\mathbb{Z}/p\mathbb{Z}$-Galois cover of smooth connected relative curves which is weakly isomorphic to $\phi_K$ on the generic fibre. Thus $\phi_K$ has good reduction. □

**Remark 3.2.6.** In trying to prove the converse of Proposition 3.2.4 in the case $G \neq \mathbb{Z}/p\mathbb{Z}$ the following problem develops. Consider a $G$-Galois cover $\phi_K^A : Y_K \to \mathbb{P}^1_K$ branched at $\infty_K$ for which the corresponding $K$-point $\gamma^A$ extends to a morphism $\bar{\gamma}^A : \mathrm{Spec}(R) \to C(I, j)$. Let $\phi_R^A : Y_R \to \mathbb{P}^1_R$ denote the normalization of $\mathbb{P}^1_R$ in $Y_K$ for $\phi_K^A$ and choose an isomorphism $R \simeq k[[t]]$. It is sufficient to show that there is a good model for the cover over $\mathrm{Spec}(R[[x^{-1}]])$ by [Ray94, 6.3.2], (also see [Saï00, 1.7]). By hypothesis the cover of $\mathrm{Spec}(K[[x^{-1}]])$ (the completion of the generic fibre) extends to an $R$-point of $C(I, j)$. However, this may not be sufficient to show that there is a good model.

To see this, let $A_{x,t} = k[[x^{-1}, t]][t^{-1}]$. Suppose (for example) that the cover $(\hat{\phi}_R)_K$ : $(\hat{Y}_R)_K \to \mathrm{Spec}(A_{x,t})$ (the generic fibre of the completion) is given by the equation $y^p - y = x^j + t^{-1}x^{-1}$. The pullback of $(\hat{\phi}_R)_K$ to $\mathrm{Spec}(k((t))[[x^{-1}]])$ is isomorphic to the cover given by the equation $y^p - y = x^j$. This is because there exists $\delta \in k((t))[[x^{-1}]]$ so that $t^{-1}x^{-1} = \delta^p - \delta$. However, $\delta \notin A_{x,t}$ so the two covers are not isomorphic over $\mathrm{Spec}(A_{x,t})$. Thus while the latter cover clearly extends to an $R$-point of $C(I, j)$ it is not clear if the former has good reduction over $\mathrm{Spec}(R[[x^{-1}]])$.

Suppose $\phi : Y_K \to \mathbb{P}^1_K$ is a $G$-Galois family of covers of smooth connected $K$-curves, branched only at $\infty_K$ with inertia $I$ and jump $j$. Proposition 3.2.4 gives a necessary condition for $\phi$ to have good reduction. In addition, Proposition 3.2.8 will give an explicit way to show that this condition is not satisfied for a given family by restricting the set of affine linear transformations which can take it to a family with a good model. To do this, the following lemma will be helpful.

**Lemma 3.2.7.** *Suppose that $A_R \in \mathbf{A}$ is an affine linear transformation over $R$. Let $\phi_K : Y_K \to \mathbb{P}^1_K$ correspond to a $K$-point $\gamma : \mathrm{Spec}(K) \to C(I,j)$. Then $\gamma$ extends to a morphism $\bar{\gamma} : \mathrm{Spec}(R) \to C(I,j)$ if and only if $\gamma^{A_R}$ extends to a morphism $\bar{\gamma}^{A_R} : \mathrm{Spec}(R) \to C(I,j)$.*

*Proof.* The affine linear transformation $A_R$ is defined by $A_R(x) = a_R x + b_R$ with $a_R \in R^*$ and $b_R \in R$. Since $a_R \in R^*$, the leading coefficients of $f_K(x)$ and $f_K(A_R(x))$ have the same valuation in $R$. Thus either both or neither vanish on the special fibre. Furthermore, the coefficients of $f_K(x)$ are in $R$ if and only if the coefficients for $f_K(A_R(x))$ are in $R$. □

**Proposition 3.2.8.** *With notation as in 3.2.1, suppose $\phi_K : Y_K \to \mathbb{P}^1_K$ corresponds to a $K$-point $\gamma : \mathrm{Spec}(K) \to C(I,j)$. Let $A_0$ be one of the finitely many affine linear transformation such that (after a finite extension of $K$) the $K$-point $\gamma^{A_0}$ corresponding to $\phi_K^{A_0}$ is in the image of the locus of $(1, 0, c_{j-2m}, \dots)$ in $\tilde{C}(I,j)$. If $\phi_K$ has good reduction then $\gamma^{A_0}$ extends to a morphism $\bar{\gamma}_0 : \mathrm{Spec}(R) \to C(I,j)$.*

*Proof.* Suppose that the family $\phi_K : Y_K \to \mathbb{P}^1_K$ has good reduction. By Proposition 3.2.4, this implies that there exists an affine linear transformation $A \in \mathbf{A}_K$ such that $\gamma^A$ extends to a morphism $\bar{\gamma}^A : \mathrm{Spec}(R) \to C(I,j)$. The goal is to show that $\gamma_0 = \gamma^{A_0}$ also extends to a morphism $\bar{\gamma}^{A_0} : \mathrm{Spec}(R) \to C(I,j)$.

Let $A' = A_0^{-1} A$. Write $A' = A'_2 A'_1$ where $A'_1(u) = au$ and $A'_2(u) = u + d$ for some $a \in K$ and $d \in K$. The first claim is that $a \in R^*$. Suppose $\gamma_0^{A'} = \gamma^A$ is given by the equation $y^p - y = a_j x^j + a_{j-m} x^{j-m} + \dots$. Note that $a_j \in R^*$ since $\gamma_0^{A'}$ extends to a morphism $\bar{\gamma}^A : \mathrm{Spec}(R) \to C(I,j)$. But then $a = a_j^{m/j}$ since the coefficient of $x^j$ for the equation for $\gamma_0$ is 1. So $a \in R^*$ also.

The second claim is that $d \in R$. If $A'_2(u) = u + d$ then $A'_2(x) = x(1 + d/u)^{1/m}$. By hypothesis, the second coefficient for $\gamma_0^{A'}$ satisfies $a_{j-m} \in R$. The cover $\phi_0^{A'_1}$ is given by the equation $y^p - y = a_j x^j + 0 x^{j-m} + \dots$. Thus $(\phi_0^{A'_1})^{A'_2}$ is given by the equation $y^p - y = a_j x^j (1 + d/u)^{j/m} + \dots$. This implies that $a_{j-m} = a_j d j/m$. But $a_j$ is a unit, so if $a_{j-m} \in R$ then $d \in R$.

Thus $A'_2$ and $A'_1$ are affine linear transformations over $R$. Thus by Lemma 3.2.7, $\gamma_0$ extends to a morphism $\bar{\gamma}_0 : \mathrm{Spec}(R) \to C(I,j)$ if and only if $(\gamma_0^{A'_1})^{A'_2} = \gamma_0^{A'}$ extends to a morphism $\bar{\gamma}_0^{A'} : \mathrm{Spec}(R) \to C(I,j)$. But $\phi_0^{A'} = (\phi^{A_0})^{A'} = \phi^{A_0 A'} = \phi^A$. Thus by hypothesis, $\gamma_0^{A'}$ extends to a morphism $\bar{\gamma}_0^{A'} : \mathrm{Spec}(R) \to C(I,j)$, which finishes the proof. □

## 3.3 Complete Families

Let $\Omega$ be a proper irreducible $k$-scheme. This section investigates proper families $\phi_\Omega : Y_\Omega \to X_\Omega$ of $G$-Galois covers of smooth connected relative curves under certain conditions. Theorem 3.3.2 shows that if the base $X_\Omega$ and the branch locus of $\phi_\Omega$ are constant then $\phi_\Omega$ is isotrivial. Theorem 3.3.4 shows that if the base is a twisted projective line and if $\phi_\Omega$ is branched at exactly one $\Omega$-point then $\phi_\Omega$ is isotrivial. The proofs use induction and the moduli space from Section 2.

**Proposition 3.3.1.** *Suppose that $\Omega$ is a proper irreducible $k$-scheme. Let $I = P \rtimes \mu_m$ where $P$ is a $p$-group and $\gcd(p,m) = 1$. Let $U_\Omega = \underline{\mathrm{Spec}}(O(\Omega)[[u^{-1}]])$. Suppose that $\hat{\phi}_\Omega : \hat{Y}_\Omega \to U_\Omega$ is an $I$-Galois cover of smooth connected germs of curves. Then there exists a finite radicial morphism $i' : \Omega' \to \Omega$ such that $i^*\hat{\phi}_\Omega$ is constant.*

*Proof.* The proof consists of induction on $\#P$. If $\#P = 1$ and the cover $\hat{\phi}_\Omega$ is prime-to-$p$ then the cover lifts to characteristic 0 and so must be constant, [Gro71, X, Corollary 3.9]. Now suppose the statement is true for any inertia group $I = P' \rtimes \mu_m$ with $\#P' < \#P$.

Consider the cover $\hat{\phi}_\Omega$ which is Galois with group $I = P \rtimes \mu_m$. By the preceding paragraph, the $\mu_m$-Galois quotient must be constant. Thus $\hat{\phi}_\Omega$ is a $P$-Galois cover of a constant germ of a curve $\underline{\mathrm{Spec}}(O(\Omega)[[x^{-1}]]) \simeq U_\Omega$. (Note that via this isomorphism the $P$-Galois cover is branched over the $\Omega$-point $u^{-1} = 0$.) Choose a normal subgroup $P'$ of $P$ with index $p$. The $P$-Galois cover of the constant germ of the curve $U_\Omega$ consists of a $\mathbb{Z}/p$-Galois quotient $\hat{\phi}_1$ and a $P'$-Galois subcover $\hat{\phi}_2$.

Let $j$ be the jump of the cover $\hat{\phi}_1$ over the $\Omega$-point $u^{-1} = 0$. Then $\hat{\phi}_1$ is an element of $F_{\mathbb{Z}/p,j}(\Omega)$. By Proposition 2.2.6, there exists a finite radicial morphism $i_0 : \Omega_0 \to \Omega$ and a unique morphism $f : \Omega_0 \to C(\mathbb{Z}/p, j)$. The space $C(\mathbb{Z}/p, j)$ is affine, but $\Omega$ and thus $\Omega_0$ are connected and proper. Thus the image of $f$ must be constant. Let $\hat{\phi}'_k : Z_k \to U_k$ be the $\mathbb{Z}/p$-Galois cover in $F_{\mathbb{Z}/p,j}(k)$ corresponding to the point in the image. Then $i_0^*\hat{\phi}_1 \simeq \hat{\phi}'_k \times_k \Omega_0$ is constant.

Since $i_0^*\hat{\phi}_1$ is constant, the cover $i_0^*\hat{\phi}_2$ is a cover of a constant germ of a curve. Since $\#P' < \#P$ the induction hypothesis applies and thus there exists a finite radicial morphism $i' : \Omega' \to \Omega_0$ so that $(i')^* i_0^*\hat{\phi}_2$ is constant. Thus there exists a finite radicial morphism $i : \Omega' \to \Omega$ such that $i^*\hat{\phi}_\Omega$ is constant. □

**Theorem 3.3.2.** *Let $\Omega$ be a proper irreducible $k$-scheme. Let $X_k$ be a smooth connected curve and let $B_k = \{\xi_i\} \subset X_k$ be a finite set of points. Let $\phi_\Omega : Y_\Omega \to X_k \times_k \Omega$ be a $G$-Galois cover of smooth connected curves branched only at $\xi_{i,\Omega} = \xi_i \times_k \Omega$ for $\xi_i \in B_k$. Then there exists a finite surjective morphism $i : \Omega' \to \Omega$ so that $i^*\phi_\Omega$ is constant. In particular $\phi_\Omega$ is isotrivial.*

*Proof.* By Lemma 2.1.4, there exists an irreducible scheme $\Omega''$ and a finite cover $i'' : \Omega'' \to \Omega$ so that the pullback $\phi_{\Omega''}$ has the following property: for any generic ramification point $\eta$ of $\phi_{\Omega''}$ the decomposition group $D_\eta$ equals the inertia group $I_\eta$. Let

$U_{\Omega''} = \underline{\mathrm{Spec}}(O(\Omega'')[[u^{-1}]])$ be the germ of the curve $X_k \times_k \Omega''$ at $\xi_{i,\Omega''}$. Let $\hat{\phi}_{i,\Omega''}$ be the germ of the cover at a point above $\xi_{i,\Omega''}$.

Fix $\xi_i \in B_k$. The cover $\hat{\phi}_{i,\Omega'}$ is Galois with group $I = P \rtimes \mu_m$. By Lemma 3.3.1, there exists a finite radicial morphism $i' : \Omega' \to \Omega''$ such that the pullback $\hat{\phi}_{\Omega'}$ is constant. By Proposition 2.1.3, if $\hat{\phi}_{i,\Omega'}$ is constant for each $\xi_i \in B_k$, then $\phi_{\Omega'}$ is constant. Since there exists a finite surjective morphism $i : \Omega' \to \Omega$ so that $i^*\phi_\Omega$ is constant, the fibres of $\phi_\Omega$ are isomorphic and thus the family is isotrivial. □

The rest of Section 3.3 answers Question 1.1.1. For any smooth connected $k$-scheme $\Omega$ let $P_\Omega$ denote a *ruled scheme* over $\Omega$. In other words, $P_\Omega$ is equipped with a flat morphism $\pi : P_\Omega \to \Omega$; the fibres of $\pi$ are all isomorphic to projective lines; and there exists a section $s : \Omega \to P_\Omega$ of $\pi$. Let $B_\Omega$ be the image of $s$.

**Lemma 3.3.3.** *Let $P_\Omega$ be a ruled scheme over $\Omega$. Let $\phi_\Omega : Y_\Omega \to P_\Omega$ be a family of $G$-Galois covers of smooth connected curves branched exactly at $B_\Omega = s(\Omega)$.*

1. *The scheme $P_\Omega$ is locally trivial over $\Omega$; (in other words there exists a finite open cover $\{V_i; 1 \le i \le t\}$ of $\Omega$ such that $P_{V_i} \simeq \mathbb{P}^1_{V_i}$).*

2. *There exist trivializations $tr_i : P_{V_i} \to \mathbb{P}^1_{V_i}$ such that the covers $\phi_i = tr_i \phi_{V_i} : Y_{V_i} \to \mathbb{P}^1_{V_i}$ are branched only at $\infty_{V_i}$ with the same inertia data as $\phi_{V_i}$.*

3. *There exist affine linear transformations $A_i \in \mathbf{A}(V_i \cap V_1)$ such that $\phi_i^{A_i}|_{V_i \cap V_1} = \phi_1|_{V_i \cap V_1}$ for $2 \le i \le t$.*

*Proof.*

1. First $P_\Omega$ must be locally free over $\Omega$. Following the proof of [Har77, Lemma 5.2.1 and Proposition 5.2.2] consider $E = \pi_* L(B_\Omega)$. Let $P_\omega$ be any fibre of $\pi$. Since $L(B_\Omega)_\omega$ is an invertible sheaf of degree 1 on $P_\omega$ the number $h^0(L(B_\Omega))_\omega = 2$ is independent of $\omega$. By Grauert's Theorem, $E$ is locally free of rank 2. The surjection $\pi^* E \to L(B_\Omega)$ determines an isomorphism $P_\Omega \simeq \mathrm{Proj}(E)$ over $\Omega$. Thus $P_\Omega$ is locally free over $\Omega$. Since $P_\Omega$ is locally free over $\Omega$ it is possible to choose a finite open cover $\{V_i\}$ of $\Omega$ and trivializations $tr_i : P_{V_i} \to \mathbb{P}^1_{V_i}$ of $P_\Omega$ over each $V_i$.

2. Furthermore, it is possible to choose $tr_i$ such that $tr_i : B_{V_i} \to \infty_{V_i}$. In other words over each $V_i$ it is possible to choose a coordinate $u_i$ for $\mathbb{P}^1$ such that the composition $tr_i(\phi_{V_i})$ is branched over $u_{V_i} = \infty_{V_i}$. If $\phi_i = tr_i(\phi_{V_i})$ then $\phi_i$ has the same inertia data over $\infty_{V_i}$ as $\phi_{V_i}$ has over $B_{V_i}$.

3. Over the (non-empty) intersection of the two open sets $V_i$ and $V_1$ of $\Omega$ this coordinate $u_i$ is determined up to a projective linear transformation of $\mathbb{P}^1_{V_i \cap V_j}$ which must fix the branch point $\infty_{V_i \cap V_1}$. Thus the choice of the coordinate $u_i$ is determined up to an affine linear transformation in $\mathbf{A}(V_i \cap V_1)$. Thus there exist $A_i$ as required.

□

The statement of Theorem 3.3.4 was suggested by M. Raynaud. The strategy to show that proper families of covers of $P_\Omega$ must be isotrivial will be similar to that of Theorem 3.3.2. However the proof will be more intricate since (when restricted to an open subspace of $\Omega$) the covers $\phi_i$ of Lemma 3.3.3 will only agree up to affine linear transformations of the base.

**Theorem 3.3.4.** *Let $\Omega$ be a proper irreducible k-scheme. Let $P_\Omega$ be a ruled scheme over $\Omega$. Let $\phi_\Omega : Y_\Omega \to P_\Omega$ be a family of G-Galois covers of smooth connected curves branched exactly above one section. Then $\phi_\Omega$ is isotrivial. In other words, the morphism $\Omega \to M_g$ taking $\omega \mapsto [Y_\omega]$ is constant.*

*Proof.* By Lemma 2.1.4, there exists an irreducible scheme $\Omega'$ and a finite surjective morphism $i : \Omega' \to \Omega$ so that for any generic ramification point $\eta$ of the pullback $\phi_{\Omega'}$ the decomposition group $D_\eta$ equals the inertia group $I_\eta$. Since $\Omega'$ is also proper and this pullback does not affect the question of isotriviality, it is possible to reduce to the case that the cover $\hat{\phi}_\Omega : \hat{Y}_\Omega \to \hat{P}_\Omega$ of germs of curves has Galois group $I = P \rtimes \mu_m$.

By Lemma 3.3.3, the scheme $P_\Omega$ is locally trivial over $\Omega$; there exists a finite open cover $\{V_i; 1 \leq i \leq t\}$ of $\Omega$ and trivializations $tr_i : P_{V_i} \simeq \mathbb{P}^1_{V_i}$ such that the covers $\phi_i = tr_i \phi_{V_i} : Y_{V_i} \to \mathbb{P}^1_{V_i}$ are branched at $\infty_{V_i}$ with the same inertia data as $\phi|_{V_i}$. Also there exist affine linear transformations $A_i \in \mathbf{A}(V_i \cap V_1)$ such that $\phi_i^{A_i}|_{V_i \cap V_1} = \phi_1|_{V_i \cap V_1}$ for $2 \leq i \leq t$.

Consider the $I$-Galois equations for the cover $\hat{\phi}_i : \hat{Y}_{\infty_{V_i}} \to \mathrm{Spec}(O(V_i)[[u^{-1}]])$ of germs of curves for each $i$ in terms of the coordinate $u_i$. The goal is to show that there exist affine linear transformations $A'_i \in \mathbf{A}(V_i)$ so that $\hat{\phi}_i^{A'_i}$ are constant.

For each $i$, let $\hat{\phi}'_i$ be the quotient of $\hat{\phi}_i$ by $P$. The covers $\hat{\phi}'_i$ are Galois with group $\mu_m$. Furthermore, the equations for $\hat{\phi}'_i$ still satisfy that $(\hat{\phi}'_i)^{A_i}|_{V_i \cap V_1} = \hat{\phi}'_1|_{V_i \cap V_1}$ for $2 \leq i \leq t$.

Such $\mu_m$-Galois covers must be constant after an affine linear transformation of the base. Namely, there exist affine linear transformations $A'_i \in \mathbf{A}(V_i)$ such that $(\hat{\phi}'_{V_i})^{A'_i}$ is constant for $1 \leq i \leq t$. Concretely, this is because the cover $\hat{\phi}'_{V_i}$ is given by an equation $x^m = a_i u$ for some $a_i \in O(V_i)^*$, and thus one can take $A'_i(u) = u/a_i$.

Note that (by Notation 2.3.1) $(\hat{\phi}'_i)^{A_i A'_1} = ((\hat{\phi}'_i)^{A_i})^{A'_1} = (\hat{\phi}'_1)^{A'_1}$ is constant. Thus it is possible to choose $A'_i = A_i A'_1$. To see this, consider the two constant covers $\sigma_i = (\hat{\phi}'_i)^{A'_i}$ and $\sigma'_i = (\hat{\phi}'_i)^{A_i A'_1}$ over $V_i \cap V_1$. Then $\sigma'_i$ is in the orbit of $\sigma_i$ under $\mathbf{A}(V_i \cap V_1)$. Thus there exists $B_i \in \mathbf{A}(V_i \cap V_1)$ such that $\sigma_i^{B_i} = \sigma'_i$. Since $\sigma_i$ and $\sigma'_i$ are constant it follows that $B_i \in \mathbf{A}(k)$ is constant. After replacing $A'_i$ by $B_i^{-1} A'_i$ for $2 \leq i \leq t$ the affine linear transformations satisfy $A'_i = A_i A'_1$.

Consider the covers $\hat{\phi}_i^{A'_i}$. Since their $\mu_m$-quotients are constant, these are $P$-Galois covers of a constant base $U_{V_i}$. Since $P$ is a $p$-group, there exists a filtration of normal subgroups of $P$ with successive quotients $\mathbb{Z}/p$. Thus by induction the proof reduces to the case that $P = \mathbb{Z}/p$.

Thus for $1 \leq i \leq t$ the covers $\hat{\phi}_i^{A'_i}$ yield $\mathbb{Z}/p$-Galois extensions $\psi_i^{A'_i}$ of $U_{V_i}$ with the property that $\psi_i^{A'_i} = \psi_i^{A_i A'_1} = \psi_1^{A'_1}$. Furthermore, since $A'_i \in \mathbf{A}(V_i)$ these covers have good reduction at every point of $V_i$. Let $j$ be the jump of these $\mathbb{Z}/p$-Galois covers. By Proposition 2.2.6, $\psi_i^{A_i'}$ determines a finite radicial morphism $V_i' \to V_i$ and an induced morphism $f_i : V_i' \to C(\mathbb{Z}/p, j)$ to the configuration space of $\mathbb{Z}/p$-Galois covers with jump $j$.

Furthermore, over $V_1 \cap V_i$ the covers $\psi_i^{A_i'} = \psi_i^{A_i A_1'} = \psi_1^{A_i'}$ are isomorphic. As a result, there exists $\Omega'$ irreducible and proper and a surjective morphism $i : \Omega' \to \Omega$ which restricts to $V_i' \to V_i$ for each $i$. Furthermore, the restrictions of the morphisms $f_1$ and $f_i$ to $i^{-1}(V_1 \cap V_i)$ must be the same. As a result, the covers $\psi_i^{A_i'}$ determine a morphism $f : \Omega' \to C(\mathbb{Z}/p, j)$.

The configuration space $C(\mathbb{Z}/p, j)$ is affine. However, $\Omega'$ is proper. Thus the image of $f$ must be constant. Thus $\psi_i^{A_i'}$ are constant. Thus the covers $\hat{\phi}_i^{A_i'}$ are constant.

By Proposition 2.1.3, for each $i$ the cover $\phi_i^{A_i'}$ is constant because it is constant near the branch point. Thus $\phi_i : Y_{V_i} \to \mathbb{P}^1_{V_i}$ are weakly constant and thus isotrivial. This implies that the morphism $\Omega \to M_g$ taking $\omega \mapsto [Y_\omega]$ is constant and $\phi_\Omega$ is isotrivial as well. □

In fact the statement that $\phi_\Omega$ is isotrivial in Theorems 3.3.2 and 3.3.4 is true even if $\phi_\Omega$ is not $G$-Galois.

**Corollary 3.3.5.** *Let $\Omega$ be a proper irreducible k-scheme. Let $\phi_\Omega : Y_\Omega \to X_\Omega$ be a family of (not necessarily Galois) covers of smooth connected curves and let $B_\Omega$ be the branch locus of $\phi_\Omega$. Suppose that $X_\Omega = X_k \times_k \Omega$ for some k-curve $X_k$ and $B_\Omega = \{\xi_{i,\Omega} = \xi_i \times_k \Omega\}$ for a finite set of k-points $\xi_i \in X_k$; (respectively suppose that $X_\Omega \simeq P_\Omega$ is a ruled scheme over $\Omega$ and $B_\Omega$ consists of exactly one section). Then $\phi_\Omega$ is isotrivial. In other words, the morphism $\Omega \to M_g$ taking $\omega \mapsto [Y_\omega]$ is constant.*

*Proof.* Let $\phi^1_\Omega : Y^1_\Omega \to X_\Omega$ be the Galois closure of $\phi_\Omega$. For some finite group $G$ the fibres of $\phi^1_\Omega$ are $G$-Galois covers of smooth proper curves with branch locus $B_\Omega$. The fibres of $Y^1_\Omega$ over $\Omega$ may not be connected if the Galois closure of the family has larger fibres than the Galois closure of the fibres. In this case, consider the Stein factorization $Y^1_\Omega \to \Omega' \to \Omega$ [Har77, III, Corollary 11.5]. Recall that $i : \Omega' \to \Omega$ is finite and surjective and thus $\Omega'$ is a proper $k$-scheme. Recall that the fibres of $Y^1_\Omega \to \Omega'$ are connected. Note that $\Omega'$ is connected (but not necessarily irreducible) since $Y_\Omega$ is connected.

Consider the fibre product $X_{\Omega'} = X_\Omega \times_\Omega \Omega'$. By the universal property of the fibre product there exists a morphism $f : Y^1_\Omega \to X_{\Omega'}$ so that the composition of $f$ with the morphism $X_{\Omega'} \to X_\Omega$ is $\phi^1_\Omega$ and the composition of $f$ with the morphism $X_{\Omega'} \to \Omega'$ is $Y^1_\Omega \to \Omega'$. Thus the cover $f$ is defined over $\Omega'$, is Galois, and has connected fibres.

Thus, over each irreducible component of $\Omega'$, the cover $f$ satisfies the hypotheses of Theorem 3.3.2 (respectively 3.3.4). The conclusion from these theorems is that $Y^1_\Omega$ is isotrivial over each irreducible component of $\Omega'$. Since $\Omega'$ is connected, $Y^1_\Omega$ is isotrivial.

There are no proper non-isotrivial families of elliptic curves since $M_1$ is affine. Thus to show that $Y_\Omega$ is isotrivial it is sufficient to assume that the genus of the fibres of $Y_\Omega$ (and thus also of $Y^1_\Omega$) satisfies $g \geq 2$. Note that $Y_\Omega$ is the quotient of $Y^1_\Omega$ by some finite group $H$. By Lemma 2.1.2, after an étale pullback the cover $Y^1_\Omega \to Y_\Omega$ is weakly constant. In particular, then $Y_\Omega$ is isotrivial. □

**Definition 3.3.6.** Let $f_k : Y_k \to \mathbb{P}^1_k$ be a $G$-Galois cover of smooth connected curves. Let $V$ be a connected scheme of finite type over $k$. A *degeneration* of $f_k$ over $V$ consists of a deformation $f_V : Y_V \to P_V$ of $f_k$ defined over $V$ together with closed $k$-points $v_0, v_1 \in V$ such that:

a) the relative curves $P_V$ and $Y_V$ are flat and proper $V$- curves;

b) the cover $f_V$ is a $G$-Galois cover of semi-stable curves and $f_k$ and $f_{v_0}$ are isomorphic as $G$-Galois covers;

c) the pullback of $f_V : Y_V \to P_V$ to $\hat{V}_{v_0}$ is a smooth deformation of $f_{v_0}$;

d) the cover $f_V$ has bad reduction at the fibre $v_1$.

Let $g$ be the genus of $Y_k$. By considering the isomorphism classes of the fibres of $Y_V \to V$, the degeneration $f_V$ induces a morphism $\tau : V \to \overline{M_g}$ with the following properties: the image of $\tau$ lies in the locus of curves with a $G$-Galois action; $\tau(v_0)$ corresponds to the isomorphism class of $Y_k$; and $\tau(v_1)$ lies in the boundary of $M_g$.

**Theorem 3.3.7.** *Let $\phi_k : Y_k \to \mathbb{P}^1_k$ be a G-Galois cover branched at only one point, over which it has inertia group $I = \mathbb{Z}/p \rtimes \mu_m$ and jump $j$. Then there exists a degeneration of $\phi$ over $\Omega$ for some proper connected variety $\Omega$ if $j \not\in j_{\min}(I)$.*

*Proof.* Let $R$ be an equal characteristic complete discrete valuation ring. By Theorem 3.1.10, the condition on $j$ insures that there exists a non-isotrivial smooth deformation $\phi_R$ of $\phi$ over $R$. By Artin's Approximation Theorem [Art70, Theorem 3.2], the cover $\phi_R$ descends to a cover $\phi_{\Omega_\circ} : Y_{\Omega_\circ} \to \mathbb{P}^1_{\Omega_\circ}$ branched over $\infty_{\Omega_\circ}$ over a variety $\Omega_\circ$ which is of finite type over $k$. Let $\Omega$ be a smooth completion of $\Omega_\circ$. Let $Y_\Omega$ be the normalization of $\mathbb{P}^1_\Omega$ in $Y_{\Omega_\circ}$. Consider the cover $\phi_\Omega : Y_\Omega \to \mathbb{P}^1_\Omega$. The family of curves $Y_\Omega$ must be non-isotrivial since $\phi_R$ was non-isotrivial. Thus by Theorem 3.3.4, the cover $\phi_\Omega$ must have bad reduction over some $k$-point $\omega \in \Omega - \Omega_\circ$. □

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