A repeatable inverse kinematics algorithm with linear invariant subspaces for mobile manipulators

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Abstract—On the basis of a geometric characterization of repeatability we present a repeatable extended Jacobian inverse kinematics algorithm for mobile manipulators. The algorithm’s dynamics have linear invariant subspaces in the configuration space. A standard Ritz approximation of platform controls results in a band-limited version of this algorithm. Computer simulations involving an RTR manipulator mounted on a kinematic car-type mobile platform are used in order to illustrate repeatability and performance of the algorithm.

Index Terms—Mobile manipulator, endogenous configuration space, inverse kinematics, extended Jacobian, repeatability.

I. INTRODUCTION

A MOBILE manipulator is a robotic device comprised of a mobile platform and a stationary onboard manipulator fixed to the platform. Mobile manipulators have recently been a subject of intense research due to their potential applications, mostly in service robotics; undoubtedly they are also challenging from the theoretical point of view. A vast literature devoted to the modeling, control, and motion planning for mobile manipulators has been thoroughly reviewed in [1], so we shall limit here to mentioning only very recent papers published after that reference. In [2] a classical model of mobile manipulator kinematics is adopted in order to define manipulability measures that have been applied to the motion planning problem. Paper [3] proposes a solution to the motion planning problem by means of the Newton algorithm augmented with control energy optimization. In [4] an optimization theory based inverse kinematics algorithm for mobile manipulators has been invented, able to avoid both singularities as well as obstacles in the taskspace. Reference [5] introduces a new adaptive control algorithm for taskspace path following in mobile manipulators with nonholonomic platform. Depending on holonomy or nonholonomy of the platform and of the manipulator, four types of mobile manipulators may be distinguished [6]. In this paper we shall deal with $(nh, h)$-type mobile manipulators composed of a nonholonomic platform and a holonomic onboard manipulator.

In [7] there has been initiated a novel approach to mobile manipulators referred to as the endogenous configuration space approach. A fundamental concept underlying this approach is the endogenous configuration that consists of all control actions exerted on the mobile manipulator, i.e., control functions of the platform, and joint positions of the onboard manipulator. The kinematics of a mobile manipulator are defined as a map from the endogenous configuration space into the taskspace containing the end effector locations. By a natural generalization of corresponding concepts for stationary manipulators the analytic Jacobian, regular and singular configurations, regular and singular Jacobian inverse kinematics algorithms, and dexterity measures for mobile manipulators have been derived [1], [8]. According to our best knowledge, the endogenous configuration space approach is so far the only framework suitable for defining repeatability for mobile manipulators [9].

It is well known that a Jacobian inverse kinematics algorithm for stationary manipulators can be represented as a dynamic system defined on the configuration space, driven by the taskspace error. An inverse kinematics algorithm for redundant manipulators that transforms closed paths in the taskspace into closed paths in the configuration space is called repeatable. The property of repeatability of Jacobian inverse kinematics algorithms for stationary manipulators is well understood and exhaustively documented in the literature. A crucial observation that the Jacobian pseudoinverse inverse kinematics algorithm is not repeatable has been made by Klein and Huang [10]. Differential geometric interpretations along with necessary and sufficient conditions for repeatability have been provided by Brockett [11], and Shamir and Yomdin [12]. The idea of obtaining repeatability by a suitable augmentation of the manipulator Jacobian can be traced back to [13], [14]. The first extended Jacobian, repeatable inverse kinematics algorithm comes from Baillieu [15]. A parameterization of extended Jacobian inverses leading to their optimal design has been proposed by Roberts and Maciejewski [16]. Two other papers [18], [19] by the same authors are devoted to the optimal synthesis of repeatable, extended Jacobian inverse kinematics algorithms with prescribed properties. The existence of stable surfaces and new aspects of repeatable behavior of inverse kinematics algorithms have been examined in [17].

Analogously as for stationary manipulators, a Jacobian inverse kinematics algorithm for mobile manipulators has also the form of a taskspace error driven dynamic system operating on the endogenous configuration space. For the endogenous configuration space appears to be an infinite dimensional Hilbert space, mobile manipulators are infinitely redundant...
robotic systems. As in the stationary case, repeatability means that the inverse kinematics algorithm transforms a closed path in the taskspace into a closed path in the endogenous configuration space. Taking into consideration the meaning of the endogenous configuration we assert that a repeatable inverse kinematics algorithm produces control functions of the platform and joint positions of the onboard manipulator dependent only on the desirable location of the end effector, irrespective of the starting point of the algorithm. The geometric interpretation of repeatability of inverse kinematics algorithms for mobile manipulators is formally the same as for stationary manipulators outside singularities the kinematics endow the endogenous configuration space with the structure of a fiber bundle over the taskspace, the fibers being self-motion manifolds, such that the connection defined by the inverse kinematics algorithm has trivial holonomy group. A difference consists in the fact that the self-motion manifolds are now infinite-dimensional. A differential geometric condition for repeatability of an inverse kinematics algorithm for mobile manipulators requires that its associated distribution should be involutive, so integrable [9]. This means that after choosing an initial condition, the algorithm evolves on a fixed integral manifold of the associated distribution. Over a simply connected region of regular endogenous configurations these integral manifolds form leaves of a foliation transverse to the fibers [20]. Similarly as in the stationary case, in order to design a repeatable inverse kinematics algorithm for mobile manipulators it suffices to find the associated distribution annihilated by an augmenting kinematics map. The level sets of this map coincide with integral manifolds of the distribution.

Assuming the endogenous configuration space approach, in this paper we shall derive and examine a specific repeatable, extended Jacobian inverse kinematics algorithm for mobile manipulators. The algorithm’s dynamics evolve on linear invariant subspaces. A Ritz approximation of platform control functions by means of truncated Fourier series yields a band-limited version of this algorithm. Computer simulations involving a mobile manipulator consisting of an RTR manipulator mounted on the kinematic car will illustrate repeatability and performance of this algorithm.

The composition of this paper is the following. In section 2 we briefly describe the concept of Jacobian extension, and propose an extended Jacobian inverse kinematics algorithm for mobile manipulators. A band-limited version of this algorithm is derived in section 3. Section 4 presents results of computer simulations. The paper is concluded with section 5.

II. BASIC CONCEPTS

We shall consider \((\mathbb{N}, h, h)\)-type mobile manipulators, consisting of a nonholonomic mobile platform and a holonomic onboard manipulator. The kinematics of such mobile manipulators will be represented in the form of a driftless control system with outputs,

\[
\dot{q} = G(q)u = \sum_{i=1}^{m} g_i(q)u_i, \quad y = k(q,x),
\]

where \(q \in \mathbb{R}^m\) denotes generalized coordinates of the platform, \(x \in \mathbb{R}^p\) describes the joint position of the manipulator, and \(y \in \mathbb{R}^r\) determines end effector coordinates in the taskspace. Variables \(u\) and \(x\) will be regarded as control inputs. Typically, the number of platform controls does not exceed the dimensionality of the taskspace, \(m \leq r\). Let \(T > 0\) denote a control time horizon. Admissible control functions \(u(\cdot)\) of the platform are assumed Lebesgue square integrable on \([0,T]\). These functions and joint positions of the onboard manipulator constitute the endogenous configuration space \(\mathcal{X} = L^2_{\text{int}}[0,T] \times \mathbb{R}^p\) that is a Hilbert space with inner product

\[
\langle (u_1(\cdot), x_1), (u_2(\cdot), x_2) \rangle = \int_0^T u_1^T(t)u_2(t)dt + x_1^T x_2.
\]

To every endogenous configuration \((u(\cdot), x) \in \mathcal{X}\) there correspond a platform trajectory \(q(t) = \varphi_{q_0,T}(u(\cdot))\) and an end effector trajectory \(y(t) = k(q(t),x)\). It will be assumed that \(q(t)\) exists for every \(t \in [0,T]\). The instantaneous kinematics \(K_{q_0,T}: \mathcal{X} \rightarrow \mathbb{R}^r\) of the mobile manipulator are defined as

\[
K_{q_0,T}(u(\cdot), x) = y(T) = k(\varphi_{q_0,T}(u(\cdot)), x).
\] (2)

It is easily seen that formula (2) determines reachable at \(T\) end effector positions and orientations of the mobile manipulator steered by the control \((u(\cdot), x)\), provided that the platform starts from \(q_0\). The analytic Jacobian

\[
J_{q_0,T}(u(\cdot), x)(v(\cdot), w) = DK_{q_0,T}(u(\cdot), x)(v(\cdot), w) = \left. \frac{d}{dt} \right|_{t=0} K_{q_0,T}(u(\cdot) + \alpha v(\cdot), x + \alpha w) = C(T,x) \int_0^T \Phi(t,s)B(s)v(s)ds + D(T,x)w,
\] (3)

is obtained by differentiation of the kinematics \(K_{q_0,T}(u(\cdot), x)\) at the endogenous configuration \((u(\cdot), x)\). The Jacobian may be identified with the output reachability map at \(T\) of the variational system

\[
\dot{\xi} = A(t)\xi + B(t)v, \quad \eta = C(t,x)\xi + D(t,x)w,
\] (4)

initialized at \(\xi_0 = 0\), whose matrices

\[
A(t) = \frac{\partial}{\partial q}(G(q(t))u(t)) , \quad B(t) = G(q(t)),
\]

\[
C(t,x) = \frac{\partial k}{\partial q}(q(t),x), \quad D(t,x) = \frac{\partial k}{\partial x}(q(t),x),
\] (5)

are computed along the control-trajectory pair \((u(t), x(t), q(t))\), and whose transition matrix \(\Phi(t,s)\) satisfies the evolution equation \(\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s)\) with initial condition \(\Phi(s,s) = I_p\). An endogenous configuration \((u(\cdot), x)\) is called regular, if the analytic Jacobian \(J_{q_0,T}(u(\cdot), x) : \mathcal{X} \rightarrow \mathbb{R}^r\) is surjective or, equivalently, when system (4) is output controllable.

For the sake of clarity of the presentation we shall restrict to a special case of \(p = r\). More general situations are manageable along the same lines of reasoning. The following derivation of Jacobian inverse kinematics algorithms, based on the continuation or homotopy method [21], [22], is standard within the endogenous configuration space approach. Consider an inverse kinematic problem for a mobile manipulator, consisting in computing an endogenous configuration \((u_d(\cdot), x_d) \in \mathcal{X}\) such that, for given \(q_0\), \(T\), and a desirable end effector location \(y_d\), the equation \(K_{q_0,T}(u_d(\cdot), x_d) = y_d\)
is satisfied. Let us suppose that a solution to this problem exists, and choose an initial configuration \((u_0(\cdot), x_0) \in \mathcal{X}\). If \(K_{q_0,T}(u_0(\cdot), x_0) \neq y_d\), we look for a smooth curve \((u_{\theta}(\cdot), x(\theta)) \in \mathcal{X}, \theta \in \mathcal{R}\), along which the taskspace error \[ e(\theta) = K_{q_0,T}(u_{\theta}(\cdot), x(\theta)) - y_d \] (6) decreases exponentially with a prescribed decay rate \(\gamma > 0\), i.e. \[ \frac{d}{d\theta} e(\theta) = -\gamma e(\theta) . \]

A differentiation of the error accompanied by the use of (3) yields \[ J_{q_0,T}(u_{\theta}(\cdot), x(\theta)) \frac{d}{d\theta} (u_{\theta}(\cdot), x(\theta)) = -\gamma J_{q_0,T}(u_{\theta}(\cdot), x(\theta)) e(\theta). \] (7)

In the region of regular endogenous configurations, where the analytic Jacobian is invertible, the Jacobian inverse kinematics algorithm takes the form of a dynamic system \[ \frac{d}{d\theta} (u_{\theta}(\cdot), x(\theta)) = -\gamma J_{q_0,T}(u_{\theta}(\cdot), x(\theta)) e(\theta), \] (8)

In order to define a repeatable inverse kinematics algorithm in the form (7) we shall follow a classic procedure of Jacobi pseudoinverse algorithm [1]. On condition that the dynamic analytic Jacobian algorithms; the most often used is the Jacobian pseudoinverse algorithm [1]. On condition that the dynamic system (7) is well defined and complete, the corresponding Jacobian inverse kinematics algorithm converges exponentially to a solution of the inverse kinematic problem, in the sense that \[ (u_d(t), x_d) = \lim_{\theta \rightarrow +\infty} (u_{\theta}(t), x(\theta)). \] (9)

To define a repeatable inverse kinematics algorithm in the form (7) we shall follow a classic procedure of Jacobian extension, modeled after [15] and [18]. We begin with augmenting the mobile manipulator kinematics in such a way that, outside a “small” singular set, the analytic Jacobian of the extended kinematics composed of the augmenting map and of the original kinematics becomes a linear isomorphism of the endogenous configuration space. To avoid algorithmic singularities [15], it is necessary that the augmenting kinematics map should be nonsingular. Under assumption that \(p = r\), we have \(\mathcal{X} = L^2_{m_r}[0, T] \times \mathcal{R}^r\), so we need an augmenting kinematics map \[ H_{q_0,T} : \mathcal{X} \rightarrow L^2_{m_r}[0, T] \] (10) that along with the kinematics (2) constitutes the extended kinematics \[ (H_{q_0,T}, K_{q_0,T}) : \mathcal{X} \rightarrow \mathcal{X}. \] (11)

Outside the singular set \[ S = \{(u(\cdot), x) \in \mathcal{X}, \quad J_{q_0,T}(u(\cdot), x) \text{ is not a linear isomorphism of } \mathcal{X}\} \] the extended Jacobian defines for \(\eta \in \mathcal{R}^r\) a right inverse \[ J_{q_0,T}(u(\cdot), x) \eta = J_{q_0,T}(u(\cdot), x)(\theta(\cdot), \eta) \] (12) of the analytic Jacobian, called an extended Jacobian inverse. It is easily checked that the extended Jacobian inverse has the following properties \[ J_{q_0,T}(u(\cdot), x) J_{q_0,T}^E(u(\cdot), x) \eta = \eta, \]
\[ DH_{q_0,T}(u(\cdot), x) J_{q_0,T}^E(u(\cdot), x) \eta = 0(\cdot). \] (13)

By the first property of (13) \(J_{q_0,T}^E(u(\cdot), x)\) is a right inverse of the analytic Jacobian, so a substitution of (12) into (7) results in an extended Jacobian inverse kinematics algorithm. Given such an algorithm, the second property of (13) means that the distribution spanned by the columns of \(J_{q_0,T}^E(u(\cdot), x)\) is annihilated by the differential of the augmenting map, hence this distribution is involutive. Referring to a result of [9] we conclude that the extended Jacobian inverse kinematics algorithm is repeatable.

In what follows we shall employ the following augmenting kinematics map \[ H_{q_0,T}(u(\cdot), x)(t) = \left( \frac{u_1(t)}{x_{i_1}}, \frac{u_2(t)}{x_{i_2}}, \ldots, \frac{u_m(t)}{x_{i_m}} \right), \] (14)

where \(x_{i_1}, \ldots, x_{i_m}\) denote selected joint variables of the onboard manipulator provided by a selector matrix \(S_{m,p} x = (x_{i_1}, \ldots, x_{i_m})\). Obviously, the map (14) is well defined provided that all \(x_{i_k} \neq 0\). The differential of (14) is equal to \[ DH_{q_0,T}(u(\cdot), x)(v(\cdot), w)(t) = \left( \begin{array}{c} x_{i_1} v_1(t) - w_{i_1} u_1(t) \\ x_{i_2} v_2(t) - w_{i_2} u_2(t) \\ \vdots \\ x_{i_m} v_m(t) - w_{i_m} u_m(t) \end{array} \right). \] (15)

The map (15), wherever defined, is surjective. The extended Jacobian inverse (12) associated with this map may be computed in the following way. First we let \(J_{q_0,T}^E(u(\cdot), x) \eta)(t) = (v(t), w)\). By (12) this implies that \[ J_{q_0,T}(u(\cdot), x)(v(\cdot), w))(t) = (0, \eta) \]

or, using (11) and (15), \[ x_{i_k} v_k(t) - w_{i_k} u_k(t) = 0, \quad k = 1, 2, \ldots, m \] (16)

and \[ C(T, x) \int_0^T \Phi(T, s) B(s) v(s) ds + D(T, x) w = \eta. \] (17)

From (16) we deduce \[ v(t) = \text{diag}(\frac{u_k(t)}{x_{i_k}}) S_{m,p} w, \]
with \(\text{diag}(a_k)\) denoting a diagonal matrix with entries \(a_k, k = 1, \ldots, m\). Next, a substitution of \(v(t)\) into (17) yields \[ C(T, x) \int_0^T \Phi(T, s) B(s) \text{diag}(\frac{u_k(s)}{x_{i_k}}) S_{m,p} + D(T, x). \]
Finally, we compute \( w = E_{q_0,T}^{-1}(u(\cdot), x) \eta \) and conclude that
\[
(J^{E_{q_0,T}}_{q_0}(u(\cdot), x) \eta)(t) = (u(t), w) = \left[ \text{diag} \left\{ \frac{u_k(t)}{x_{ik}} \right\}_m \, S_{m,p}, I_r \right] E_{q_0,T}^{-1}(u(\cdot), x) \eta. \]

Assuming that the map (14) is well defined, the singular set of the extended Jacobian is equal to
\[
S = \{ (u(\cdot), x) \in \mathcal{X} | \det E_{q_0,T}(u(\cdot), x) = 0 \}.
\]

Given a desirable point \( y_d \in \mathcal{X}^t \) in the taskspace, the inverse (18) determines an extended Jacobian inverse kinematics algorithm
\[
\frac{d}{dt} (u_0(t), x(\theta)) = -\gamma \left[ \text{diag} \left\{ \frac{u_k(t)}{x_{ik}} \right\}_m S_{m,p}, I_r \right] E_{q_0,T}^{-1}(u(\cdot), x(\theta)) e(\theta),
\]
where \( e(\theta) \) is defined by (6). The solution of the inverse kinematics problem is obtained as the limit (8). Thanks to the second property of (13) all trajectories of dynamic system (19) initialized at \( (u_0(\cdot), x_0) \) satisfy \( H_{q_0,T}(u(\cdot), x) = H_{q_0,T}(u_0(t), x_0) \), i.e. they lie in the invariant set
\[
V(u_0(\cdot), x_0) = \left\{ (u(\cdot), x) \in \mathcal{X} | u_k(t) = \frac{u_{0k}(t)}{x_{0ik}}, k = 1, 2, \ldots, m \right\}.
\]

It is not hard to check that \( V(u_0(\cdot), x_0) \) is a linear subspace of the endogenous configuration space (compare the concept of stable surfaces [16], [17]). Within this space variables \( u_k(t) \) and \( x_{ik} \) evolve in a coordinated way. The choice of coordinated joint positions on the selector matrix \( S_{m,p} \).

Relying on (20) we can simplify the algorithm (19) to the following form
\[
\left\{ \begin{array}{l}
\frac{d}{dt} x(\theta) = -\gamma F_{q_0,T}(u_0(\cdot), x_0, x(\theta)) e(\theta) \\
u_0(t) = \text{diag} \left\{ \frac{u_{0k}(t)}{x_{0ik}} \right\}_m S_{m,p}, x(\theta),
\end{array} \right.
\]

where
\[
F_{q_0,T}(u_0(\cdot), x_0, x) = E_{q_0,T} \left( \text{diag} \left\{ \frac{u_{0k}(\cdot)}{x_{0ik}} \right\}_m S_{m,p}, x, x \right).
\]

We notice that in the algorithm (21) only joint positions of the onboard manipulator need to be updated dynamically.

### III. BAND-LIMITED EXTENDED JACOBIAN ALGORITHM

For computational reasons it is useful to choose a finite-dimensional representation of platform control functions in the form of truncated Fourier series, \( u(t) = \lambda_0 + \sum_{k=1}^{s_1} (\lambda_{2k-1}^0 \sin \omega t + \lambda_{2k}^0 \cos \omega t) \), \( i = 1, 2, \ldots, m, \omega = 2\pi/T \), written briefly as \( u(t) = P(t) \lambda \), where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m, s = m + 2 \sum_{i=1}^{s_1} s_i \), and
\[
P(t) = \text{block diag} \{ P_1(t), P_2(t), \ldots, P_m(t) \},
\]
is a block diagonal matrix whose \( i \)th block entry
\[
P_i(t) = [\sin \omega t, \cos \omega t, \ldots, \sin s_i \omega t, \cos s_i \omega t].
\]

This procedure is known as the Ritz approximation [23]. The use of truncated Fourier series restricts the bandwidth of control signals, so we shall speak of band-limited controls [24]. In the band-limited case the endogenous configuration gets parameterized by \( (\lambda, x) \in \mathbb{R}^{s+1} \). The band-limited kinematics (2) will be denoted by \( \tilde{K}_{q_0,T} (\lambda, x) \). The band-limited analytic Jacobian becomes a Jacobian matrix
\[
\tilde{J}_{q_0,T}(\lambda, x) = \left[ C_{\lambda}(T, x) \right]^T \Phi_s(T, s) B_s(s) P(s) ds, \quad D_{\lambda}(T, x) \right],
\]
where subscript \( \lambda \) refers to matrices (5) computed for \( u(t) = P(t) \lambda \). The augmenting kinematics map \( \tilde{H}_{q_0,T} : R^{s+1} \rightarrow R^s \), corresponding to (14), takes the form
\[
\tilde{H}_{q_0,T}(\lambda, x) = \left( \lambda_1^x, \ldots, \lambda_1^x x_{i1}, \lambda_2^x x_{i2}, \ldots, \lambda_m^x x_{im} \right).
\]

Letting \( u_d(t) = P(t) \lambda(\theta) \) and \( x = x(\theta) \), we obtain the following band-limited extended Jacobian inverse kinematics algorithm (19)
\[
\frac{d}{dt} (\lambda(\theta), x(\theta)) = -\gamma \left[ \text{block diag} \left\{ \frac{\lambda_k^x(\theta)}{x_{0ik}} \right\}_m S_{m,p}, I_r \right] \tilde{E}_{q_0,T}(\lambda(\theta), x(\theta)) \tilde{e}(\theta),
\]
where \( \tilde{e}(\theta) = \tilde{K}_{q_0,T}(\lambda(\theta), x(\theta)) - y_d \) is the taskspace error, and the matrix
\[
\tilde{E}_{q_0,T}(\lambda, x) = D_{\lambda}(T, x) + C_{\lambda}(T, x) \int_0^T \Phi_s(T, t) B_s(s) P(s) ds \, \text{block diag} \left\{ \frac{\lambda_k^x(\theta)}{x_{0ik}} \right\}_m S_{m,p}.
\]

It is easily observed that the dynamics of (22) have linear invariant subspaces defined for \( k = 1, 2, \ldots, m \) by
\[
\lambda_k^x = \frac{\lambda_k^x(0)}{x_{0ik}} x_{ik}.
\]

Each invariant subspace determines a relationship between the motion of the platform and of the onboard manipulator. By a proper choice of the invariant subspaces one may achieve a prescribed motion coordination of a mobile manipulator while accomplishing its task.

The existence of invariant subspaces allows us to transform the inverse kinematics algorithm (22) to the following equivalent form
\[
\left\{ \begin{array}{l}
\frac{dx(\theta)}{d\theta} = -\gamma \tilde{E}_{q_0,T}^{-1}(\lambda(0), x_0, x(\theta)) \tilde{e}(\theta) \\
\lambda_k^x(\theta) = \frac{\lambda_k^x(0)}{x_{0ik}} x_{ik}(\theta), k = 1, 2, \ldots, m,
\end{array} \right.
\]
where
\[
\tilde{E}_{q_0,T}(\lambda(0), x_0, x) = \tilde{E}_{q_0,T} \left( \text{block diag} \left\{ \frac{\lambda_k^x(0)}{x_{0ik}} \right\}_m S_{m,p}, x, x \right).
\]
IV. SIMULATIONS

The inverse kinematics algorithm (23) will be applied to a mobile manipulator composed of a kinematic car platform equipped with an RTR onboard manipulator, portrayed in figure 1. Vector \( q = (q_1, q_2, q_3, q_4) = (x, y, \varphi, \psi) \in \mathbb{R}^4 \) of platform coordinates comprises position and orientation of the platform, and heading angle of its front wheels. Vector \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) denotes joint positions of the onboard manipulator. Cartesian positions of the end effector \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \) serve as taskspace coordinates. The length \( l \) of the car is chosen as a measure unit. In computer simulations link lengths of the onboard manipulator will be set to \( l_2 = 0.5 \), \( l_3 = 1 \). The resulting control system representation (1) of kinematics, excluding side-slip of platform wheels, takes the following form

\[
\begin{align*}
q_1 &= u_1 \cos q_3 \cos q_4, \\
q_2 &= u_1 \sin q_3 \cos q_4, \\
q_3 &= u_1 \sin q_4, \\
q_4 &= u_2, \\
y &= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} q_1 + (l_2 + l_3 \cos x_3) \cos(q_3 + x_1) \\ q_2 + (l_2 + l_3 \cos x_3) \sin(q_3 + x_1) \\ x_2 + l_3 \sin x_3 \end{pmatrix}.
\end{align*}
\]

We shall apply very simple platform controls \( u_i(t) = \lambda_0^i + \lambda_1^i \sin 2\pi t + \lambda_2^i \cos 2\pi t, \ i = 1, 2 (T = 1) \), and use a discrete version of the algorithm (23). An advantage of linear invariant subspaces is that they are preserved after discretization. This being so, in computer simulations we use a discrete inverse kinematics algorithm

\[
\begin{align*}
x_{\theta + 1} &= x_\theta + \frac{1}{T} \tilde{\varphi}_{q_0}^{-1} (\lambda_0, x_0, x_\theta) \tilde{e}_\theta, \\
\lambda_{\theta + 1}^i &= \lambda_{\theta + 1}^i x_{\theta + 1} \theta + 1, \quad \lambda_{\theta + 1}^2 = \frac{\lambda_{\theta + 1}^2}{x_{\theta + 1}^2} x_{\theta + 1} \theta + 1,
\end{align*}
\]

where \( \theta = 0, 1, \ldots \).

For illustration of repeatability of this algorithm we examine two sequences of inverse kinematic problems, each consisting of three problems of reaching first a taskspace destination \( y_{d1} \), then \( y_{d2} \), and finally again \( y_{d1} \). At each run of the algorithm the solution of the previous problem is taken as an initial point for the current one. The repeatability test is shown schematically in figure 2. The inverse kinematics algorithm starts from \( (x_0, y_0) \) and returns as a solution of problem 1 a configuration \((x_{1f}, \lambda_{1f}) \). Next problem 2 is solved starting from \((x_{1f}, \lambda_{1f}) \) and resulting in \((x_{2f}, \lambda_{2f}) \). Finally, the algorithm is initiated at \((x_{2f}, \lambda_{2f}) \) and delivers a solution \((x_{3f}, \lambda_{3f}) \) of problem 3. Equalities \( x_{1f} = x_{3f} \) and \( \lambda_{1f} = \lambda_{3f} \) confirm repeatability of the algorithm. In sequence 1 the initial platform coordinates equal \( q_0 = (-3, 0, 0, 0) \), in sequence 2 we choose \( q_0 = (-3, 0, \pi/6, -\pi/12) \). Taskspace destinations in sequence 1 are \( y_{d1} = (0, 0, 1) \) and \( y_{d2} = (2, -0.1, 1.5) \), in sequence 2 they are set to \( y_{d1} = (0, 0, 1) \) and \( y_{d2} = (-0.5, -0.5, 1.5) \). Numerical results of computations have been collected in tables 1 and 2. It is easily observed that rows II and IV of the tables are identical, which means repeatability of the examined inverse kinematics algorithm.

**TABLE I**

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Initial Configuration</th>
<th>Final Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( x_0 = 1.5708 )</td>
<td>( y_0 = 1.5708 )</td>
</tr>
<tr>
<td>II</td>
<td>( x_{1f} = 4.3697 )</td>
<td>( y_{1f} = 3.4873 )</td>
</tr>
<tr>
<td>III</td>
<td>( x_{2f} = 7.7164 )</td>
<td>( y_{2f} = 1.7366 )</td>
</tr>
<tr>
<td>IV</td>
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<td>( y_{3f} = 3.4873 )</td>
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**TABLE II**

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<td>I</td>
<td>( x_0 = 1.5708 )</td>
<td>( y_0 = 1.5708 )</td>
</tr>
<tr>
<td>II</td>
<td>( x_{1f} = 5.7883 )</td>
<td>( y_{1f} = 3.7152 )</td>
</tr>
<tr>
<td>III</td>
<td>( x_{2f} = 5.3098 )</td>
<td>( y_{2f} = 4.4900 )</td>
</tr>
<tr>
<td>IV</td>
<td>( x_{3f} = 5.7883 )</td>
<td>( y_{3f} = 3.7152 )</td>
</tr>
</tbody>
</table>

For performance assessment of the algorithm we have examined end effector and platform trajectories as well as control functions resulting from solving the inverse kinematics problems. Plots for sequence 2 are shown in figures 3-5.

V. CONCLUSION

Relying on the endogenous configuration space characterization of repeatability we have derived an extended Jacobian inverse kinematics algorithm for \((nh,h)\)-type mobile manipulators, and adopted this algorithm to a finite-dimensional endogenous configuration space by means of the Ritz approximation of platform controls. Both the band-unlimited
and the band-limited versions of the algorithm possess linear invariant subspaces in the configuration space. The existence of invariant subspaces not only makes the algorithm repeatable, but also simplifies the execution of the algorithm. By a proper choice of these subspaces one should be able to achieve a prescribed motion coordination between the platform and the onboard manipulator. Computer simulations have verified repeatability, and demonstrated fast convergence and excellent accuracy of the algorithm. Also the shapes of resulting taskspace trajectories are smooth and intuitive.

The following questions might be addressed as a subject of further research on extended Jacobian inverse kinematics algorithms for mobile manipulators:

- more advanced augmenting kinematics maps,
- completeness of associated dynamic systems,
- well posedness of the algorithms and robustification against singularities,
- motion coordination by proper design of invariant manifolds,
- extensions toward (nh, 0) (no onboard manipulator) and (nh, nh) (onboard manipulator nonholonomic) mobile manipulators.

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**REFERENCES**


Fig. 5. Performance, sequence 2: desirable taskspace point $y_{d1} = (0, 0, 1)$, number of iterations 103, final error 8.1176e − 13


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