An interior point approach to postoptimal and parametric analysis in linear programming

Report 92-21

B. Jansen
C. Roos
T. Terlaky

TU Delft
Technische Universiteit Delft
Delft University of Technology

Faculteit der Technische Wiskunde en Informatica
Faculty of Technical Mathematics and Informatics

ISSN 0922-5641

Copies of these reports may be obtained from the bureau of the Faculty of Technical Mathematics and Informatics, Julianalaan 132, 2628 BL Delft, phone +3115784568.
A selection of these reports is available in PostScript form at the Faculty's anonymous ftp-site. They are located in the directory /pub/publications/tech-reports at ftp.twi.tudelft.nl

Optimal Bases versus Optimal Partitions for Postoptimal Analysis in Linear Programming

B. Jansen[*] C. Roos T. Terlaky[+]

Revised version
May 12, 1993

[*]Supported by the Dutch Organization for Scientific Research (NWO) under grant 611-304-028.
[+]On leave from Eötvös University, Budapest, and partially supported by OTKA No. 2115 and 2116.

Abstract: In practice, understanding the behavior of the solution of the linear programming problem due to changes in the data is often as important as obtaining the optimal solution itself. Postoptimal analysis based on the simplex method by using an optimal basis is well established and widely used. However, in case of degenerate optimal solutions, due to nonunicity of optimal bases, problems arise in correct interpretation of the results of the analysis; partial, in a certain sense erroneous, information is e.g. provided by commercial packages for linear programming. We discuss the problems in this approach and the proposals that have been made to resolve the difficulties. Then we investigate postoptimal analysis in linear programming from an interior point of view. We make use of the partition of the variables induced by a pair of strictly complementary solutions (the *optimal partition*), which is uniquely determined and arises as a natural concept in interior point methods. We give examples to compare the information obtained from both the approach using bases and the approach using optimal partitions.

Keywords: linear programming, postoptimal analysis, sensitivity analysis, shadow prices, parametric analysis, interior point methods, degeneracy.

Address of the authors:

Faculty of Technical Mathematics and Computer Science
Delft University of Technology
Mekelweg 4, 2628 CD, Delft, The Netherlands.
e-mail: b.jansen@math.tudelft.nl, c.roos@math.tudelft.nl, t.terlaky@math.tudelft.nl.

1 Introduction

In 1984 Karmarkar [16] proposed his projective algorithm for linear programming (LP), which he showed to be a polynomial time algorithm. This gave the impuls to an enormous amount of research on interior point methods (IPM's), however, almost exclusively dealing with algorithmic aspects and implementation. Surely, it is also of interest whether other subjects related to linear programming can be treated with the interior point approach. Recently, Güler et al. [14] have shown that the fundamental results of the theory of linear inequalities and linear programming can be derived from the interior point of view.

Another important topic in the field of linear programming is *postoptimal analysis* (often referred to as *sensitivity analysis*), which means the study of the behavior of the optimal solution with respect to changes in the LP model after an initial model has been solved. This is closely related to *parametric programming*, where one (several) parameter(s) is (are) included in the model and the optimal value is computed for all values of the parameter(s). These approaches are of tremendous importance in practice, where parameter values may be estimates and questions of type "What if..." are frequently encountered.

The classical way to perform postoptimal analysis is based on the Simplex method, and is founded on the concept of *optimal basis* (see Gal [7] for an extensive treatise). Until recently, many people believed that postoptimal analysis could not be handled with IPM's. It was argued that for this an optimal basic solution is needed. A concept naturally arising in IPM's is that of *strictly complementary solutions* (see Goldman and Tucker [11]) and the associated *optimal partition* of the variables. As was first shown by Adler and Monteiro [1] it is possible to perform parametric (and hence postoptimal) analysis in LP using optimal partitions. Moreover, the optimal partition appears to be a very natural tool in parametric analysis.

In this paper we survey the topic of postoptimal analysis from both the classical approach based on optimal bases and the new approach based on optimal partitions. In case of degeneracy the differences between these approaches become quite apparent. As many authors have noticed, the analysis using optimal bases may yield incomplete or even erroneous information in case of a degenerate optimal solution, see e.g. [3,5,8,9,12,17,23]. Unfortunately, most textbooks on (applied) linear programming do not mention this phenomenon and practitioners often are unaware of it. Rubin and Wagner [23, p.150] state: "Managers who build their own microcomputer linear programming models are apt to misuse the resulting shadow prices and shadow costs. Fallacious interpretations of these values can lead to expensive mistakes, especially unwarranted capital investments." As an example, we solved the following simple transportation problem with some LP packages and compared the results.

Example 1 Consider the unbalanced transportation-problem with three suppliers and three markets:

$$\min\{ \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} \ : \ \sum_{j=1}^{3} x_{ij} + s_i = a_i, \ \ \sum_{i=1}^{3} x_{ij} - d_j = b_j, \ \ x_{ij}, \ s_i, \ d_j \geq 0 \quad i,j = 1,2,3 \}.$$

Let the transportation costs be $c_{ij} = 1 \ \ (i,j = 1,2,3)$; the supplies $a_i \ \ (i = 1,2,3)$ are 2, 6 and 5 respectively, while all demands are equal to 3, that is $b_j = 3 \ \ (j = 1,2,3)$. We solved this problem with five LP packages and used the options for postoptimal analysis. The results are summarized in Tables 1 and 2.

LP package	Primal solution									Dual solution					
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	y_1	y_2	y_3	y_4	y_5	y_6
CPLEX	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
LINDO	2	0	0	0	0	2	1	3	1	0	0	0	1	1	1
OSL	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
PC-PROG	0	0	0	0	3	1	3	0	2	0	0	0	1	1	1
XMP	0	0	2	3	3	0	0	0	1	0	0	0	1	1	1

Table 1: Primal and dual solutions Example 1.

LP package	COST-intervals								
	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}
CPLEX	$[1,\infty)$	$(-\infty,1]$	$[1,\infty)$	[1,1]	[1,1]	[0,1]	[1,1]	$[1,\infty)$	$[1,\infty)$
LINDO	$(-\infty,1]$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	[1,1]	[1,1]	[0,1]	[1,1]
OSL	$[1,\infty)$	[1,1]	$[1,\infty)$	[1,1]	[1,1]	[1,1]	[1,1]	$[1,\infty)$	$[1,\infty)$
PC-PROG	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	[0,1]	[1,1]	[0,1]	$[1,\infty)$	[1,1]
XMP			$(-\infty,1]$	[1,1]	[0,1]		[1,1]		[1,1]

LP package	RHS-intervals					
	a_1	a_2	a_3	b_1	b_2	b_3
CPLEX	[0,3]	[4,7]	$[1,\infty)$	[2,7]	[2,5]	[2,5]
LINDO	[1,3]	$[2,\infty)$	[4,7]	[2,4]	[1,4]	[1,7]
OSL	[0,3]	[4,7]	$[1,\infty)$	[2,7]	[2,5]	[2,5]
PC-PROG	$[0,\infty)$	$[4,\infty)$	[3,6]	[2,5]	[0,5]	[2,5]
XMP	[0,3]	[3,6]	$[1,\infty)$	[3,7]	[3,6]	[2,7]

Table 2: COST-intervals and RHS-intervals Example 1.

Even in this simple example, no two packages yield the same result. Four different primal solutions are obtained and the intervals for the coefficients in the objective function all differ; even though the dual solution is equal for all packages, the intervals for the right-hand-side coefficients are not the same. $\diamond$

Some proposals have been made to resolve the difficulties encountered in the classical analysis in case of degeneracy. Unfortunately, it seems that the impact these proposals have made in the scientific world as well as in practice is small. In the first part of this paper we give a survey of these proposals and show what information can be obtained from them. The second part of this paper is devoted to the interior approach. As mentioned, the foundation of this approach is the use of the unique optimal partition of the LP problem. As an optimal basis is related to an optimal basic solution, the optimal partition is related to a strictly complementary solution. One of the strictly complementary solutions is the so-called *central solution*, which is the limit point of the central path of the LP problem; this path plays an important role in IPM's (see e.g. Megiddo [19]). In case of nondegeneracy, the two approaches coincide; however, as will be shown, in case of degeneracy, the approach using optimal partitions gives a natural way of doing postoptimal analysis, contrary to the analysis using bases. This is due to the unicity of the optimal partition for any LP problem.

It should be stressed that the approaches using either bases or optimal partitions are not necessarily connected to the method by which the initial LP problem is solved. However, it is very natural to connect the basis-approach with the Simplex method and the partition-approach with IPM's. We mention also that given the final (interior) iterate in IPM's an optimal basic solution can be obtained (see e.g. Charnes and Kortanek [4]); given the optimal partition and a strictly complementary solution an optimal basic solution can be obtained in strongly polynomial time (see Megiddo [20]). The reverse implications are, however, not true.

In this paper, we restrict ourselves to changes in a single element of the right-hand-side (RHS) and the objective (COST) function and in the combination of the two. The analysis is split in two parts. We consider *sensitivity analysis*, by which we mean the computation of intervals (critical regions, ranges); secondly, we consider *shadow prices* and *shadow costs*, which are the rates at which the optimal value changes as a consequence of changes in elements of the RHS- respectively COST-vector.

This paper is built up as follows. In Section 2 we describe the classical postoptimal analysis using bases and the modifications that have been proposed to handle postoptimal analysis in case of degeneracy. We give some examples where the information obtained is not complete. In Section 3 we derive properties of the optimal partition and indicate its use in parametric and postoptimal analysis. In this section we give simple proofs of results that were first obtained by Adler and Monteiro [1]. Our proofs make use of results from the theory of IPM's whereas [1] uses the polyhedral geometry present in linear programming. In Section 4 we show how complete information for Example 1 and the examples in Section 2 is obtained from the partition approach and compare the (theoretical) computational cost for obtaining the same information with the classical approach. Finally we make some concluding remarks.

2 Postoptimal Analysis using Bases

Since the early 1950s it has been noticed that it is of much interest to know how the optimal solution of the LP problem changes due to (small) perturbations in the problem data. In this section we discuss the way postoptimal analysis has been performed on basis of the simplex method.

We introduce some notation. We consider the following primal-dual pair of LP problems

$$\begin{array}{ll} (P) & \min\{\, c^T x : \; Ax = b, \; x \geq 0\} \\ (D) & \max\{\, b^T y : \; A^T y + s = c, \; s \geq 0\}, \end{array}$$

where A is an $m \times n$ matrix, $c, x, s \in \mathbb{R}^n$ and $b, y \in \mathbb{R}^m$. We make the assumption that $\text{rank}(A) = m$ $(m < n)$. We feel free to refer to any dual feasible pair (y, s) by y or by s. We denote by z^* the common optimal value of problems (P) and (D). Finally, we denote by e_k the $k - th$ unit vector of appropriate dimension.

We restrict ourselves to changes in a single RHS-coefficient $b_i = \beta$, for some $1 \leq i \leq m$, in a single COST-coefficient $c_j = \gamma$ for some $1 \leq j \leq n$ and in the combination of the two. Let β^0 and γ^0 be some initial values of the parameters β and γ respectively. Corresponding to these values, let x^* be an optimal basic solution of (P), y^* an optimal basic solution of (D). A basis is denoted by the index set $\mathcal{B}$; the indices of the nonbasic variables are in $\mathcal{N}$. Given this partition of the index set $\{1, \ldots, n\}$, the vector x (matrix A) can be written as $x^T = (x_{\mathcal{B}}^T, \; x_{\mathcal{N}}^T)$ (resp. $A = (A_{\mathcal{B}}, \; A_{\mathcal{N}})$). So a primal basic solution is given by $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b$, a dual basic solution is $y = A_{\mathcal{B}}^{-T} c_{\mathcal{B}}$; when $x_{\mathcal{B}} \geq 0$ then $\mathcal{B}$ is a primal feasible basis, when $A_{\mathcal{N}}^T y \leq c_{\mathcal{N}}$ then $\mathcal{B}$ is dual feasible. We call a basis *optimal* if it is both primal and dual feasible; a basis is called *primal optimal* if it is associated with an optimal solution of (P) and is primal but not necessarily dual feasible; analogously, a basis is called *dual optimal* if it is associated with an optimal solution of (D) but not necessarily primal feasible.

2.1 Sensitivity analysis

The classical approach to sensitivity analysis is to pose the following typical question:

(Q1) In what interval may β (or γ) vary such that $\mathcal{B}$ remains an optimal basis?

The question is inspired by the desire to know the interval where the current dual (resp. primal) optimal solution remains optimal. We already stress at this moment that (Q1) is *not* the adequate question for this inspiration, as will soon become clear. We briefly discuss how the answer to (Q1) is obtained. Letting

$$\gamma = \gamma^0, \quad \beta = \beta^0 + \lambda,$$

the basis $\mathcal{B}$ remains primal feasible (hence optimal, since dual feasibility is not affected by λ) for all λ such that

$$A_{\mathcal{B}}^{-1}(b + \lambda e_i) \geq 0.$$

This condition determines an interval Λ for λ, where the interval may be unbounded. Analogously, when

$$\beta = \beta^0, \quad \gamma = \gamma^0 + \tau,$$

an interval T is determined from

$$\gamma^0 + \tau \geq a_j^T y^* \quad \text{if } j \in \mathcal{N}$$

and from

$$A_{\mathcal{N}}^T A_{\mathcal{B}}^{-T}(c_{\mathcal{B}} + \tau (e_j)_{\mathcal{B}}) \leq c_{\mathcal{N}} \quad \text{if } j \in \mathcal{B},$$

such that $\mathcal{B}$ remains dual feasible (and optimal) for all $\tau \in T$. Note that these intervals can be obtained at low computational cost: for each interval, m, 1 or $n - m$ respectively quotients have to be computed and compared. Also, both β and γ may simultaneously be varied within these intervals without changing the optimality of the basis $\mathcal{B}$.

Indeed, in case of nondegeneracy, the respective interval obtained in this way is exactly the interval where the optimal dual resp. primal solution does not change. But, as is well-known, in case of primal degeneracy the optimal basis $\mathcal{B}$ is not necessarily unique. In case of dual degeneracy, the optimal solution x^* itself may not be unique. This can have a significant influence on the intervals obtained, as already appeared in Example 1. In fact the intervals given there, were obtained in the way described above. Although from a theoretical point of view this behavior is known, to the best of our knowledge, all commercial LP packages offering the opportunity of doing sensitivity analysis take this approach, independently of whether degeneracy is present or not; also this approach is standard in textbooks often without referring to degeneracy problems. In the literature attempts have been made to circumvent the obvious shortcomings of the classical approach, see e.g. [5,17,12,8,9]. So (Q1) has been sharpened to

(Q2) In what interval may β (or γ) vary such that at least one of the optimal bases associated with x^* remains optimal?

We discuss this proposal in some more detail (cf. Gal [8, 9]). Denote

$$F(x^*) := \{\mathcal{B} \ : \ \mathcal{B} \text{ is primal and dual feasible and } A_{\mathcal{B}}^{-1}b = x^*_{\mathcal{B}}\}$$

as the set of optimal bases associated with x^*. Each basis $\mathcal{B}^k \in F(x^*)$ $(k = 1, \dots, K)$ yields an interval Λ^k for λ where $\mathcal{B}^k$ remains an optimal basis. Obviously the overall critical region for λ given by the approach (Q2) is

$$\overline{\Lambda} = \bigcup_{k=1}^{K} \Lambda^k.$$

Similarly, each basis $\mathcal{B}^k$ yields an interval T^k for τ, and the overall critical region for τ is given by

$$\overline{T} = \bigcup_{k=1}^{K} T^k.$$

It should be noted that $\overline{\Lambda}$ and $\overline{T}$ depend on x^*. As is clear, determination of $\overline{\Lambda}$ and $\overline{T}$ is more expensive than computing the interval for a single basis: more optimal bases have to be generated. Evans and Baker [5] suggest to solve a sequence of LP problems to find the intervals. Knolmayer [17] proposes an algorithm which does not need to generate all optimal bases associated with x^*; however, the statement of his algorithm is not quite clear nor complete. Gal [9] provides a parametric algorithm inspired by [18] that does not need all optimal bases associated with x^*. We illustrate the approach according to (Q2) with the following example.

Example 2 Consider the following pair of primal-dual LP problems containing a parameter γ:

$$\begin{aligned}
\min\{-2x_2 - \gamma x_3 + 4x_4 + 5x_5 + 6x_6 \ : \ & -x_1 - 2x_2 + x_4 + x_5 = 0, \\
& -x_2 - x_3 - x_4 + x_6 = -1, \ x \geq 0\}
\end{aligned}$$

$$\max\{-y_2 \ : \ -y_1 \leq 0, -2y_1 - y_2 \leq -2, \ -y_2 \leq -\gamma, \ y_1 - y_2 \leq 4, \ y_1 \leq 5, \ y_2 \leq 6 \ \}.$$

In Figure 1 the dual feasible region is depicted. Solving these problems for the initial value $\gamma^0 = -1$, the unique optimal solution of the primal problem is given by $x_3^* = 1,\ x_i^* = 0,\ \ i \neq 3$; the optimal basic solutions of the dual problem are $y^* = (1.5, -1)^T$ and $\bar{y}^* = (3, -1)^T$. The set of optimal bases associated with x^* is

$$F(x^*) = \{\ \{2,3\},\ \{3,4\}\ \}.$$

It is easily seen that the bases in $F(x^*)$ are optimal in the following intervals for γ (cf. Figure 1):

$$\begin{array}{lclcl} \mathcal{B}^1 = \{2,3\} & \longrightarrow & T^1 = [-1,3] & \longrightarrow & \gamma \in [-2,2] \\ \mathcal{B}^2 = \{3,4\} & \longrightarrow & T^2 = [-1,2] & \longrightarrow & \gamma \in [-2,1]. \end{array}$$

So $\overline{T} = [-1,3]$ and (Q2) gives the answer $\gamma \in [-2,2]$. In this case $\mathcal{B}^1$ would be sufficient for determination of this interval. ◇

At this stage it can be made clear that also (Q2) is not the appropriate question: in Example 2 the primal solution x^* remains optimal for $\gamma \in [-2,6]$, a much larger interval than the interval $[-2,2]$ obtained from (Q2). So we conclude that the answer to (Q2) does not provide the desired complete information. Ward and Wendell [26] propose to deal with the following question.

(Q3) In what interval may β (or γ) vary such that at least one of the dual (primal) optimal bases associated with x^* is optimal?

We discusss this approach for the case that γ varies (for varying β the results are analogous). We need some concepts from parametric linear programming here. We introduce for $\tau \in \mathbb{R}$

$$(\mathrm{P}_\tau) \qquad z(\tau) = \min\{(c + \tau e_j)^T x \ :\ Ax = b,\ x \geq 0\ \}.$$

It is well-known that the *optimal value function* $z(\tau)$ is a piecewise linear concave function (cf. Gal [7]). The intervals where $z(\tau)$ is linear are called the *linearity intervals*; the points where the slope of the function changes are the *breakpoints* of $z(\tau)$. Note that the optimal value function for γ is just a translation of $z(\tau)$, since $z(\tau) = z(\gamma - \gamma^0)$. Ward and Wendell [26] define the *optimal coefficient set* of a primal solution x as

$$T(x) := \{\tau\ :\ x \text{ is an optimal solution of } (\mathrm{P}_\tau)\ \}.$$

We also define

$$R(x) := \{\tau\ :\ z(\tau) = z(0) + \tau\, x_j\}.$$

So $R(x)$ is either a linearity interval of $z(\tau)$ with slope x_j, or the set $\{0\}$; in the latter case $\tau = 0$ is a breakpoint of $z(\tau)$. The set of primal optimal bases associated with x^* is denoted by $S(x^*)$. The following result is given by Ward and Wendell [26, Th. 17].

Lemma 1 *Let x^* be an optimal basic solution of (P_0), then*

$$T(x^*) = \bigcup_{k \in H} T^k, \text{ where } H = \{k\ :\ \mathcal{B}^k \in S(x^*)\}.$$

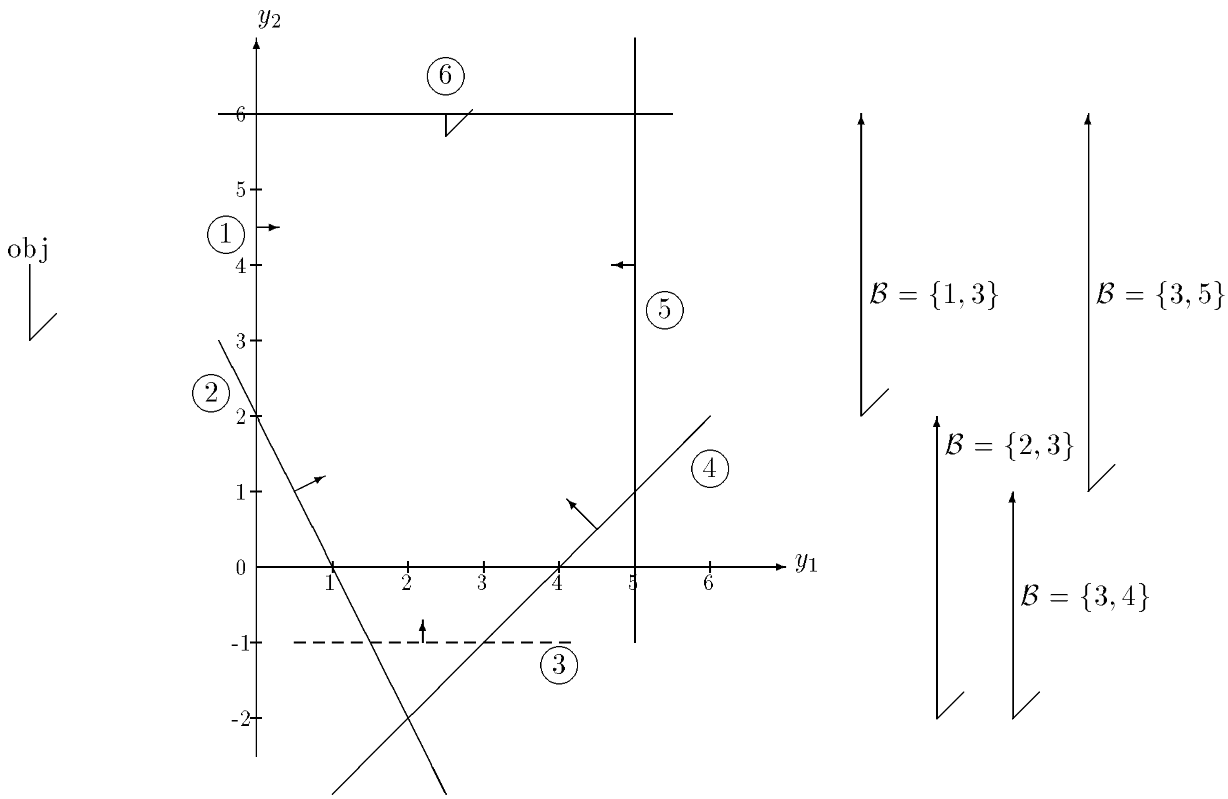

Figure 1: Feasible region of the dual problem in Examples 2 and 3; constraint 3 is shifted. The intervals where the indicated bases are optimal are denoted by the arrows.

Proof:
If $\tau \in \bigcup_{k \in H} T^k$, then clearly x^* is optimal in (P_τ), so $\tau \in T(x^*)$. Conversely, if $\tau \in T(x^*)$ then x^* is optimal in (P_τ). Clearly there is an optimal basis $\mathcal{B}^k$ with respect to (P_τ), which is associated with x^*. So $\tau \in T^k$. We will show that $k \in H$. Since $\mathcal{B}^k$ is primal feasible for (P_τ), it is primal feasible for (P_0). So $\mathcal{B}^k \in S(x^*)$ and $k \in H$. □

We proceed by showing that the interval obtained from (Q3) is exactly the interval where $z(\tau)$ is determined by x_j. This is the contents of the next lemma.

Lemma 2 *Let x^* be an optimal basic solution of (P_0), then $T(x^*) = R(x^*)$.*

Proof:
Clearly, $T(x^*) \subseteq R(x^*)$. Suppose that $\tau \in R(x^*)$, $\tau \notin T(x^*)$. Let $\tilde{x}$ be the optimal solution of (P_τ). Because of the linearity of $z(\tau)$ on $R(x^*)$, it must hold that $\tilde{x}_j = x^*_j$. Since by assumption

x^* is not optimal for τ, we have

$$(c + \tau e_j)^T \tilde{x} < (c + \tau e_j)^T x^*,$$

which implies that $c^T \tilde{x} < c^T x^*$, contradicting the optimality of x^* for $\tau = 0$. □

From Lemma 2 we can conclude that either the optimal basic solution is only optimal in the breakpoint, or it corresponds to a linearity interval of the optimal value function (in the sense that for each value of the parameter in this interval this solution is an optimal solution of the corresponding problem). If $\tau = 0$ is a breakpoint of $z(\tau)$ then obviously there must exist more than one optimal basic solutions of (P_0). The following lemma implies that when the intersection of optimal coefficient sets corresponding to different optimal basic solutions is nontrivial, then the sets are equal.

Lemma 3 *Let x^* and $\tilde{x}$ both be optimal basic solutions of (P_0) and let $T(x^*) \cap T(\tilde{x}) \neq \{0\}$, then $T(x^*) = T(\tilde{x})$.*

Proof:
Assume that $\tau \in T(x^*)$ and $\tau \notin T(\tilde{x})$. Then

$$(c + \tau e_j)^T x^* < (c + \tau e_j)^T \tilde{x}.$$

However, this inequality cannot hold, since the optimality at $\tau = 0$ implies $c^T x^* = c^T \tilde{x}$ and the linearity of $z(\tau)$ implies $x_j^* = \tilde{x}_j$. □

Ward and Wendell [26] argue that $T(x^*)$ can be obtained from the application of a parametric programming type algorithm. We conclude this subsection by continuing Example 2.

Example 3 Consider the LP problems from Example 2 with $\gamma^0 = -1$. The set of primal optimal bases associated with the optimal primal solution x^* is given by

$$S(x^*) = \{\ \{1,3\},\ \{2,3\},\ \{3,4\},\ \{3,5\}\ \}.$$

The intervals for γ are now

$$\begin{array}{lclcl}
\mathcal{B}^1 = \{1,3\} & \longrightarrow & T^1 = [3,7] & \longrightarrow & \gamma \in [2,6] \\
\mathcal{B}^2 = \{2,3\} & \longrightarrow & T^2 = [-1,3] & \longrightarrow & \gamma \in [-2,2] \\
\mathcal{B}^3 = \{3,4\} & \longrightarrow & T^3 = [-1,2] & \longrightarrow & \gamma \in [-2,1] \\
\mathcal{B}^4 = \{3,5\} & \longrightarrow & T^4 = [2,7] & \longrightarrow & \gamma \in [1,6]\ ,
\end{array}$$

as can be seen from Figure 1. So the overall critical region is $T(x^*) = [-1, 7]$ and the answer to (Q3) is $\gamma \in [-2, 6]$. Indeed, the optimal value function is linear on $T(x^*)$; a breakpoint occurs at $\gamma = -2$, while for $\gamma > 6$ the primal problem is unbounded. See Figure 2. ◇

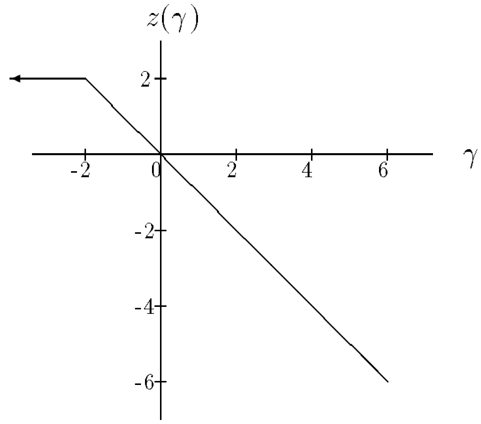

Figure 2: Optimal value function Example 2 and 3.

Note that in (Q3) we consider primal optimal bases associated with the given optimal solution x^*. Evans and Baker [5] suggest to take the union of the intervals of all primal optimal bases, also using the bases associated with different optimal solutions. Obviously, this may yield an overall critical region containing points of two linearity intervals, and so containing points where the current optimal solution x^* is not optimal (see also [25] and [6]). This suggests that the decision maker has all optimal basic solutions available and desires the interval where at least one of them remains optimal. As Steinberg and Aucamp [25] notice, it is questionable whether this is useful information from a practical point of view.

2.2 Shadow prices and shadow costs

The second important aspect of postoptimal analysis is the determination of shadow prices (shadow costs), that is, the rate at which the optimal objective value changes as a result of a small change in an element of the RHS-vector b (or the COST-vector c). Considering a nondegenerate optimal solution, it is well-known that the shadow prices for the constraints in (P) are given by the optimal dual solution y^*. In case of primal degeneracy, there may be several optimal dual basic solutions and the usual interpretation is not valid; instead, one has to distinguish between the rates of change as a consequence of an increase and as a consequence of a decrease in the RHS-coefficient (for a short history on this, see Gal [8]). Rubin and Wagner [23] indicate the traps and give a number of tips for correct interpretation of results of the dual problem in practice.

Gauvin [10] used *left and right shadow prices* to distinguish between increasing and decreasing the RHS-coefficient. In the case of varying the coefficient $b_i = \beta$, the right shadow price p_i^+ for the $i-th$ constraint is the rate at which the objective value changes as a consequence of an increase in λ; the left shadow price p_i^- gives the rate when λ decreases. Gauvin proves that

$$p_i^- \quad = \quad \min\{\, y_i \;:\; y \text{ is optimal in (D)}\} \qquad (1)$$

$$p_i^+ \quad = \quad \max\{\, y_i \;:\; y \text{ is optimal in (D)}\}, \tag{2}$$

and moreover that p_i^- and p_i^+ are the left respectively right derivatives of the (piecewise linear) optimal value function at the current value of the parameter. From this it is easy to see that

$$\begin{aligned} p_i^- &= \min_k\{\, y_i^k \}, \\ p_i^+ &= \max_k\{\, y_i^k \}, \end{aligned}$$

where y^k $(k = 1, \ldots, K)$ are the optimal dual basic solutions. Analogous results have been derived in [2,3,12,17]. As may be clear, the values of p_i^+ and p_i^- can be found by applying a parametric programming algorithm. Knolmayer [17] states that his algorithm for determining critical regions can also be applied to the computation of shadow prices.

We illustrate the notion of left and right shadow prices with the following example.

Example 4 Let the following primal-dual pair of LP problems be given:

$$\begin{aligned} &\min\{-2x_2 + 2x_3 + 4x_4 + 5x_5 + 6x_6 \;:\; -x_1 - 2x_2 + x_4 + x_5 = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_2 - x_3 - x_4 + x_6 = -1,\; x \geq 0\} \\ &\max\{y_1 - y_2 \;:\; -y_1 \leq 0, -2y_1 - y_2 \leq -2,\; -y_2 \leq 2,\; y_1 - y_2 \leq 4,\; y_1 \leq 5,\; y_2 \leq 6\,\}, \end{aligned}$$

(so the LP problems as in Example 2 with $\gamma^0 = -2$ and a slightly different dual objective). The optimal dual basic solutions are

$$y^1 = (2, -2)^T \quad y^2 = (5, 1)^T.$$

So the left and right shadow prices are given by

$$\begin{aligned} y_1^- &= 2 & y_1^+ &= 5 \\ y_2^- &= -2 & y_2^+ &= 1 \end{aligned}$$

The optimal objective value is 4. From y^1 it could erroneously be concluded that a unit increase in b_1 (from 1 to 2) would yield a value 6, whereas the correct answer is 9; also y^2 suggests that decreasing b_2 from -1 to -2 gives an optimal value 3 instead of the true value 6. $\diamond$

Obviously an analogous derivation can be applied to shadow costs (cf. Greenberg [12]). This yields left and right shadow costs q_i^- and q_i^+, which represent the rate of change in the optimal value when the parameter $c_j = \gamma$ decreases respectively increases. So

$$\begin{aligned} q_i^- &= \max\{\, x_j \;:\; x \text{ is optimal in (P)}\} \\ q_i^+ &= \min\{\, x_j \;:\; x \text{ is optimal in (P)}\}. \end{aligned}$$

3 Parametric Analysis using Optimal Partitions

In the previous section we made clear that the phenomenon of nonuniqueness of an optimal basis in degenerate problems causes troubles when performing postoptimal analysis. The aim of this section is to show that the use of *optimal partitions* is not affected by such phenomena, cf. [1,15]. The results to be presented in this section were first obtained by [1], who use the geometry of the linear programming problem in their proofs. Here we present new and elegant proofs, which use concepts from the interior point methodology and all essentially have the same structure.

3.1 The optimal partition

An early result in the theory of linear programming concerns the existence of a *strictly complementary (primal-dual) optimal pair* (see [11]). This is an optimal solution pair (x^c, s^c) satisfying $x^c + s^c > 0$. We call x (s or y) a strictly complementary solution if either belongs to a strictly complementary optimal pair. For any vector $x \in \mathbb{R}^n$ its *support* is the set of coordinate indices i for which $x_i > 0$. Obviously a strictly complementary optimal pair (x, s) uniquely determines a partition (B, N) of the index set $\{1, \ldots, n\}$ via the support:

$$\begin{aligned} x_i > 0 &\iff i \in B \\ s_i > 0 &\iff i \in N. \end{aligned}$$

This partition is called the *optimal partition* of the problems (P) and (D). Using the optimal partition we can write the optimal faces $\mathcal{P}^*$ and $\mathcal{D}^*$ of (P) and (D) as

$$\begin{aligned} \mathcal{P}^* &= \{x \; : \; Ax = b, \; x_B \geq 0, \; x_N = 0\} \\ \mathcal{D}^* &= \{(y, s) \; : \; A^T y + s = c, \; s_B = 0, \; s_N \geq 0\}. \end{aligned}$$

Note that N is the set containing the indices of the variables which are zero in any optimal solution of (P). Likewise, B contains the indices of the constraints in (D) which are binding in any optimal solution of (D). So the knowledge of a strictly complementary solution of the pair (P) and (D) gives much more information than only one optimal basic solution does (see Greenberg [13] for some practical examples where this information is useful or necessary).

We now point out the relationship between the optimal partition and IPM's. When $\mathcal{P}^*$ and $\mathcal{D}^*$ are both bounded sets, the analytic centers (Sonnevend [24]) of the optimal faces $\mathcal{P}^*$ and $\mathcal{D}^*$ are the points determined by

$$\begin{aligned} \hat{x} &= \operatorname{argmax} \{ \prod_{i \in B} x_i \; : \; x \in \mathcal{P}^* \} \\ \hat{s} &= \operatorname{argmax} \{ \prod_{i \in N} s_i \; : \; s \in \mathcal{D}^* \}. \end{aligned}$$

This optimal solution $\hat{x}$ is called the *central solution* of (P); likewise $\hat{s}$ is the central solution of (D). Since the central solutions form a strictly complementary pair they determine the optimal partition. It is well-known that the *central path* of the LP problem ends up in the central solution (see e.g. Megiddo [19]). Moreover, in many of the existing IPM's the *central path* of the LP problem is approximately followed, and these methods are able to generate strictly complementary optimal pairs (see e.g. Mehrotra and Ye [21]).

In this section we investigate the effect of changes in b and c on the optimal value of (P). Therefore, we denote (P) as (P(b,c)) and (D) as (D(b,c)). The optimal value is denoted as $z(b,c)$, and the optimal partition as $\pi(b,c)$. We call the pair (b,c) a *feasible pair* if the problems (P(b,c)) and (D(b,c)) are both feasible. If the problem (P(b,c)) is unbounded then we define $z(b,c)=-\infty$, and if its dual (D(b,c)) is unbounded then we define $z(b,c)=\infty$. If both (P(b,c)) and (D(b,c)) are infeasible then $z(b,c)$ is undefined.

In fact we are mainly interested in parametric perturbations of b and c, as considered in the classical topics of parametric and postoptimal analysis. So we want to study the behavior of $z(b+\lambda\Delta b, c+\tau\Delta c)$, where Δb and Δc are given *perturbation* vectors, as a function of the parameters λ and τ. It is well known, and will be proved below, that $z(b+\lambda\Delta b,c)$ is a piecewise linear convex function of λ and $z(b,c+\tau\Delta c)$ is a piecewise linear concave function of τ. In practice one is interested in the *slopes* of these functions for given values of b and c, at $\lambda=0$ and $\tau=0$ respectively (cf. shadow prices and shadow costs), and also in the interval of the optimal value function to which b and c belong (sensitivity analysis). The main result is that along the linearity intervals of the optimal value functions $z(b+\lambda\Delta b,c)$ and $z(b,c+\tau\Delta c)$ the optimal partitions of the underlying problems remain constant. We also show that in a breakpoint the optimal partition differs from the partitions of the 'surrounding' linearity intervals. In all proofs we use a similar structure: we construct a solution pair that is feasible for the given problem pair and then show that this pair is strictly complementary and renders the required optimal partition.

The first result we prove relates having the same optimal partition for different values of the parameter to linearity of the optimal value function.

Lemma 4 *Let (b^0,c^0) and (b^1,c^1) be feasible pairs such that the partitions $\pi(b^0,c^0)$ and $\pi(b^1,c^1)$ are equal. If b is any convex combination of b^0 and b^1, and c of c^0 and c^1, then $\pi(b,c)$ is the same partition. Moreover, $z(b,c)$ is linear on the line segment from (b^0,c) to (b^1,c) and also on the line segment from (b,c^0) to (b,c^1).*

Proof:
Let $\pi=(B,N)$ denote the common partition for the feasible pairs (b^0,c^0) and (b^1,c^1). So we have $\pi=\pi(b^0,c^0)=\pi(b^1,c^1)$. Let $0\le\mu_1,\mu_2\le 1$ be such that

$$\begin{aligned} b &= \mu_1 b^1+(1-\mu_1)b^0, \\ c &= \mu_2 c^1+(1-\mu_2)c^0. \end{aligned}$$

Furthermore, let x^0,s^0,y^0 denote the central solutions of (P) and (D) for the pair (b^0,c^0), and, similarly, x^1,s^1,y^1 for the pair (b^1,c^1). Now define

$$\begin{aligned} x &:= \mu_1 x^1+(1-\mu_1)x^0, \\ y &:= \mu_2 y^1+(1-\mu_2)y^0, \\ s &:= \mu_2 s^1+(1-\mu_2)s^0. \end{aligned}$$

Then one easily checks that x is feasible for $(\mathrm{P}(b,c))$, and (y,s) for $(\mathrm{D}(b,c))$. Since $\pi(b^1,c^1) = \pi(b^0,c^0) = (B,N)$, it follows that $x_k > 0$ if and only if $k \in B$, and $s_k > 0$ if and only if $k \in N$. Therefore, the pair (x,s) is strictly complementary (and hence optimal) for the pair (b,c). The partition determined by this pair being (B,N), the first part of the lemma follows.

Now let us deal with the proof of the second part. Since x is optimal for $(\mathrm{P}(b,c))$, we have $z(b,c) = c^T x$. So

$$z(b,c) = (\mu_2 c^1 + (1-\mu_2)c^0)^T(\mu_1 x^1 + (1-\mu_1)x^0).$$

This makes clear that $z(b,c)$ is a bilinear function of μ_1 and μ_2. Now fixing c, i.e. fixing μ_2, this expression becomes linear in μ_1, and hence $z(b,c)$ is linear on the line segment from (b^0,c) to (b^1,c). Similarly, fixing b, i.e., fixing μ_1, the expression for $z(b,c)$ becomes linear in μ_2, so $z(b,c)$ is also linear on the line segment from (b,c^0) to (b,c^1). □

In the next subsections we deal with the converse implication. We will show that if $z(b,c)$ is linear on the line segment from (b^0,c) to (b^1,c) then the partition is constant on this line segment with a possible exception in the end points of the line segment; a similar result holds if $z(b,c)$ is linear on the line segment from (b,c^0) to (b,c^1).

3.2 Perturbations in the right hand side vector

To facilitate the discussion we introduce the notation

$$b(\lambda) := \lambda b^1 + (1-\lambda)b^0,$$

and

$$\psi(\lambda) := z(b(\lambda),c),$$

where b^0, b^1 and c are such that the pairs (b^0,c) and (b^1,c) are feasible. For each λ we denote the corresponding optimal partition by $\pi(\lambda) = (B^\lambda, N^\lambda)$, and the corresponding central solutions by $x^\lambda, y^\lambda, s^\lambda$. Note that the feasible region of the dual problem $(\mathrm{D}(b(\lambda),c))$ is independent of λ. It is simply given by $\{\, y \;:\; A^T y \le c\}$. So the dual problem is feasible for each λ and $\psi(\lambda)$ is well defined for each $\lambda \in \mathbb{R}$. The following result is well-known; we give a simple proof using optimal partitions.

Theorem 1 *The value function $\psi(\lambda)$ is piecewise linear and convex.*

Proof:
Let λ move from $-\infty$ to ∞. Then $\pi(\lambda)$ runs through a set of partitions of the index set $\{1,\ldots,n\}$. Since the number of such partitions is finite, we can get a partition of the real line into a finite number of nonoverlapping intervals by 'starting' a new interval every time the partition $\pi(\lambda)$ changes. Now we know from Lemma 4 that on each subinterval the value function $\psi(\lambda)$ is linear. To prove the convexity let $(\mathrm{P}(b(\lambda_1),c))$ and $(\mathrm{P}(b(\lambda_2),c))$ have optimal solutions x^1 and x^2 respectively; let $0 < \mu < 1$ and $\lambda = \mu\lambda_1 + (1-\mu)\lambda_2$. It is easily verified that

$$x := \mu x^1 + (1-\mu)x^2$$

is feasible for $(\mathrm{P}(b(\lambda), c))$, so $c^T x \geq \psi(\lambda)$. Since

$$c^T x = \mu c^T x^1 + (1-\mu) c^T x^2 = \mu \psi(\lambda_1) + (1-\mu)\psi(\lambda_2)$$

the convexity of $\psi(\lambda)$ follows. This proves the theorem. □

We proceed by showing that in the open part of a linearity interval the optimal partition is constant. In other words, the linearity intervals are precisely the intervals constructed in the proof of Theorem 1.

Lemma 5 *Let for some b^0, b^1, c the value function $\psi(\lambda)$ be linear for $0 \leq \lambda \leq 1$. So*

$$\psi(\lambda) = \lambda \psi(1) + (1-\lambda)\psi(0).$$

Then $\pi(\lambda)$ is independent of λ for all values of λ in the open interval $(0, 1)$.

Proof:
For $0 \leq \lambda \leq 1$, let the triple $(x^\lambda,\, y^\lambda,\, s^\lambda)$ be the central solution for the pair $(b(\lambda), c)$, and (B^λ, N^λ) the corresponding optimal partition. Since y^λ is feasible for both $(\mathrm{D}(b^0, c))$ and $(\mathrm{D}(b^1, c))$ we have, $(b^0)^T y^\lambda \leq (b^0)^T y^0 = \psi(0)$ and $(b^1)^T y^\lambda \leq (b^1)^T y^1 = \psi(1)$. Using this and the definition of $b(\lambda)$ we may write

$$\psi(\lambda) = \lambda \psi(1) + (1-\lambda)\psi(0) \geq \lambda (b^1)^T y^\lambda + (1-\lambda)(b^0)^T y^\lambda = b(\lambda)^T y^\lambda = \psi(\lambda).$$

For $0 < \lambda < 1$, this implies that

$$(b^0)^T y^\lambda = \psi(0) \text{ and } (b^1)^T y^\lambda = \psi(1).$$

Now let $0 \leq \alpha \leq 1$. Then we may write

$$b(\alpha)^T y^\lambda = (\alpha b^1 + (1-\alpha) b^0)^T y^\lambda = \alpha \psi(1) + (1-\alpha)\psi(0) = \psi(\alpha).$$

This proves that y^λ is optimal for all problems $(\mathrm{D}(b(\alpha), c))$, with $0 \leq \alpha \leq 1$. From this we derive that $(x^\alpha)^T s^\lambda = 0$. Therefore, $N^\lambda \cap B^\alpha = \emptyset$. This is equivalent to $N^\lambda \subseteq N^\alpha$ and $B^\alpha \subseteq B^\lambda$. Now taking $0 < \alpha < 1$, we can apply the same argument to α and obtain the converse inclusions $N^\alpha \subseteq N^\lambda$ and $B^\lambda \subseteq B^\alpha$. We conclude that $B^\lambda = B^\alpha$ and $N^\lambda = N^\alpha$. This proves the lemma. □

From this lemma (and its proof) we draw an important conclusion, which we state as a corollary.

Corollary 1 *The set of optimal solutions of (D($b(\lambda), c$)) is constant on the interval $0 < \lambda < 1$ and, moreover, each of these solutions is also optimal for the extreme cases $\lambda = 0$ and $\lambda = 1$.*

We have now shown that on the open part of the linearity interval the partition does not change; moreover, we have that a linearity interval corresponds to the situation that the optimal dual solution remains optimal in case of varying λ. This is exactly the issue of interest in postoptimal analysis. We still have to answer the question how to perform the postoptimal analysis, that is, how to find the slopes of the optimal value function when we are in a breakpoint and how to find the linearity interval otherwise. This is closely related to the question how to find the 'surrounding' partitions of a given partition. We show that these partitions can be found by solving two appropriate linear programming problems. These problems are stated in terms of the given partition and the perturbation vector. We need to distinguish between the case that the given partition belongs to a breakpoint and the case that the given partition belongs to a linearity interval.

To simplify the presentation we assume for the moment that $\lambda = 0$ and $\lambda = 1$ are two consecutive breakpoints of $\psi(\lambda)$. We denote the perturbation vector by $\Delta b := b^1 - b^0$. As a consequence we have

$$b(\lambda) = b^0 + \lambda \Delta b.$$

The partitions $(B^\lambda, N^\lambda), 0 < \lambda < 1$, are all equal, so we simply denote them by $\bar{\pi} = (\bar{B}, \bar{N})$. Similarly, the central solutions $y^\lambda, 0 < \lambda < 1$, being all equal, are denoted by $\bar{y}$. From the proof of Lemma 5 we deduce the relation

$$\psi(\lambda) = \psi(0) + \lambda(\Delta b)^T \bar{y},\ 0 \leq \lambda \leq 1.$$

We now show how the optimal partition in the linearity interval $0 < \lambda < 1$ can be determined from the optimal partition at the breakpoint $\lambda = 0$. To this end we define the following pair of dual linear programming problems:

$$(P_{\mapsto}^{\Delta b}) \quad \min\left\{c^T x : Ax = \Delta b,\ x_{N^0} \geq 0\right\},$$

$$(D_{\mapsto}^{\Delta b}) \quad \max\{(\Delta b)^T y \ :\ A^T y + s = c,\ s_{B^0} = 0,\ s_{N^0} \geq 0\}.$$

Note that in ($\mathrm{P}_{\mapsto}^{\Delta b}$) the variables x_i, $i \in B^0$, are free. As a consequence we need to define the concepts of strictly complementary solution and optimal partition for this particular case. This can be done in an obvious way. Replace the free part x_{B^0} of x by $x_{B^0}^+ - x_{B^0}^-$, with $x_{B^0}^+$ and $x_{B^0}^-$ nonnegative, and we are again in the standard format. As a consequence we obtain that there exists a solution triple $(\tilde{x},\ \tilde{y},\ \tilde{s})$ such that for each $i \in N^0$ one has $\tilde{x}_i > 0$ if and only if $\tilde{s}_i = 0$. In this way we obtain a (unique) partition $(\tilde{B},\ \tilde{N})$ of the set N^0, namely by taking for $\tilde{B}$ the subset of N^0 for which the coordinates $\tilde{x}_i$ are positive and for $\tilde{N}$ the subset of N^0 for which the coordinates $\tilde{s}_i$ are positive. Then we call $(B^0 \cup \tilde{B},\ \tilde{N})$ the optimal partition of the problem. Any such solution with the above property is called strictly complementary. It is clear that strict complementarity implies optimality.

It may be worthwhile to indicate that the feasible region of the dual problem ($\mathrm{D}_{\mapsto}^{\Delta b}$) is the optimal face for the dual problem ($\mathrm{D}(b^0, c)$). As a consequence it admits the analytic center $(y^0,\ s^0)$ of this face as a feasible solution. Recall that also $(\bar{y},\ \bar{s})$ is in this face.

Theorem 2 *The optimal partition for the pair of dual problems ($P_{\mapsto}^{\Delta b}$) and ($D_{\mapsto}^{\Delta b}$) is just $(\bar{B}, \bar{N})$. Furthermore, $\bar{y}$ is the central solution of ($D_{\mapsto}^{\Delta b}$).*

Proof:
Let $0 < \lambda < 1$, and consider

$$x := \frac{x^\lambda - x^0}{\lambda}.$$

Since $x^0_{N^0} = 0$ one has $x_{N^0} \geq 0$. Obviously $Ax = \Delta b$. So x is feasible for ($\mathrm{P}^{\Delta b}_{\mapsto}$). We already observed that the dual problem ($\mathrm{D}^{\Delta b}_{\mapsto}$) admits $(\bar{y}, \bar{s})$ as a feasible solution. So we have found a pair of feasible solutions for ($\mathrm{P}^{\Delta b}_{\mapsto}$) and ($\mathrm{D}^{\Delta b}_{\mapsto}$). We conclude the proof by showing that this pair is strictly complementary and that it determines $\bar{\pi} = (\bar{B}, \bar{N})$ as the optimal partition. Recall that the support of x^λ is $\bar{B}$ and the support of x^0 is B^0. So, for $i \in N^0$, we have $x_i > 0$ if and only if $i \in N^0 \setminus \bar{N}$. On the other hand, if $i \in N^0$, then we have $\bar{s}_i > 0$ if and only if $i \in \bar{N}$. This proves that the given pair of solutions is strictly complementary and that the optimal partition is just $\bar{\pi} = (\bar{B}, \bar{N})$. □

An immediate consequence of the theorem is that the optimal value of the problem ($\mathrm{D}^{\Delta b}_{\mapsto}$) gives the slope of the optimal value function when the parameter increases. The partition belonging to the left of a breakpoint can be obtained by solving a similar pair of linear programming problems, namely:

$$(P^{\Delta b}_{\leftarrow}) \quad \min\left\{c^T x : Ax = -\Delta b,\, x_{N^1} \geq 0\right\},$$

$$(D^{\Delta b}_{\leftarrow}) \quad \max\{-(\Delta b)^T y \ : \ A^T y + s = c,\, s_{B^1} = 0,\, s_{N^1} \geq 0\}.$$

Without further proof we state

Theorem 3 *The optimal partition for the pair of dual problems ($P^{\Delta b}_{\leftarrow}$) and ($D^{\Delta b}_{\leftarrow}$) is just $(\bar{B}, \bar{N})$. Furthermore, $\bar{y}$ is the central solution of ($D^{\Delta b}_{\leftarrow}$).*

For future use we include the following

Lemma 6 $(\Delta b)^T(\bar{y} - y^0) > 0$ *and* $(\Delta b)^T(y^1 - \bar{y}) > 0$.

Proof:
Recall that y^0 is the central solution (hence the analytic center of the optimal (dual) face) for the case that $\lambda = 0$. Now Theorem 2 makes clear that when we maximize $(\Delta b)^T y$ over this face then $\bar{y}$ is the central solution. Also, as a consequence of Theorem 3, if we minimize $(\Delta b)^T y$ over the optimal (dual) face for the case that $\lambda = 0$, then the optimal partition is the optimal partition associated to the linearity interval to the left of the breakpoint $\lambda = 0$; let $\bar{\bar{y}}$ denote the central solution for this problem. Since the value function $\psi(\lambda)$ has a breakpoint at $\lambda = 0$, its left and right derivatives to λ are different at $\lambda = 0$. Since these derivatives are given by $(\Delta b)^T \bar{\bar{y}}$ and $(\Delta b)^T \bar{y}$ respectively, we conclude that $(\Delta b)^T \bar{\bar{y}}$ and $(\Delta b)^T \bar{y}$ are different. This implies that $(\Delta b)^T \bar{\bar{y}} < (\Delta b)^T \bar{y}$, because $\bar{y}$ maximizes and $\bar{\bar{y}}$ minimizes $(\Delta b)^T y$ over the optimal dual face for $\lambda = 0$, so it follows that $(\Delta b)^T y$ is not constant in this face. Since y^0 is the analytic center of this face, we conclude that $(\Delta b)^T \bar{\bar{y}} < (\Delta b)^T y^0 < (\Delta b)^T \bar{y}$. This implies the first inequality of the lemma; the second follows in a similar way. □

Finally we consider the case that the optimal partition $\bar{\pi} = (\bar{B}, \bar{N})$ associated with some given linearity interval is known. We show how the optimal partitions at the surrounding breakpoints can be found from this partition and the perturbation vector Δb. In the analysis below it will be convenient to assume that the $\lambda = 0$ belongs to the linearity interval under consideration, and that the surrounding breakpoints (if they exist) occur at $\lambda^- < 0$ and $\lambda^+ > 0$ respectively. For the present purpose we consider the following pair of dual problems.

$$(P^{\Delta b}_{\leftarrow}) \quad \min\{\lambda \;:\; Ax = b(\lambda),\, x_{\bar{B}} \geq 0,\, x_{\bar{N}} = 0\},$$
$$(D^{\Delta b}_{\leftarrow}) \quad \max\{(b^0)^T y \;:\; A^T y + s = 0,\, (\Delta b)^T y = -1,\, s_{\bar{B}} \geq 0\}.$$

Since these problems do not have the standard format we have to discuss the meaning of strictly complementary solution and optimal partition for these problems. Note that ($P^{\Delta b}_{\leftarrow}$) can be brought in the standard format by omitting the variables x_i, $i \in \bar{N}$. By omitting also the constraints in ($D^{\Delta b}_{\leftarrow}$) which are indexed by the indices in $\bar{N}$, we obtain a completely equivalent pair of dual problems, namely

$$\min\{\lambda \;:\; A_{\bar{B}} x_{\bar{B}} = b(\lambda),\, x_{\bar{B}} \geq 0\},$$
$$\max\{(b^0)^T y \;:\; A_{\bar{B}}^T y + s_{\bar{B}} = 0,\, (\Delta b)^T y = -1,\, s_{\bar{B}} \geq 0\}.$$

Let $(\tilde{B}, \tilde{N})$ be the optimal partition for this pair. Then this is a partition of the set $\bar{B}$ with the property that there exists a solution triple $(\tilde{x}, \tilde{y}, \tilde{s})$ for the original pair of problems such that for each $i \in \bar{B}$ one has $\tilde{x}_i > 0$ if and only if $\tilde{s}_i = 0$. Now it becomes natural to define $(\tilde{B}, \tilde{N} \cup \bar{N})$ as the optimal partition of the original problems. Any solution triple $(\tilde{x}, \tilde{y}, \tilde{s})$ with the above property is called strictly complementary. It is clear that strict complementarity implies optimality.

Theorem 4 *The optimal partition for the pair of dual problems ($P^{\Delta b}_{\leftarrow}$) and ($D^{\Delta b}_{\leftarrow}$) is just $\pi(\lambda^-)$. Furthermore, y^{λ^-} is the central solution of ($D^{\Delta b}_{\leftarrow}$).*

Proof:
Observe that

$$x := x^{\lambda^-},\; \lambda := \lambda^-$$

is feasible for the primal problem, since x^{λ^-} is the central solution of the problem (P(b(λ^-),c). We show that

$$y := \frac{y^{\lambda^-} - \bar{y}}{(\Delta b)^T(\bar{y} - y^{\lambda^-})}$$

is feasible for the second problem. First we deduce from Lemma 6, that $(\Delta b)^T(\bar{y} - y^{\lambda^-})$ is positive, so y is well defined. Clearly $(\Delta b)^T y = -1$. Furthermore, one has

$$\left((\Delta b)^T(\bar{y} - y^{\lambda^-})\right) A^T y = A^T(y^{\lambda^-} - \bar{y}) = \bar{s} - s^{\lambda^-}.$$

Since $\bar{s}_{\bar{B}} = 0$ and $s^{\lambda^-} \geq 0$, it follows that $\bar{s}_{\bar{B}} - s^{\lambda^-}_{\bar{B}} = -s^{\lambda^-}_{\bar{B}} \leq 0$. So the given y is feasible for the dual problem. The given pair (x, y, s) is strictly complementary with the partition $(B^{\lambda^-}, N^{\lambda^-})$, since for $i \in \bar{B}$ we have $x_i > 0$ if and only if $i \in B^{\lambda^-}$, and $s_i = 0$ if and only if $i \in B^{\lambda^-}$. Hence the theorem is proved. □

An immediate consequence of this theorem is that the optimal value of $(\mathrm{P}^{\Delta b}_{\leftarrow})$ equals λ^-, so we also find the breakpoint to the left. To obtain the breakpoint λ^+ to the right of $\lambda = 0$ and its optimal partition, a similar pair of linear programming problems is needed:

$$(P^{\Delta b}_{\rightarrow}) \quad \max\{\lambda \;:\; Ax = b(\lambda),\, x_{\bar{B}} \geq 0,\, x_{\bar{N}} = 0\},$$
$$(D^{\Delta b}_{\rightarrow}) \quad \min\{(b^0)^T y \;:\; A^T y + s = 0,\, (\Delta b)^T y = 1,\, s_{\bar{B}} \geq 0\}.$$

Without further proof we state

Theorem 5 *The optimal partition for the pair of dual problems $(P^{\Delta b}_{\rightarrow})$ and $(D^{\Delta b}_{\rightarrow})$ is just $\pi(\lambda^+)$. Furthermore, y^{λ^+} is the central solution of $(D^{\Delta b}_{\rightarrow})$.* □

3.3 Perturbations in the objective vector

In this section we consider the effect of variations in the vector c on the value function. It turns out that by "dualizing" the results of the previous section we obtain the appropriate results. The proofs are based on the same idea as for their dual counterparts. One checks that some natural candidate solutions for both problems indeed are feasible, and then shows that these solutions are strictly complementary with the correct partition. Therefore, in this section we state these results without proofs. The discussion below is facilitated by using the notation

$$\Delta c := c^1 - c^0,$$

$$c(\tau) := c^0 + \tau \Delta c = \tau c^1 + (1 - \tau) c^0,$$

and

$$\chi(\tau) := z(b, c(\tau)),$$

where b and c^0, c^1 are such that the pairs (b, c^0) and (b, c^1) are feasible. For each τ we denote the corresponding optimal partition by $\pi(\tau) = (B^\tau, N^\tau)$.

Theorem 6 *The value function $\chi(\tau)$ is piecewise linear and concave.*

Lemma 7 *Let for some b, c^0, c^1 the value function $\chi(\tau)$ be linear for $0 \leq \tau \leq 1$. So*

$$\chi(\tau) = \tau \chi(1) + (1 - \tau) \chi(0).$$

Then $\pi(\tau)$ is independent of τ for all values of τ in the open interval $(0, 1)$.

Corollary 2 *The set of optimal solutions of $(P(b, c(\tau))$ is constant on the interval $0 < \tau < 1$ and, moreover, each of these solutions is also optimal for the extreme cases $\tau = 0$ and $\tau = 1$.*

Just as in the previous section we conclude with the problem of finding the 'surrounding' partitions of a given partition. Also in the present case these partitions can be found by solving appropriate linear programming problems. These problems are formulated in terms of the given partition and the direction Δc of the perturbation in c.

We start with the case that the given partition belongs to a breakpoint. Without loss of generality we assume again that $\tau = 0$ and $\tau = 1$ are two consecutive breakpoints of $\chi(\tau)$, and that we have given either the partition $\pi(0) = (B^0, N^0)$ or the partition $\pi(1) = (B^1, N^1)$. We show that either of these partitions, together with the perturbation vector Δc, determines the partition associated to the interval between these breakpoints, which will be denoted by $\bar{\pi} = (\bar{B}, \bar{N})$. The central solution $x^\tau, 0 < \tau < 1$, is denoted by $\bar{x}$. As a consequence we have

$$\chi(\tau) = \chi(0) + \tau(\Delta c)^T \bar{x},\; 0 \leq \tau \leq 1.$$

We consider the following pair of dual linear programming problems:

$$\begin{array}{ll} (P_{\mapsto}^{\Delta c}) & \min\left\{(\Delta c)^T x \;:\; Ax = b,\, x_{B^0} \geq 0,\, x_{N^0} = 0\right\}, \\ (D_{\mapsto}^{\Delta c}) & \max\left\{b^T y \;:\; A^T y + s = \Delta c,\, s_{B^0} \geq 0\right\}. \end{array}$$

Theorem 7 *The optimal partition for the pair of dual problems $(P_{\mapsto}^{\Delta c})$ and $(D_{\mapsto}^{\Delta c})$ is just $(\bar{B}, \bar{N})$. Furthermore, $\bar{x}$ is the central solution of $(P_{\mapsto}^{\Delta c})$.*

A similar result can be obtained for the optimal partition at $\tau = 1$. Defining the pair of dual linear programming problems

$$\begin{array}{ll} (P_{\mapsfrom}^{\Delta c}) & \max\left\{(\Delta c)^T x \;:\; Ax = b,\, x_{B^1} \geq 0,\, x_{N^1} = 0\right\}, \\ (D_{\mapsfrom}^{\Delta c}) & \max\left\{b^T y \;:\; A^T y + s = -\Delta c,\, s_{B^1} \geq 0\right\}, \end{array}$$

one has

Theorem 8 *The optimal partition for the pair of dual problems $(P_{\mapsfrom}^{\Delta c})$ and $(D_{\mapsfrom}^{\Delta c})$ is just $(\bar{B}, \bar{N})$. Furthermore, $\bar{x}$ is the central solution of $(P_{\mapsfrom}^{\Delta c})$.*

Using these results one easily derives that

Corollary 3 $(\Delta c)^T(\bar{x} - x^0) < 0$ *and* $(\Delta c)^T(x^1 - \bar{x}) < 0$.

Yet we turn to the case that the optimal partition $\bar{\pi} = (\bar{B}, \bar{N})$ associated to a linearity interval is given. This is the contents of the last two results. It is convenient to assume that the vector c^0 belongs to the linearity interval under consideration, and that the surrounding breakpoints, if they exist, occur at τ^- and τ^+ respectively. We consider the following pair of dual problems.

$$\begin{array}{ll} (P_{\rightarrow}^{\Delta c}) & \max\{\tau \;:\; A^T y + s = c(\tau),\, s_{\bar{B}} = 0,\, s_{\bar{N}} \geq 0\}, \\ (D_{\rightarrow}^{\Delta c}) & \min\{(c^0)^T x \;:\; Ax = 0,\, (\Delta c)^T x = -1,\, x_{\bar{N}} \geq 0\}. \end{array}$$

From the discussion in the previous section it will be clear how to define in a natural way the notions of strictly complementary solution and optimal partition for these problems. We now may state

Theorem 9 *The optimal partition for the pair of dual problems* $(P_{\rightarrow}^{\Delta c})$ *and* $(D_{\rightarrow}^{\Delta c})$ *is just* $\pi(\tau^{+})$. *Furthermore,* $x^{\tau^{+}}$ *is the central solution of* $(D_{\rightarrow}^{\Delta c})$.

A similar result can be obtained for the pair of dual linear programming problems given by:

$$\begin{array}{ll} (P_{\leftarrow}^{\Delta c}) & \min\{\tau \ : \ A^T y + s = c(\tau),\, s_{\bar{B}} = 0,\, s_{\bar{N}} \geq 0\}, \\ (D_{\leftarrow}^{\Delta c}) & \min\{(c^0)^T x \ : \ Ax = 0,\, (\Delta c)^T x = 1,\, x_{\bar{N}} \geq 0\}. \end{array}$$

The following theorem will be no surprise.

Theorem 10 *The optimal partition for the pair of dual problems* $(P_{\leftarrow}^{\Delta c})$ *and* $(D_{\leftarrow}^{\Delta b})$ *is just* $\pi(\tau^{-})$. *Furthermore,* $x^{\tau^{-}}$ *is the central solution of* $(D_{\leftarrow}^{\Delta b})$. □

3.4 Example

We conclude this section by presenting an example which demonstrates the results obtained so far, and elucidates the relationship between optimal partitions and linearity of the optimal value function.

Example 5 Consider the following pair of LP problems:

$$\begin{array}{l} \min\{\ 2x_1 + 2x_3 + \gamma x_5 + 3x_6 \ : \ x_1 - x_2 + x_5 + x_6 = 1,\ x_3 - x_4 + x_5 + x_6 = \beta,\ x \geq 0\} \\ \max\{\ y_1 + \beta y_2 \ : \ y_1 \leq 2,\ -y_1 \leq 0,\ y_2 \leq 2,\ -y_2 \leq 0,\ y_1 + y_2 \leq \gamma,\ y_1 + y_2 \leq 3\}, \end{array}$$

being parametrized by β and γ. The feasible region of the dual problem is drawn in Figure 3. Note that the dual problem is feasible for $\gamma \geq 0$, and bounded for all values of β. Figure 4 shows the optimal partitions for all feasible pairs, and in Figure 5 the surface of the optimal value function $z(\beta, \gamma)$ is depicted. Note the correspondence between the regions where the optimal partition does not change and the areas where z is (bi)linear. ◇

4 Interior Point Approach to Postoptimal Analysis

Using the results of the previous section we reconsider postoptimal analysis in linear programming from the interior point of view. As is clear by now, we can use the optimal partition to obtain the linearity intervals and the slopes of the optimal value function by solving suitable linear programming problems. This yields clear and unambiguous results, not apt with the possible interpretation problems as in the analysis using bases. In postoptimal analysis we are interested in changes in single coefficients of the right-hand-side or the objective function. So we have $\Delta b = e_i$ for some $1 \leq i \leq m$ and $\Delta c = e_j$ for some $1 \leq j \leq n$.

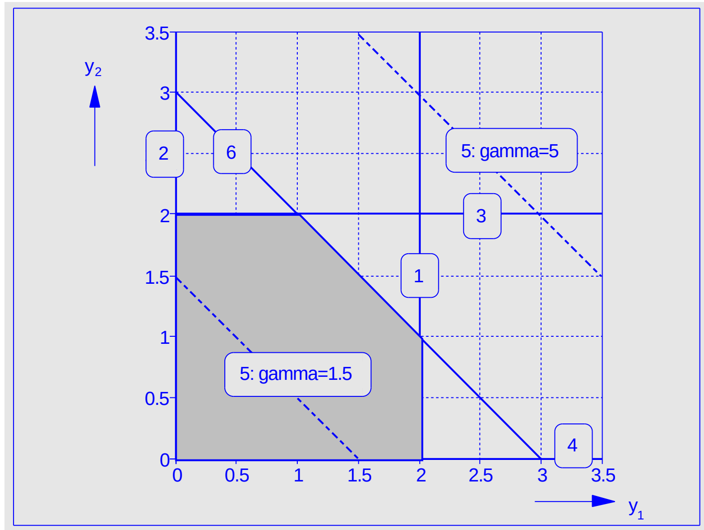

Figure 3: Feasible region for Example 5.

4.1 Sensitivity analysis

We propose the following question as being relevant in sensitivity analysis:

(Q4) In what interval may β (or γ) vary such that the optimal partition does not change?

Note that in case β (or γ) is not a breakpoint of the optimal value function (Q3) and (Q4) yield the same answer. When it is a breakpoint, then (Q4) surely gives the interval consisting of the breakpoint only, while (Q3) may give either the breakpoint, the linearity interval to the left, or the linearity interval to the right of this breakpoint.

We first consider the case where the RHS coefficient b_i is changed and c_j is fixed. Let $(\beta^0, \gamma^0) = (0, 0)$ be an initial feasible pair with partition $\pi^0 = \pi(\beta^0, \gamma^0) = (B^0, N^0)$ and let x^0, y^0 and s^0 be strictly complementary solutions.

Assume for the moment that $\lambda = 0$ does not correspond to a breakpoint of the optimal value function. We will answer (Q4) by solving the appropriate LP problems. They can be derived from Theorems 4 and 5:

$$\min_{\lambda, x_{B^0}} \{ \lambda \ : \ A_{B^0} x_{B^0} = b + \lambda e_i, \ x_{B^0} \geq 0 \} \tag{3}$$

$$\max_{\lambda, x_{B^0}} \{ \lambda \ : \ A_{B^0} x_{B^0} = b + \lambda e_i, \ x_{B^0} \geq 0 \}. \tag{4}$$

The solutions of these problems give the required interval and also imply the optimal partitions in the breakpoints. Also for all λ in the interval the value of z is now known to be

$$\psi(\lambda) = \psi(\beta^0) + \lambda y_i^0.$$

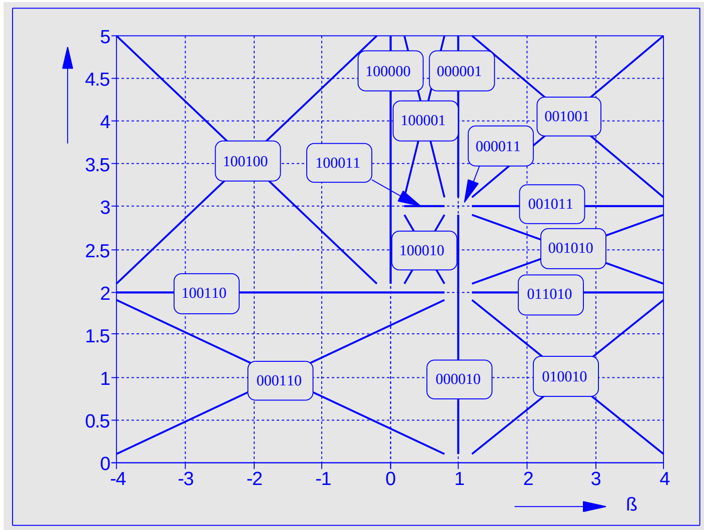

Figure 4: Optimality sets for Example 5.

For each λ in the interval it also is not difficult to construct a pair of optimal solutions for the corresponding perturbed problems. Obviously y^0 is optimal for each of the dual problems. Now let u be any vector such that $A_{B^0}u = e_i$, then $x(\lambda) = x^0 + \lambda u$ is an optimal solution for the corresponding primal problem (P($b(\lambda), c$)). This condition of u being in the columnspace of the matrix A_{B^0} appears explicitly in Adler and Monteiro [1], where it is needed to derive the theory; note that we did not need to use it in the previous section.

We will briefly discuss the differences with postoptimal analysis in the classical sense. When the optimal solution associated with the feasible pair (β^0, γ^0) is nondegenerate, it is a unique solution, equal to the optimal basic solution and to the central solution. In that case it holds that $|B| = m$ and A_B is nonsingular. Hence the interval obtained with (Q4) can be determined in the same way as in the classical analysis using (Q1).

The interesting differences occur when the optimal solution is degenerate. When the complete linearity interval need to be known, using optimal partitions the two LP problems (3) and (4) have to be solved, which can be done in polynomial time. Note that the size of these problems is in general smaller than the size of (P). Using bases would imply that all dual optimal bases might have to be used. Although some algorithms have been developed which do not need the derivation of all these bases (see [9,26], cf. Section 2), it is still not clear what is the theoretical complexity of this approach (also efficient bases enumeration techniques are needed). Extensive testing would be needed to compare these two approaches; an important aspect in this would be whether the LP problems (3) and (4) can be solved efficiently from a warm start using IPM's.

When the user is satisfied with partial information he can proceed in several ways. Using bases he can restrict himself to answer the question (Q1) and use only one optimal basis, which is cheap; alternatively he can use optimal bases to handle (Q2). Using optimal partitions partial information can be obtained by computing an optimal basis (this can be done in strongly

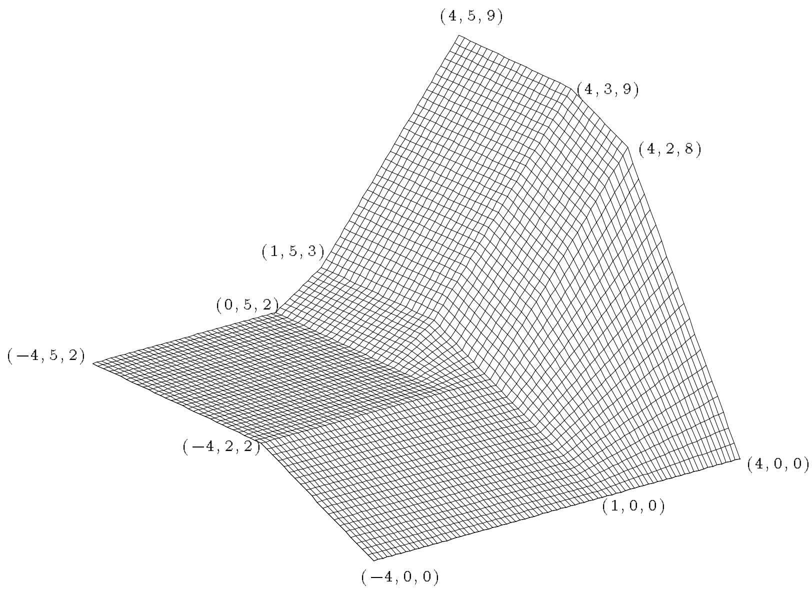

Figure 5: Optimal value function in Example 5. The triples represent values of β, γ and $z(\beta, \gamma)$ respectively.

polynomial time, see Megiddo [20]) and proceeding as above. A different way is to take an arbitrary feasible solution of either of the LP problems (3) and (4) by using an arbitrary u satisfying $A_{B^0}u = e_i$ and computing the interval where $x^0 + \lambda u \geq 0$; this is just a ratio test, hence a cheap operation (cf. Mehrotra and Monteiro [22]).

We will illustrate the primal degenerate RHS-case with the following example.

Example 6 Consider the following primal-dual pair of LP problems containing a parameter β:

$$\min\{x_3 + x_4 + 2x_5 \;:\; -x_1 + x_3 + x_5 + x_6 = 1,\; -x_2 + x_4 + x_5 - x_6 = 2 + \beta,\; x \geq 0\}$$
$$\max\{y_1 + (2+\beta)y_2 \;:\; -y_1 \leq 0,\; -y_2 \leq 0,\; y_1 \leq 1,\; y_2 \leq 1,\; y_1 + y_2 \leq 2,\; y_1 - y_2 \leq 0\}$$

Let $\beta^0 = 0$, then the optimal dual solution is $y^0 = (1,1)^T$. The optimal partition has $B^0 = \{3,4,5,6\}$. The critical region is found by solving

$$\min\{\;\lambda \;:\; x_3 + x_5 + x_6 = 1,\; x_4 + x_5 - x_6 = 2 + \lambda,\; x_{B^0} \geq 0\} \tag{5}$$

and the corresponding maximization problem. It is easily derived that this yields the interval $\lambda \in (-3, \infty)$, so $\beta \in (-3, \infty)$. The approach using bases gives the following. There are three optimal bases, each yielding a different interval for β:

$$\mathcal{B}^1 = \{3,4\} \quad \longrightarrow \quad \beta \in [-2, \infty)$$
$$\mathcal{B}^2 = \{4,5\} \quad \longrightarrow \quad \beta \in [-1, \infty)$$
$$\mathcal{B}^3 = \{4,6\} \quad \longrightarrow \quad \beta \in [-3, \infty).$$

In this example the optimal bases are sufficient for determination of the complete interval, even the basis $\mathcal{B}^3$ suffices. The dual optimal bases $\{3,5\}$ and $\{3,6\}$ need not be considered; unfortunately, this is not known beforehand. Using the optimal partition, incomplete (and cheap) information can be obtained as follows. An optimal primal solution with partition B^0 is given by $x_1^0 = x_2^0 = 0,\ x_3^0 = 1/3,\ x_4^0 = 2,\ x_5^0 = x_6^0 = 1/3$; the vector $u = (0,1,0,0)^T$ satisfies $A_{B^0}u = e_2$. So, the interval where $x^0 + \lambda u > 0$ is given by $\lambda \in (-2,\infty)$. ◇

Consider now the case where b_i is fixed and the objective coefficient c_j varies. Assume that γ^0 does not correspond to a breakpoint. Then one is interested in the interval where the optimal primal solution remains optimal, so in the linearity interval. Using optimal partitions this amounts to solving two auxiliary LP problems again which are given by

$$\min_{\tau,y}\{\ \tau\ :\ A_B^T y = c_B + \tau(e_j)_B,\ A_N^T y \le c_N + \tau(e_j)_N\}$$
$$\max_{\tau,y}\{\ \tau\ :\ A_B^T y = c_B + \tau(e_j)_B,\ A_N^T y \le c_N + \tau(e_j)_N\}.$$

The new partitions are obtained and for all γ in the interval the optimal value of the corresponding pair of problems is known to be

$$\chi(\tau) = \chi(\gamma^0) + \tau x_j^0.$$

Also a pair of optimal solutions for this pair can be constructed by using a vector v in the row space of A_{B^0} (cf. [1]). For any τ in the interval x^0 is optimal in the primal problem $(\mathrm{P}(b,c(\tau)))$, whereas $y = y^0 + \tau v$ is optimal in the corresponding dual problem $(\mathrm{D}(b,c(\tau)))$.

With respect to the computation time required in the approaches using optimal paritions and using optimal bases, almost the same remarks apply as in the RHS-case. One exception is when the degeneracy is solely caused by weakly redundant constraints: Gal [8,9] shows that these constraints may be omitted without influencing the critical region. We continue with looking again at Examples 2 and 3. Recall that the overall critical region could only be obtained as the union of the intervals of the primal optimal bases (see Figure 1).

Example 7 Consider again the primal-dual pair of LP problems from Examples 2 and 3.

$$\min\{-2x_2 + \gamma x_3 + 4x_4 + 5x_5 + 6x_6\ :\ -x_1 - 2x_2 + x_4 + x_5 = 0,$$
$$-x_2 - x_3 - x_4 + x_6 = -1,\ x \ge 0\}$$
$$\max\{-y_2\ :\ 0 \le y_1, -2y_1 - y_2 \le -2,\ \gamma \le y_2,\ y_1 - y_2 \le 4,\ y_1 \le 5,\ y_2 \le 6\ \}$$

The feasible region for the dual problem was depicted in Figure 1. Letting $\gamma^0 = 0$, the optimal partition has $B = \{3\}$, which remains the same for all $\gamma \in (-2,6)$. This interval is immediately determined from

$$\min\{\gamma\ :\ y_2 = \gamma,\ 0 \le y_1,\ -2y_1 - y_2 \le -2,\ y_1 - y_2 \le 4,\ y_1 \le 5,\ y_2 \le 6\},$$

and the corresponding maximization problem. ◇

4.2 Shadow prices

We now turn to the case where the initial parameter β^0 is a breakpoint of the optimal value function. This implies by Theorems 2 and 3 that the optimal partitions for $\beta \neq \beta^0$ are different from π^0. The left and right shadow price for the i-th constraint are the solutions of the problems

$$\min\{\ y_i\ :\ A^T y + s = c,\ s_{B^0} = 0, s_{N^0} \geq 0\ \} \tag{6}$$
$$\max\{\ y_i\ :\ A^T y + s = c,\ s_{B^0} = 0,\ s_{N^0} \geq 0\}. \tag{7}$$

These problems also imply the new partitions. Obviously, the optimal dual basic solutions are the vertices of the optimal dual set. So, the results (1) and (2) concerning the left and right shadow prices are immediately obtained. Approximate shadow prices can be obtained by taking any feasible solution of (6) or (7). Using an optimal basis different situations may occur. The interval where the given optimal basis is optimal may consist of β^0 only; in this case the corresponding dual variable will satisfy $p_i^- < y_i^0 < p_i^+$. When an interval to the left of the breakpoint is obtained then one is sure that $y_i^0 = p_i^-$, but the interval need not be the complete linearity interval to the left of the breakpoint; a similar situation happens to the right of the breakpoint.

We illustrate the RHS-case with the following example.

Example 8 We continue with Example 6. Now let $\beta^0 = -3$, which is a breakpoint of the optimal value function. The optimal partition has $B^0 = \{6\}$. The dual optimal solutions are represented by $0 \leq y_1^0 = y_2^0 \leq 1$; the unique primal solution is given by $x_6^0 = 1,\ x_i^0 = 0\ \ i \neq 6$. Solving the problem

$$\min\{\ y_2\ :\ y_1 - y_2 = 0,\ 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1,\ y_1 + y_2 \leq 2\}$$

renders the left shadow price $y_2^- = 0$ and $B^- = \{1, 2, 6\}$. The corresponding maximization problem gives the right shadow price $y_2^+ = 1$ and $B^+ = \{3, 4, 5, 6\}$. In Figure 6 we depict the optimal value function and the optimal partitions for all values of β. Using bases we see that there are five optimal bases:

$$\begin{array}{rrcl}
y^1 = (0,0)^T, & \mathcal{B}^1 = \{1,6\} & \longrightarrow & \beta \in (-\infty, -3] \\
 & \mathcal{B}^2 = \{2,6\} & \longrightarrow & \beta \in (-\infty, -3] \\
y^2 = (1,1)^T, & \mathcal{B}^3 = \{3,6\} & \longrightarrow & \beta \in [-3, -2] \\
 & \mathcal{B}^4 = \{4,6\} & \longrightarrow & \beta \in [-3, \infty) \\
 & \mathcal{B}^5 = \{5,6\} & \longrightarrow & \beta \in [-3, -1]
\end{array}$$

Using y^1 with any of its optimal bases we find the complete linearity interval to the left and the left shadow price; using y^2 we find the right shadow price and part of the linearity interval with $\mathcal{B}^3$ or $\mathcal{B}^5$ and the complete interval with $\mathcal{B}^4$. ◇

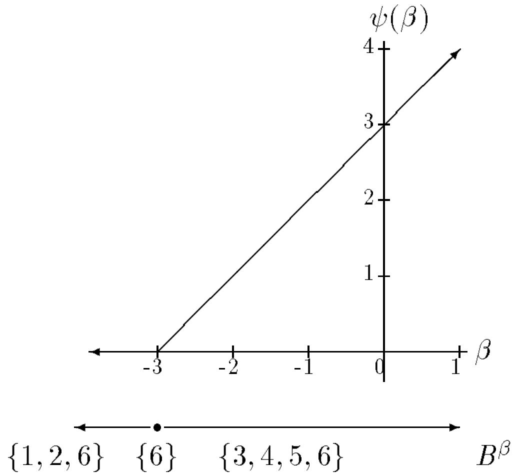

Figure 6: Optimal value function and optimal partitions Example 8.

The reader will understand that for the computation of the shadow costs a similar procedure applies as for the computation of shadow prices, so we only state that to obtain the left and right shadow cost two LP problems need to be solved whose feasible regions are the optimal set of the initial primal problem (cf. Section 2.2).

4.3 Example

At the end of this section we give the reader the appropriate answers for the postoptimal analysis of the transportation problem given in the Introduction. This makes the differences between the use of bases and partitions most apparent. We provide the answers from the interior approach using (Q4); we also give the answers using (Q3) with the optimal solutions provided by the Simplex codes CPLEX and LINDO. Note that for the COST-intervals, there are still differences in the answers obtained with the Simplex-solutions.

Approach	Optimal solution									Shadow prices					
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	(1)	(2)	(3)	(4)	(5)	(6)
CPLEX	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
LINDO	2	0	0	0	0	2	1	3	1	0	0	0	1	1	1
IPM	0.62	0.62	0.62	1.45	1.45	1.45	0.93	0.93	0.93	0	0	0	1	1	1
shadow costs	-2,0	-2,0	-2,0	-3,0	-3,0	-3,0	-3,0	-3,0	-3,0						

Table 3: Optimal solution, shadow prices and shadow costs in Example 1.

Approach	COST-intervals								
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
(Q3)-CPLEX	$[1,\infty)$	$(-\infty,1]$	$[1,\infty)$	[1,1]	[1,1]	$(-\infty,1]$	[1,1]	$[1,\infty)$	$[1,\infty)$
(Q3)-LINDO	$(-\infty,1]$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	[1,1]	[1,1]	$(-\infty,1]$	[1,1]
(Q4)-IPM	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]

Approach	RHS-intervals					
	(1)	(2)	(3)	(4)	(5)	(6)
(Q3)-CPLEX	$[0,\infty)$	$[2,\infty)$	$[1,\infty)$	[0,7]	[0,7]	[0,7]
(Q3)-LINDO	$[0,\infty)$	$[2,\infty)$	$[1,\infty)$	[0,7]	[0,7]	[0,7]
(Q4)-IPM	$[0,\infty)$	$[2,\infty)$	$[1,\infty)$	[0,7]	[0,7]	[0,7]

Table 4: Ranges in Example 1.

5 Concluding Remarks

We have shown that postoptimal analysis in linear programming can be performed by using bases as is the classical approach, but also by using optimal partitions. The use of bases is suggested by using the Simplex method to solve an LP problem. The use of optimal partitions is naturally connected to interior point methods, although the topic of how to find the optimal partition best is not completely resolved yet.

In the nondegenerate case, the desired information can be obtained at the same cost in both approaches. However, in case of degeneracy the situation is quite different. To obtain complete information either (possibly all) dual or primal optimal bases have to be computed or two auxiliary LP problems have to be solved.

An interesting topic is how postoptimal analysis based on both approaches compare in theory and in practice with respect to computation time and delivered information; this is subject of current research.

Acknowledgement: the authors gratefully acknowledge the many helpful remarks and suggestions of the referees that improved this paper significantly.

References

[1] I. Adler and R.D.C. Monteiro, "A geometric view of parametric linear programming," *Algorithmica* 8 (1992) pp. 161-176.

[2] M. Akgül, "A note on shadow prices in linear programming," *J. Opl. Res. Soc.* 35 (1984) pp. 425-431.

[3] D.C. Aucamp and D.I. Steinberg, "The computation of shadow prices in linear programming," *J. Opl. Res. Soc.* 33 (1982) pp. 557-565.

[4] A. Charnes and K.O. Kortanek, "An opposite sign algorithm for purification to an extreme point solution," Office of Naval Research, Memorandum No. 129, Northwestern University (Evanston, Illinois, 1963).

[5] J.R. Evans and N.R. Baker, "Degeneracy and the (mis)interpretation of sensitivity analysis in linear programming," *Decision Sciences* 13 (1982) pp. 348-354.

[6] J.R. Evans and N.R. Baker, "Reply to *"On ranging cost coefficients in dual degenerate linear programming problems"* ", *Decision Sciences* 14 (1983) pp. 442-443.

[7] T. Gal, *Postoptimal analyses, parametric programming and related topics* (Mac-Graw Hill Inc., New York/Berlin, 1979).

[8] T. Gal, "Shadow prices and sensitivity analysis in linear programming under degeneracy, State-of-the-art-survey," *OR Spektrum* 8 (1986) pp. 59-71.

[9] T. Gal, "Weakly redundant constraints and their impact on postoptimal analyses in LP," Diskussionsbeitrag Nr. 151, FernUniversität Hagen, (Hagen, W. Germany, 1990).

[10] J. Gauvin, "Quelques precisions sur les prix marginaux en programmation lineaire", *INFOR* 18 (1980) pp. 68-73 (in French).

[11] A.J. Goldman and A.W. Tucker, "Theory of linear programming," in: H.W. Kuhn and A.W. Tucker, eds., *Inequalities and Related Systems* (Annals of Mathematical Studies, No. 38, Princeton University Press, Princeton, New Jersey, 1956) pp. 53-97.

[12] H.J. Greenberg, "An analysis of degeneracy," *Naval Research Logistics Quarterly* 33 (1986) pp. 635-655.

[13] H.J. Greenberg, "The use of the optimal partition in a linear programming solution for postoptimal analysis," Working Paper, Mathematics Department, University of Colorado at Denver, (Denver, USA, 1993).

[14] O. Güler, C. Roos, T. Terlaky and J.-Ph. Vial, "Interior point approach to the theory of linear programming," Technical Report 1992.3, Department of Management Studies, Faculté des SES, University of Geneva, (Geneva, Switzerland, 1992).

[15] B. Jansen, C. Roos and T. Terlaky, "An interior point approach to postoptimal and parametric analysis in linear programming," Report 92-21, Faculty of Technical Mathematics and Computer Science, Delft University of Technology (Delft, The Netherlands, 1992).

[16] N. Karmarkar, "A new polynomial time algorithm for linear programming," *Combinatorica* 4 (1984) pp. 373-395.

[17] G. Knolmayer, "The effects of degeneracy on cost-coefficient ranges and an algorithm to resolve interpretation problems," *Decision Sciences* 15 (1984) pp. 14-21.

[18] T.L. Magnanti and J.B. Orlin, "Parametric linear programming and anti-cycling pivoting rules," *Mathematical Programming* 41 (1988) pp. 317-325.

[19] N. Megiddo, "Pathways to the optimal set in linear programming," in: N. Megiddo, ed., *Progress in Mathematical Programming: Interior Point and Related Methods* (Springer Verlag, New York, 1989) pp. 131-158.

[20] N. Megiddo, "On finding primal and dual optimal bases," *ORSA Journal on Computing* 3 (1991) pp. 63-65.

[21] S. Mehrotra and Y. Ye, "On finding the optimal facet of linear programs," Technical Report 91-10, Department of Management Sciences, The University of Iowa (Iowa City, IA 52242, USA, 1991).

[22] S. Mehrotra and R.D.C. Monteiro, "Parametric and Range Analysis for Interior Point Methods," Working Paper, Department of IE/MS, Northwestern University, (Evanston IL, 1992).

[23] D.S. Rubin and H.M. Wagner, "Shadow prices: tips and traps for managers and instructors," *Interfaces* 20 (1990) pp. 150-157.

[24] G. Sonnevend, "An "analytical centre" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming," in: *System modelling and optimization: Proceedings of the 12th IFIP-conference, Budapest, Hungary* (volume 84 of Lecture Notes in Control and Information Sciences, Springer Verlag, Berlin, 1985) pp. 866-876.

[25] D.I. Steinberg and D.C. Aucamp, "On ranging cost coefficients in dual degenerate linear programming problems," *Decision Sciences* 14 (1983) pp. 440-441.

[26] J.E. Ward and R.E. Wendell, "Approaches to sensitivity analysis in linear programming," *Annals of Operations Research* 27 (1990) pp. 3-38.