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MAPPING PROPERTIES OF SOME CLASSES OF ANALYTIC FUNCTIONS UNDER A GENERAL INTEGRAL OPERATOR DEFINED BY THE HADAMARD PRODUCT

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Abstract. In this paper, we consider certain subclasses of analytic functions with bounded radius and bounded boundary rotation and study the mapping properties of these classes under a general integral operator defined by the Hadamard product.

1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

A function $f \in \mathcal{A}$ is said to be spiral-like if there exists a real number λ $\left(|\lambda| < \frac{\pi}{2}\right)$ such that

$$\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

The class of all spiral-like functions was introduced by L. Spacek [16] in 1933 and we denote it by $\mathcal{S}_\lambda^*$. Later in 1969, Robertson [15] considered the class $\mathcal{C}_\lambda$ of analytic functions in $\mathbb{U}$ for which $zf'(z) \in \mathcal{S}_\lambda^*$.

Let $\mathcal{P}_k^\lambda(\delta)$ be the class of functions $h(z)$ analytic in $\mathbb{U}$ with $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re e^{i\lambda} h(z) - \delta\cos\lambda}{1-\delta} \right| d\theta \le k\pi\cos\lambda, \quad z = re^{i\theta}, \tag{1.2}$$

where $k \ge 2$, $0 \le \delta < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$.

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For $\lambda = 0$, this class was introduced in [12] and for $\delta = 0$, see [13]. For $k = 2$, $\lambda = 0$ and $\delta = 0$, the class $\mathcal{P}_2^0(0)$ reduces to the class $\mathcal{P}$ of functions $h(z)$ analytic in $\mathbb{U}$ with $h(0) = 1$ and whose real part is positive.

Definition 1.1. (Hadamard product or convolution) Given two functions f and g in the class $\mathcal{A}$, where f is given by (1.1) and g is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}_k^\lambda(\delta, b; g)$ if and only if

$$1 + \frac{1}{b}\left(\frac{z(f * g)'(z)}{(f * g)(z)} - 1\right) \in \mathcal{P}_k^\lambda(\delta) \tag{1.4}$$

where $(f * g)(z)/z \neq 0$ $(z \in \mathbb{U})$, $k \geq 2$, $0 \leq \delta < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$, $b \in \mathbb{C} - \{0\}$ and $g \in \mathcal{A}$.

Remark 1.3. (i) If we set

$$g(z) = z + \sum_{n=2}^{\infty} z^n \qquad \text{and} \qquad g(z) = z + \sum_{n=2}^{\infty} n z^n$$

in Definition 1.2, then we obtain the classes

$$\mathcal{R}_k^\lambda\left(\delta, b; z + \sum_{n=2}^{\infty} z^n\right) := \mathcal{R}_k^\lambda(\delta, b) = \left\{f \in \mathcal{A} : 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \in \mathcal{P}_k^\lambda(\delta)\right\}$$

and

$$\mathcal{R}_k^\lambda\left(\delta, b; z + \sum_{n=2}^{\infty} n z^n\right) := \mathcal{V}_k^\lambda(\delta, b) = \left\{f \in \mathcal{A} : 1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\delta)\right\},$$

respectively. For $\lambda = 0$, these classes were studied by Noor et al. [10].

(ii) If we set $b = 1$ in (i), then we have the classes

$$\mathcal{R}_k^\lambda\left(\delta, 1; z + \sum_{n=2}^{\infty} z^n\right) = \mathcal{R}_k^\lambda(\delta) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k^\lambda(\delta)\right\}$$

and

$$\mathcal{R}_k^\lambda\left(\delta, 1; z + \sum_{n=2}^{\infty} n z^n\right) = \mathcal{V}_k^\lambda(\delta) = \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\delta)\right\},$$

respectively, studied by Noor et al. [11] and Moulis [9].

(iii) For $k=2$ and $\lambda=0$, we have the class

$$\mathcal{R}_2^0(\delta,b;g)=\mathcal{S}_\delta(g,b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(\frac{z\,(f*g)'(z)}{(f*g)(z)}-1\right)\right\}>\delta\right\}$$

defined by Prajapat [14].

(iv) If we set

$$g(z)=z+\sum_{n=2}^{\infty}z^n \qquad \text{and} \qquad g(z)=z+\sum_{n=2}^{\infty}nz^n$$

in (iii), then we have the classes

$$\mathcal{R}_2^0\left(\delta,b;z+\sum_{n=2}^{\infty}z^n\right)=\mathcal{S}_\delta^*(b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>\delta\right\}$$

and

$$\mathcal{R}_2^0\left(\delta,b;z+\sum_{n=2}^{\infty}nz^n\right)=\mathcal{C}_\delta(b)=\left\{f\in\mathcal{A}:\Re\left\{1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right\}>\delta\right\},$$

respectively, introduced by Frasin [6].

Definition 1.4. [7] Given $f_j,g_j\in\mathcal{A}$, $\alpha_j\in\mathbb{C}$ for all $j=1,2,\dots,n$, $n\in\mathbb{N}$. We let $\mathcal{I}:\mathcal{A}^n\to\mathcal{A}$ be the integral operator defined by

$$\mathcal{I}(f_1,\dots,f_n;g_1,\cdots,g_n)=\mathcal{F},$$
$$\mathcal{F}(z)=\int_0^z\left(\frac{(f_1*g_1)(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{(f_n*g_n)(t)}{t}\right)^{\alpha_n}dt, \tag{1.5}$$

where $(f_j*g_j)(z)/z\neq 0$ $(z\in\mathbb{U},1\le j\le n)$.

Remark 1.5. The integral operator $\mathcal{F}$ generalizes many operators which were introduced and studied recently.

(i) For $g_j(z)=z+\sum_{n=2}^{\infty}z^n$ with $\alpha_j>0$ $(1\le j\le n)$, we have the integral operator

$$\mathcal{F}_n(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{t}\right)^{\alpha_n}dt \tag{1.6}$$

and for $g_j(z)=z+\sum_{n=2}^{\infty}nz^n$ with $\alpha_j>0$ $(1\le j\le n)$, we have the integral operator

$$\mathcal{F}_{\alpha_1,\dots,\alpha_n}(z)=\int_0^z(f_1'(t))^{\alpha_1}\cdots(f_n'(t))^{\alpha_n}\,dt, \tag{1.7}$$

recently studied by Breaz and Breaz [2], Breaz et al. [4], Breaz and Güney [3] and Bulut [5].

(ii) For $n=1$, $\alpha_1=\alpha\in[0,1]$, $\alpha_2=\cdots=\alpha_n=0$ and $f_1=f\in\mathcal{S}$, $g_1(z)=g(z)=z+\sum_{n=2}^{\infty}z^n$, we have the integral operator

$$\mathcal{F}(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\beta}dt$$

studied in [8].

(iii) For $n=1$, $\alpha_1=1$, $\alpha_2=\cdots=\alpha_n=0$ and $f_1=f\in\mathcal{A}$, $g_1(z)=g(z)=z+\sum_{n=2}^{\infty}z^n$, we have the integral operator of Alexander

$$\mathcal{F}(z)=\int_0^z\frac{f(t)}{t}\,dt$$

introduced in [1].

For other examples, see Frasin [7].

In this paper, we investigate some properties of the integral operator $\mathcal{F}$ defined by (1.5) for the class $\mathcal{R}_k^{\lambda}(\delta,b;g)$.

2. Main results

Theorem 2.1. *Let $f_j\in\mathcal{R}_k^{\lambda}(\delta_j,b;g_j)$ for $1\le j\le n$ with $k\ge 2$, $0\le\delta_j<1$, $b\in\mathbb{C}-\{0\}$. Also let λ is real with $|\lambda|<\frac{\pi}{2}$, $\alpha_j>0$ $(1\le j\le n)$. If*

$$0\le 1+\sum_{j=1}^{n}\alpha_j(\delta_j-1)<1,$$

then the integral operator $\mathcal{F}$ defined by (1.5) is in the class $\mathcal{V}_k^{\lambda}(\gamma,b)$ with

$$\gamma=1+\sum_{j=1}^{n}\alpha_j(\delta_j-1). \tag{2.1}$$

Proof. Since $f_j,g_j\in\mathcal{A}$ $(1\le j\le n)$, by (1.3), we have

$$\frac{(f_j*g_j)(z)}{z}=1+\sum_{n=2}^{\infty}a_{n,j}b_{n,j}z^{n-1}$$

and $\frac{(f_j*g_j)(z)}{z}\neq 0$ for all $z\in\mathbb{U}$. By (1.5), we get

$$\mathcal{F}'(z)=\left(\frac{(f_1*g_1)(z)}{z}\right)^{\alpha_1}\cdots\left(\frac{(f_n*g_n)(z)}{z}\right)^{\alpha_n}.$$

This equality implies that

$$\ln\mathcal{F}'(z)=\alpha_1\ln\frac{(f_1*g_1)(z)}{z}+\cdots+\alpha_n\ln\frac{(f_n*g_n)(z)}{z}$$

or equivalently

$$\ln\mathcal{F}'(z)=\alpha_1\left[\ln(f_1*g_1)(z)-\ln z\right]+\cdots+\alpha_n\left[\ln(f_n*g_n)(z)-\ln z\right].$$

By differentiating above equality, we get

$$\frac{\mathcal{F}''(z)}{\mathcal{F}'(z)}=\sum_{j=1}^{n}\alpha_j\left(\frac{(f_j*g_j)'(z)}{(f_j*g_j)(z)}-\frac{1}{z}\right).$$

Hence, we obtain from this equality that

$$\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}=\sum_{j=1}^{n}\alpha_j\left(\frac{z(f_j*g_j)'(z)}{(f_j*g_j)(z)}-1\right).$$

Then by multiplying the above relation with $1/b$, we have

$$\begin{aligned}\frac{1}{b}\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} &= \sum_{j=1}^{n}\alpha_j\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\\ &= \sum_{j=1}^{n}\alpha_j\left[1+\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\right]-\sum_{j=1}^{n}\alpha_j\end{aligned}$$

or equivalently

$$e^{i\lambda}\left(1+\frac{1}{b}\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)=\left(1-\sum_{j=1}^{n}\alpha_j\right)e^{i\lambda}+\sum_{j=1}^{n}\alpha_j e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\right].$$

Subtracting and adding $\left(\cos\lambda\sum_{j=1}^{n}\alpha_j\delta_j\right)$ on the left hand side and then taking real part, we have

$$\begin{aligned}&\Re\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)-\gamma\cos\lambda\right\}\\ &\qquad=\sum_{j=1}^{n}\alpha_j\Re\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\right]-\delta_j\cos\lambda\right\},\end{aligned}\tag{2.2}$$

where γ is given by (2.1). Integrating (2.2) and then using (2.1), we have

$$\begin{aligned}&\int_0^{2\pi}\left|\Re\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)-\gamma\cos\lambda\right\}\right|d\theta\\ &\quad\leq\sum_{j=1}^{n}\alpha_j\int_0^{2\pi}\left|\Re\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\right]-\delta_j\cos\lambda\right\}\right|d\theta.\end{aligned}\tag{2.3}$$

Since $f_j\in\mathcal{R}_k^{\lambda}\left(\delta_j,b;g_j\right)$ $(1\leq j\leq n)$, we get

$$\begin{aligned}&\int_0^{2\pi}\left|\Re\left\{e^{i\lambda}\left[1+\frac{1}{b}\left(\frac{z\left(f_j * g_j\right)'(z)}{\left(f_j * g_j\right)(z)}-1\right)\right]-\delta_j\cos\lambda\right\}\right|d\theta\\ &\qquad\leq\left(1-\delta_j\right)k\pi\cos\lambda\end{aligned}\tag{2.4}$$

for $1\leq j\leq n$. Using (2.4) in (2.3), we obtain

$$\begin{aligned}\int_0^{2\pi}\left|\Re\left\{e^{i\lambda}\left(1+\frac{1}{b}\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right)-\gamma\cos\lambda\right\}\right|d\theta&\leq k\pi\cos\lambda\sum_{j=1}^{n}\alpha_j\left(1-\delta_j\right)\\ &=k\pi\cos\lambda\left(1-\gamma\right).\end{aligned}$$

Hence, we obtain $\mathcal{F}\in\mathcal{V}_k^{\lambda}\left(\gamma,b\right)$ with γ is given by (2.1). ■

By setting $g_j\left(z\right)=z+\sum_{n=2}^{\infty}z^n$ $(1\leq j\leq n)$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let $f_j\in\mathcal{R}_k^{\lambda}\left(\delta_j,b\right)$ for $1\leq j\leq n$ with $k\geq 2$, $0\leq\delta_j<1$, $b\in\mathbb{C}-\{0\}$. Also let λ is real with $|\lambda|<\frac{\pi}{2}$, $\alpha_j>0$ $(1\leq j\leq n)$. If*

$$0\leq 1+\sum_{j=1}^{n}\alpha_j\left(\delta_j-1\right)<1,$$

then the integral operator $\mathcal{F}_n$ *defined by* (1.6) *is in the class* $\mathcal{V}_k^{\lambda}(\gamma, b)$, *where* γ *is defined by* (2.1).

Remark 2.3. If we set $k = 2$ and $\lambda = 0$ in Corollary 2.2, then we have [5, Theorem 1].

By setting $g_j(z) = z + \sum_{n=2}^{\infty} nz^n$ $(1 \leq j \leq n)$ in Theorem 2.1, we obtain the following result.

Corollary 2.4. *Let* $f_j \in \mathcal{V}_k^{\lambda}(\delta_j, b)$ *for* $1 \leq j \leq n$ *with* $k \geq 2$, $0 \leq \delta_j < 1$, $b \in \mathbb{C} - \{0\}$. *Also let* λ *is real with* $|\lambda| < \frac{\pi}{2}$, $\alpha_j > 0$ $(1 \leq j \leq n)$. *If*

$$0 \leq 1 + \sum_{j=1}^{n} \alpha_j (\delta_j - 1) < 1,$$

then the integral operator $\mathcal{F}_{\alpha_1,\dots,\alpha_n}$ *defined by* (1.7) *is in the class* $\mathcal{V}_k^{\lambda}(\gamma, b)$, *where* γ *is defined by* (2.1).

Remark 2.5. If we set $k = 2$ and $\lambda = 0$ in Corollary 2.4, then we have [5, Theorem 3].

Letting $k = 2$ and $\lambda = 0$ in Theorem 2.1, we have [7, Theorem 2.1] as follows.

Corollary 2.6. *Let* $f_j \in \mathcal{S}_{\delta_j}(g_j, b)$ *for* $1 \leq j \leq n$ *with* $0 \leq \delta_j < 1$, $b \in \mathbb{C} - \{0\}$. *Also let* $\alpha_j > 0$ $(1 \leq j \leq n)$. *If*

$$0 \leq 1 + \sum_{j=1}^{n} \alpha_j (\delta_j - 1) < 1,$$

then the integral operator $\mathcal{F}$ *defined by* (1.5) *is in the class* $\mathcal{C}_{\gamma}(b)$, *where* γ *is defined by* (2.1).

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