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Asymptotic analysis on nonlinear vibration of axially accelerating viscoelastic strings with the standard linear solid model

Li-Qun Chen · Hao Chen

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Abstract Nonlinear parametric vibration of axially accelerating viscoelastic strings is investigated via an approximate analytical approach. The standard linear solid model using the material time derivative is employed to describe the string viscoelastic behaviors. A coordinate transformation is introduced to derive Mote's model of transverse motion from the governing equation of the stationary string. Mote's model leads to Kirchhoff's model by replacing the tension with the averaged tension over the string. An asymptotic perturbation approach is proposed to study principal parametric resonance based on the two models. The amplitude and the existence conditions of the steady-state responses are determined by locating the nonzero fixed points in the modulation equations resulting from the solvability condition. Numerical results are presented to highlight the effects of the material parameters, the axial-speed fluctuation amplitude, and the initial stress on steady-state responses.

Keywords Asymptotic perturbation · Axially accelerating string · Nonlinearity · Parametric vibration · Viscoelasticity

1 Introduction

Transverse vibrations of axially moving strings occur in many branches of engineering such as serpentine belts, fiber windings, magnetic tapes and thread lines. Much research has been devoted to these phenomena [1]. Because increasingly used materials such metallic or ceramic reinforced materials exert inherently viscoelastic behaviors, researchers investigate free vibration [2,3], forced vibration [4], and parametric vibrations [5–23] of axially moving viscoelastic strings. The viscoelastic strings they studied are differential-type materials described by the Kelvin model [2,4,7,8,11,12,14,15,18,20,21,23] or the standard linear solid model [3,6,17,22], as well as integral-type materials defined by the stress relaxation as an exponential function [5,9,10,16,19] or a power function [13]. Actually, the standard linear solid model, which can describe the behavior of linear viscoelastic materials of solid type with limited creep deformation, covers the Kelvin model and the integral-type constitution relation with an exponential

L.-Q. Chen
Department of Mechanics, Shanghai University, Shanghai 200444, China

L.-Q. Chen (✉) · H. Chen
Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 200072, China
e-mail: lqchen@staff.shu.edu.cn

relaxation function as special cases. When differential-type constitutive laws are incorporated, some investigators used the partial time derivative in the viscoelastic constitutive relations [2–4,6–8,11,12,14,15,20,21,23,24]. However, Mochensturm and Guo [18] demonstrated that the material time derivative should be used to account for the additional "steady state" dissipation of an axially moving viscoelastic string. Actually the material time derivative was also used in other works on axially moving viscoelastic strings [6,12,22] and beams [25–28]. In the present investigation, a coordinate transform will be used to develop the governing equations. The approach can introduce naturally the material time derivative in the standard linear solid model.

Exact solutions are usually unavailable for nonlinear partial differential equations. Hence numerical methods [5,6,9,11–14,16,17,21–23] and approximate analytical methods [2–4,7,10,15,18–20] were applied to investigate transverse vibration of axially moving viscoelastic strings. The approximate analytical method used is the method of multiple scales. In addition to the method of multiple scales, which is widely applied in dynamic analysis of structures [29], the method of asymptotic perturbation is also an effective approach to treat nonlinear vibration [30,31]. The asymptotic approach has been applied to nonlinear continuous systems. Boertjens and van Horssen [32] constructed analytically approximate solutions in the cases of internal resonances for a beam with quadratic and cubic nonlinearities. Maccari [33] determined external force-response and frequency-response curves in the cases of primary resonance and subharmonic resonance for a weakly periodically forced beam with quadratic and cubic nonlinearities. Boertjens and van Horssen demonstrated the existence and uniqueness of solutions and the asymptotic validity of the approximation for a beam with a quadratic nonlinearity [34]. They also studied the interactions of modes for a weakly forced beam with a geometric nonlinearity [35]. Maccari [36] proposed an asymptotic method for a bar with nonlinear boundary conditions. Andrianov and Danishevs'kyy developed the asymptotic approach to locate the periodic response of a beam with a cubic nonlinearity [37]. Chen [38] used an asymptotic perturbation technique to derive a new family of two-dimensional nonlinear dispersive equation for a pre-stressed hyperelastic compressible plate. However, those investigators focused on conservative linear systems with nonlinear disturbances. Chen et al. [39] applied the asymptotic analysis to an axially accelerating viscoelastic string constituted by the Kelvin model. In the present investigation, the string material obeys the standard linear solid model, which in not only more general, but also more difficult, especially when the material time derivative is used in the constitutive relation. Some mathematical techniques will be developed to overcome the difficulties.

As observed in [19], there are two kinds of nonlinear transverse models of axially moving beams. Kirchhoff [40] proposed the first nonlinear model for transverse motion of a string, which is a nonlinear integro-partial-differential equation. Kirchhoff's nonlinear string model can be derived from the governing equations of in-plan motion, based on the assumption that both the transverse displacement and the longitudinal displacement are finite but small so that the Lagrangian strain can account for the geometric nonlinearity, and the string stretches in a quasi-static manner. Wickert [41] extended Kirchhoff's equation to an elastic string moving at a constant axial speed. Kirchhoff's model was applied to free [42], forced [43] and parametric [44] vibrations of axially moving elastic strings. Nevertheless, the applications of Kirchhoff's model in transverse motion of axially moving viscoelastic strings were rather limited. Mote [45] proposed another model, viz. a nonlinear partial differential equation, for nonlinear transverse motion of axially moving strings. Mote's model can be derived from the governing equations of in-plan motion by omitting all longitudinal terms and higher order nonlinear terms. Almost all investigations on transverse vibration of axially moving viscoelastic strings were based on Mote's model [2–23]. The steady-state responses based on two models were compared in [19] for axially accelerating viscoelastic strings constituted by the Boltzmann superposition principle. Numerical investigations on free vibration of stationary elastic strings [46], free vibration of axially moving elastic beams [47], and forced vibration of axially moving viscoelastic beams [48] indicated that the nonlinear integro-partial-differential equation is superior to the partial differential equation, in the sense that it approximates the coupled governing equation of planar motion better. However, since there is no decisive evidence to favor any of the models for axially moving viscoelastic strings, both models are treated in this investigation. Although the subject addressed here seems similar to that in [19], it is different in the following two aspects. The constitutive relation here is the standard linear solid model, which can lead to the Boltzmann superposition principle used in [19] as a special case and is mathematically more difficult to tackle. The approach used here is asymptotic, while that in [19] uses the method of multiple scales.

The present paper is organized as follows. Section 2 introduces a coordinate transformation to derive the governing equations, a nonlinear partial differential equation and a nonlinear integro-partial-differential equation. Section 3 proposes an asymptotic perturbation approach to the nonlinear integro-partial-differential equations in Sect. 2. Section 4 examines steady-state responses based on the analysis in Sect. 3. Section 5 presents the corresponding results of the nonlinear partial differential equation. Section 6 ends the paper with concluding remarks.

2 Problem formulation

Consider an axially moving string of density ρ, area of cross-section A, initial tension T traveling between two eyelets separated by a distance l in a uniform axial speed $\gamma(t)$ that is a prescribed function of time t. The string is subjected to no external loads. Assume that the transverse motion is not coupled with the longitudinal motion. The transverse motion of the string is specified by the transverse displacement $v(x,t)$.

Study the motion of the string in a reference frame moving in the axial direction and at the speed $\gamma(t)$. The equation of motion in the transverse direction can be derived from Newton's second law

$$\rho A\frac{\mathrm{d}^2v}{\mathrm{d}t^2} = \frac{\partial}{\partial x}\left[(T+A\sigma)\,v_{,x}\right] \tag{1}$$

where $\sigma(x,t)$ denotes the disturbed string stress, and $v_{,x}=\partial v/\partial x$. The standard linear solid model is adopted to describe the viscoelastic property of the string material. Lagrangian strain is employed as a finite measure to account for geometric nonlinearity due to finite stretching. For one-dimensional problems, the stress–strain relationship of the model is expressed in a differential form as [3,6,22]

$$(E_1+E_2)\sigma(x,t)+\eta\frac{\mathrm{d}}{\mathrm{d}t}\sigma(x,t)=E_1E_2\varepsilon_L(x,t)+E_1\eta\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_L(x,t), \tag{2}$$

where E_1 and E_2 are the stiffness constants, η is the dynamic viscosity, and $\varepsilon_L(x,t)$ is the Lagrangian strain,

$$\varepsilon_L=\frac{1}{2}v_{,x}^2 \tag{3}$$

In the noninertial reference frame, the boundaries are moving and the boundary conditions are

$$v\,(s,t)=v\,(s+l,t)=0, \tag{4}$$

where the frame displacement s satisfies $\mathrm{d}s/\mathrm{d}t=\gamma$. Introduce the transformation of the coordinates,

$$x\leftrightarrow x+s,\quad t\leftrightarrow t. \tag{5}$$

In the new coordinate system, the boundary conditions are

$$v(0,t)=v(l,t)=0. \tag{6}$$

In the new coordinate system, the partial derivatives with respect to x and t remain invariant, and the total time derivative changes due to the fact $s=s(t)$ and $\mathrm{d}s/\mathrm{d}t=\gamma$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\leftrightarrow\gamma\frac{\partial}{\partial x}+\frac{\partial}{\partial t} \tag{7}$$

Substitution of Eq. 7 in (1) and (2), respectively, yields

$$\rho A\left(v_{,tt}+\dot{\gamma}v_{,x}+2\gamma v_{,xt}+\gamma^2v_{,xx}\right) = \frac{\partial}{\partial x}\left[(T+A\sigma)\,v_{,x}\right], \tag{8}$$

$$(E_1+E_2)\,\sigma+\eta\,(\gamma\sigma_{,x}+\sigma_{,t})=E_1E_2\varepsilon_{\mathrm{L}}+E_1\eta\left(\gamma\varepsilon_{\mathrm{L},x}+\varepsilon_{\mathrm{L},t}\right), \tag{9}$$

where $v_{,tt}=\partial^2v/\partial t^2$, $v_{,xt}=\partial^2v/\partial x\partial t$, $v_{,xx}=\partial^2v/\partial x^2$, $\sigma_{,x}=\partial\sigma/\partial x$, $\sigma_{,t}=\partial\sigma/\partial t$, $\varepsilon_{\mathrm{L},x}=\partial\varepsilon_{\mathrm{L}}/\partial x$, and $\varepsilon_{\mathrm{L},t}=\partial\varepsilon_{\mathrm{L}}/\partial t$. Eliminating ε_{L} from (9) via (3) leads to:

$$(E_1+E_2)\,\sigma+\eta\,(\gamma\sigma_{,x}+\sigma_{,t})=\frac{1}{2}E_1E_2v_{,x}^2+E_1\eta v_{,x}\,(\gamma v_{,xx}+v_{,xt})\,. \tag{10}$$

Equation 8, together with (10), is a governing equation for the transverse motion of an axially accelerating viscoelastic string. Equation 8 is referred as Mote's model because it is the generalization of Mote's Eq. [43] of transverse motion of an axially moving elastic string to the case of an axially accelerating viscoelastic string.

If the spatial variation of the tension is fairly small compared with the initial tension, the exact form of the disturbed tension $A\sigma$ can be replaced by its spatially averaged value $\frac{1}{l}\int_0^l A\sigma \mathrm{d}x$. Then Eq. 8 leads to

$$\rho A\left(v_{,tt} + \dot{\gamma} v_{,x} + 2\gamma v_{,xt} + \gamma^2 v_{,xx}\right) = \left(T + \frac{1}{l}\int_0^l A\sigma \mathrm{d}x\right) v_{,xx} . \tag{11}$$

Equation 11, together with Eq. 10, is also a governing equation of transverse motion of an axially accelerating viscoelastic string. Equation 11 is referred as Kirchhoff's model because it is the generalization of Kirchhoff's equation [39] of transverse motion of an elastic string to the case of an axially accelerating viscoelastic string.

In the present investigation, the axial speed is assumed to be a small simple harmonic fluctuation about the constant mean speed,

$$\gamma(t) = \gamma_0 + \varepsilon\gamma_1 \sin\Omega t, \tag{12}$$

where γ_0 and γ_1 are positive coefficients and ε is a dimensionless parameter, as a book-keeping device, to indicate the fact that the axial speed fluctuation is small compared with the mean axial speed. Substitution of Eq. 12 in (8), (9), and (11) yields

$$\rho A\left(v_{,tt} + 2\gamma_0 v_{,xt} + \gamma_0^2 v_{,xx}\right) - T\ v_{,xx} = \frac{\partial}{\partial x}\left(A\sigma v_{,x}\right) - \varepsilon\gamma_1\Omega v_{,x}\cos\Omega t - 2\varepsilon\gamma_1\left(\gamma_0 v_{,xx} + v_{,xt}\right)\sin\Omega t - \varepsilon^2\gamma_1^2\sin^2\Omega t, \tag{13}$$

$$(E_1 + E_2)\sigma + \eta\left(\gamma_0\sigma_{,x} + \varepsilon\gamma_1\sigma_{,x}\sin\Omega t + \sigma_{,t}\right) = \frac{1}{2}E_1E_2 v_{,x}^2 + E_1\eta v_{,x}\left(\gamma_0 v_{,xx} + \varepsilon\gamma_1 v_{,xx}\sin\Omega t + v_{,xt}\right), \tag{14}$$

$$\rho A\left(v_{,tt} + 2\gamma_0 v_{,xt} + \gamma_0^2 v_{,xx}\right) - T v_{,xx} = \frac{v_{,xx}}{l}\int_0^l A\sigma \mathrm{d}x - \varepsilon\gamma_1\Omega v_{,x}\cos\Omega t - 2\varepsilon\gamma_1\left(\gamma_0 v_{,xx} + v_{,xt}\right)\sin\Omega t - \varepsilon^2\gamma_1^2\sin^2\Omega t. \tag{15}$$

respectively.

Introduce the dimensionless variables

$$v \leftrightarrow \frac{v}{\sqrt{\varepsilon}l}, \quad x \leftrightarrow \frac{x}{l}, \quad t \leftrightarrow \frac{t}{l}\sqrt{\frac{T}{\rho A}}, \quad \varsigma = \frac{A\sigma}{\varepsilon T} \tag{16}$$

and parameters

$$c = \gamma_0\sqrt{\frac{\rho A}{T}}, \quad c_1 = \gamma_1\sqrt{\frac{\rho A}{T}}, \quad \omega = \Omega l\sqrt{\frac{\rho A}{T}}. \tag{17}$$

The appearance of the book-keeping device ε in the first and last terms of (16) indicates that the transverse displacement is very small and the disturbed stress $\sigma(x,t)$ is much smaller than the initial stress $\sigma_0 = T/A$. Expressed in the dimensionless variables and parameters, Eqs. 13 and 15 become:

$$v_{,tt} + 2cv_{,xt} + (c^2-1)v_{,xx} = \varepsilon\frac{\partial}{\partial x}\left[\varsigma v_{,x}\right] - \varepsilon\omega c_1 v_{,x}\cos\omega t - 2\varepsilon c_1\left(c_0 v_{,xx} + v_{,xt}\right)\sin\omega t - \varepsilon^2 c_1^2\sin^2\omega t, \tag{18}$$

$$v_{,tt} + 2cv_{,xt} + (c^2-1)v_{,xx} = \varepsilon v_{,xx}\int_0^1 \varsigma \mathrm{d}x - \varepsilon\omega c_1 v_{,x}\cos\omega t - 2\varepsilon c_1\left(c_0 v_{,xx} + v_{,xt}\right)\sin\omega t - \varepsilon^2 c_1^2\sin^2\omega t. \tag{19}$$

Equation 14 can also been cast into the dimensionless form

$$\varsigma_{,t} + c\varsigma_{,x} + \varepsilon c_1 \sin\omega t\,\varsigma_{,x} + \eta_\alpha \varsigma = E v_{,x}\left(\frac{1}{2}v_{,x} + E_\alpha v_{,xt} + E_0 v_{,xx} + E_3 \varepsilon \sin\omega t\, v_{,xx}\right), \tag{20}$$

where

$$\eta_\alpha = \frac{E_1 + E_2}{\eta}\sqrt{\frac{\rho A}{T}},\quad E = \frac{A E_1 E_2}{T\eta}\sqrt{\frac{\rho A}{T}},\quad E_\alpha = \frac{\eta}{E_2}\sqrt{\frac{T}{\rho A}},\quad E_0 = \frac{\eta\gamma_0}{E_2},\quad E_3 = \frac{\eta\gamma_1}{E_2}. \tag{21}$$

The dimensionless boundary conditions are

$$v(0,t) = v(1,t) = 0. \tag{22}$$

3 Asymptotic analysis

An asymptotic perturbation method will be developed to analyze Kirchhoff's model, namely (19) and (20), subject to the boundary conditions (22). If $\varepsilon = 0$ in (19), the resulting equation,

$$v_{,tt} + 2cv_{,xt} + (c^2 - 1)v_{,xx} = 0, \tag{23}$$

is on longer coupled with (20). Under boundary conditions (22), Eq. 23 has the natural frequencies [44]

$$\omega_j = j\pi(1 - c^2)\quad (j = 1, 2, \ldots). \tag{24}$$

If the dimensionless speed-variation frequency ω approaches twice a natural frequency, principal parametric resonance may occur. A detuning parameter μ is introduced to quantify the deviation of ω from $2\omega_j$, and ω is described by

$$\omega = 2\omega_j + \varepsilon\mu. \tag{25}$$

In the jth principal parametric resonance, the jth frequency component dominates the response. Therefore, the solution of (19) is assumed to be given by

$$v(x,t) = \Theta_1(x,\tau;\varepsilon)\exp\left(\mathrm{i}\omega_j t\right) + \mathrm{NST} + \mathrm{cc}, \tag{26}$$

where NST stands for the terms that do not contain the jth frequency component, cc represents the complex conjugate of all preceding terms on the right-hand side of an equation, $\tau = \varepsilon t$ is the slowly varying time, and the function $\Theta_1(x,\ \tau;\varepsilon)$ can be expanded in a power series of ε:

$$\Theta_1(x,\tau;\varepsilon) = \Psi_1(x,\tau) + \varepsilon\Psi_1^{(1)}(x,\tau) + O(\varepsilon^2) \tag{27}$$

Notice that ς appears in (19) in the form $\varepsilon v_{,xx}\varsigma$. Hence, according to Eq. 26, the time-independent component and double the jth frequency component yield the jth frequency component in $\varepsilon v_{,xx}\varsigma$. Besides, only the zero order of ε need to be considered. So the solution of (20) can be assumed as

$$\varsigma(x,t) = \Psi_0(x,\tau) + \Psi_2(x,\tau)\exp(2\mathrm{i}\omega_j t) + O(\varepsilon) + \mathrm{cc} + \mathrm{NST}. \tag{28}$$

The chain rule of partial derivatives leads to

$$\frac{\partial}{\partial t}\left[\Psi_m \mathrm{e}^{(\pm m\mathrm{i}\omega_j t)}\right] = \left(\pm\mathrm{i}m\omega_j\Psi_0 + \varepsilon\frac{\partial\Psi_m}{\partial\tau}\right)\mathrm{e}^{(\pm\mathrm{i}m\omega_j t)}\quad (m = 1, 2). \tag{29}$$

Inserting (26)–(29) into (19) and equating the coefficients of $\exp(-\mathrm{i}\omega_j t)$ at the order ε^0 and ε^1 in the resulting equation yield

$$-\omega_j^2\Psi_1 + 2\mathrm{i}c\omega_j\Psi_{1,x} + (c^2 - 1)\Psi_{1,xx} = 0, \tag{30}$$

$$-\omega_j^2\Psi_1^{(1)} + 2\mathrm{i}c\omega_j\Psi_1^{(1)}{}_{,x} + (c^2 - 1)\Psi_1^{(1)}{}_{,xx} = \left[\Psi_{1,xx}\int_0^1 \mathfrak{Re}(\Psi_0)\,\mathrm{d}x + \bar{\Psi}_{1,xx}\int_0^1 \Psi_2\mathrm{d}x\right] - 2\mathrm{i}\omega_j\Psi_{1,\tau} - 2c\Psi_{1,x\tau} + \mathrm{i}\delta c^2\bar{\Psi}_{1,xx}\mathrm{e}^{\mathrm{i}\mu\tau}, \tag{31}$$

where $\delta = c_1/c$. Substitution of (26) and (27) in (22) leads to

$$\Psi_1(0,t) = \Psi_1(1,t) = 0, \quad \Psi_1^{(1)}(0,t) = \Psi_1^{(1)}(1,t) = 0. \tag{32,33}$$

Inserting (28) and (29) into (20) and equating the coefficients of the time-independent term and $\exp(-\mathrm{i}\omega_j t)$ at the order ε^0 in the resulting equation yield

$$c\mathfrak{Re}\left(\Psi_{0,x}\right) + \eta_\alpha \mathfrak{Re}\left(\Psi_0\right) = E\left(\Psi_{1,x}\bar{\Psi}_{1,x} + E_0\bar{\Psi}_{1,x}\Psi_{1,xx} + E_0\Psi_{1,x}\bar{\Psi}_{1,xx}\right), \tag{34}$$

$$c\Psi_{2,x} + \left(2\mathrm{i}\omega_j + \eta_\alpha\right)\Psi_2 = E\Psi_{1,x}\left[\left(\frac{1}{2} + \mathrm{i}\omega_j E_\alpha\right)\Psi_{1,x} + E_0\Psi_{1,xx}\right]. \tag{35}$$

Subject to the boundary condition (32), Eq. 30 has the solution [45]:

$$\Psi_1(x,\tau) = \phi_j(x)q(\tau), \tag{36}$$

where

$$\phi_j(x) = \sqrt{2}\sin(j\pi x)\,\mathrm{e}^{\mathrm{i}j\pi cx} \tag{37}$$

and $q(\tau)$ is an arbitrary function with respect to the slowly varying time, which will be determined later. Substitution of (36) in (34) and (35) gives

$$c\mathfrak{Re}\left(\Psi_{1,x}\right) + \eta_\alpha \mathfrak{Re}\left(\Psi_1\right) = Eq\bar{q}\left(\phi_j'\bar{\phi}_j' + E_0\bar{\phi}_j'\phi_j'' + E_0\phi_j'\bar{\phi}_j''\right), \tag{38}$$

$$c\Psi_{2,x} + \left(2\mathrm{i}\omega_j + \eta_\alpha\right)\Psi_2 = Eq^2\phi_j'\left[\left(\frac{1}{2} + \mathrm{i}\omega_j E_\alpha\right)\phi_j' + E_0\phi_j''\right]. \tag{39}$$

Equations 38 and 39 are nonhomogeneous constant-coefficient linear ordinary differential equations about $\mathfrak{Re}(\Psi_1)$ and Ψ_2, respectively. Substitution of (37) yields the following particular solutions

$$\mathfrak{Re}\left(\Psi_1\right) = g_1(x)\,q(\tau)\,\bar{q}(\tau), \quad \Psi_2(x,\tau) = g_2(x)\,q^2(\tau), \tag{40,41}$$

where

$$g_1(x) = \pi^2 n^2 E\left\{\frac{c^2+1}{\eta_\alpha} - \frac{\left(c^2-1\right)\left[\left(4n^2\pi^2 cE_0 + \eta_\alpha\right)\cos(2n\pi x) - 2n\pi\left(E_0\eta_\alpha - c\right)\sin(2n\pi x)\right]}{4n^2\pi^2c^2 + \eta_\alpha^2}\right\}, \tag{42}$$

$$\begin{aligned} g_2(x) = \frac{\pi^2}{4}n^2 E\left\{\frac{(c-1)^2\left[2\pi(c-1)E_0 n + 2E_\alpha\omega_n - \mathrm{i}\right]}{2n\pi(c-1)c + 2\omega_n - \mathrm{i}\eta_\alpha}\mathrm{e}^{2n\pi\mathrm{i}(c-1)x} - \frac{2\left(c^2-1\right)\left(n\pi cE_0 + 2E_\alpha\omega_n - \mathrm{i}\right)}{2n\pi c^2 + 2\omega_n - \mathrm{i}\eta_\alpha}\mathrm{e}^{2n\pi\mathrm{i}cx}\right. \\ \left. + \frac{(c+1)^2\left[2n\pi(c+1)E_0 + 2E_\alpha\omega_n - \mathrm{i}\right]}{2n\pi(c+1)c + 2\omega_n - \mathrm{i}\eta_\alpha}\mathrm{e}^{2n\pi\mathrm{i}(c+1)x}\right\}. \end{aligned} \tag{43}$$

Substitution of (36), (40), and (41) in (31) leads to

$$\begin{aligned} -\omega_j^2\Psi_1^{(1)} + 2\mathrm{i}c\omega_j\Psi_1^{(1)}{}_{,x} + (c^2-1)\Psi_1^{(1)}{}_{,xx} = q^2\bar{q}\left[\bar{\phi}_j''(x)\int_0^1 g_2(x)\mathrm{d}x + \phi_j''(x)\int_0^1 g_1(x)\mathrm{d}x\right] \\ - \left(2\mathrm{i}\omega_j\phi_j + 2c\phi_j'\right)\frac{\mathrm{d}q}{\mathrm{d}\tau} + \mathrm{i}\delta c^2\bar{\phi}_j''\bar{q}\mathrm{e}^{\mathrm{i}\mu\tau}. \end{aligned} \tag{44}$$

Equation 44 has a bounded solution only if a solvability condition is satisfied. Under boundary condition (33), the solvability condition [39] is the orthogonality of the all terms at the right-hand side of (44) and the linear modal function ϕ_j with the inner product defined by

$$\langle f_1, f_2\rangle = \int_0^1 f_1(x)\bar{f}_2(x)\mathrm{d}x \tag{45}$$

for complex functions f_1 and f_2 on [0,1]. Therefore, the solvability condition is

$$\left\langle q^2\bar{q}\left[\bar{\phi}_j''(x)\int_0^1 g_2(x)\mathrm{d}x + \phi_j''(x)\int_0^1 g_1(x)\mathrm{d}x\right] - \left(2\mathrm{i}\omega_j\phi_j + 2c\phi_j'\right)\frac{\mathrm{d}q}{\mathrm{d}\tau} + \mathrm{i}\delta c^2\bar{\phi}_j''\bar{q}\mathrm{e}^{\mathrm{i}\mu\tau}, \phi_j\right\rangle = 0. \tag{46}$$

Equation 46 can be rewritten as

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} + k_1\bar{q}\mathrm{e}^{\mathrm{i}\mu\tau} + k_2 q^2\bar{q} = 0, \tag{47}$$

where

$$k_1 = \frac{\mathrm{i}}{4}\delta c(\mathrm{e}^{-2j\pi\mathrm{i}c} - 1), \quad k_2 = -\frac{1}{2\mathrm{i}j\pi}\int_0^1\left[\bar{\phi}_j''(x)\int_0^1 g_2(x)\mathrm{d}x + \phi_j''(x)\int_0^1 g_1(x)\mathrm{d}x\right]\bar{\phi}_j\mathrm{d}x. \tag{48}$$

Obviously, Eq. 47 has a zero solution $q(\tau) = 0$. The nonzero solution is expressed in the polar form

$$q(\tau) = \alpha_j(\tau)\exp\left(\mathrm{i}\beta_j(\tau)\right), \tag{49}$$

where both α_j and β_j are real functions with respect to τ. Inserting (49) into (47), separating the resulting equation into real and imaginary parts and solving the derivatives of α_j and β_j with respect to τ from the resulting equations yield the modulation equations

$$\begin{aligned}\frac{\mathrm{d}\alpha_j}{\mathrm{d}\tau} &= \alpha_j\left[\mathfrak{Im}(k_1)\sin\theta_j - \mathfrak{Re}(k_1)\cos\theta_j\right] - \mathfrak{Re}(k_2)\alpha_j^3,\\ \frac{\mathrm{d}\theta_j}{\mathrm{d}\tau} &= \mu + 2\left[\mathfrak{Re}(k_1)\sin\theta_j + \mathfrak{Im}(k_1)\cos\theta_j\right] + 2\mathfrak{Im}(k_2)\alpha_j^2,\end{aligned} \tag{50}$$

where

$$\theta_j = \mu\tau - 2\beta_j. \tag{51}$$

After α_j and θ_j are solved from the modulation equation (50), the approximate solution of 19 is

$$v(x,t) = \alpha_j(\varepsilon t)\cos\left(\frac{\omega}{2}t + \theta_j(\varepsilon t)\right) + O(\varepsilon). \tag{52}$$

Equation 52 is derived from Eqs. 25–27, (49) and (51).

4 Steady-state responses

The steady-state periodic response in transverse vibration of the string is with constant amplitude α_j and phase angle θ_j. Therefore the steady-state response corresponds to the nonzero fixed points of (50). Setting α_j and θ_j to be constant in (50) yields:

$$\alpha_j\left[\mathfrak{Im}(k_1)\sin\theta_j - \mathfrak{Re}(k_1)\cos\theta_j\right] - \mathfrak{Re}(k_2)\alpha_j^3 = 0,\ \mu + 2\left[\mathfrak{Re}(k_1)\sin\theta_j + \mathfrak{Im}(k_1)\cos\theta_j\right] + 2\mathfrak{Im}(k_2)\alpha_j^2 = 0. \tag{53}$$

It can be proved that $\mathfrak{Re}(k_2) > 0$ and $\mathfrak{Im}(k_2) < 0$. Eliminating θ_j from (43), for positive $\mathfrak{Re}(k_2)$ and negative $\mathfrak{Im}(k_2)$, one can solve the amplitudes of the steady-state periodic response from the resulting equation as

$$\alpha_{j1,2} = \frac{\sqrt{-\mu\mathfrak{Im}(k_2) \pm \sqrt{4|k_1|^2|k_2|^2 - \mu^2\mathfrak{Re}^2(k_2)}}}{\sqrt{2}|k_2|}, \tag{54}$$

where the positive and the minus signs correspond, respectively, to the larger and the smaller amplitude steady-state response. Equation 54 yields the existence condition of steady-state responses

$$\mp 2|k_1| \le \mu \le \frac{2|k_1||k_2|}{\mathfrak{Re}(k_2)}. \tag{55}$$

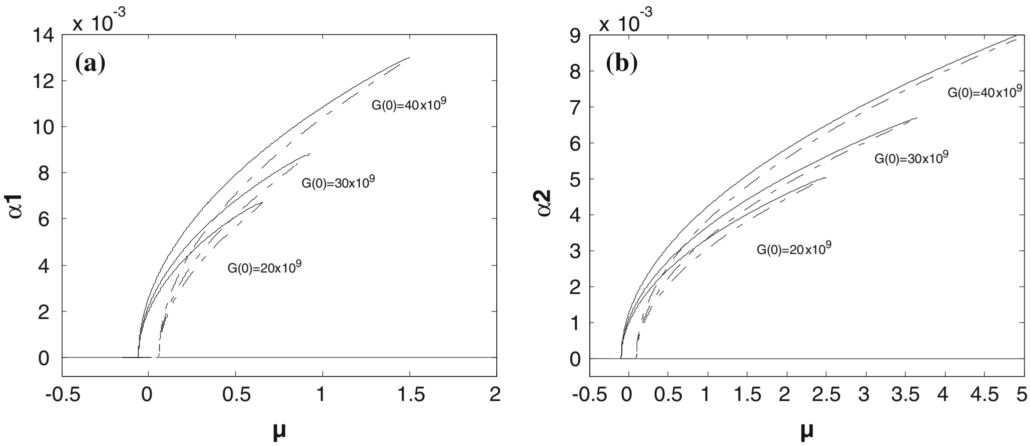

Fig. 1 Effect of initial stress-relaxation function value $G(0)$ **a** the first principal parametric resonance; **b** the second principal parametric resonance

It can be proved, [19], that the larger (smaller) amplitude response is stable (unstable), and the lower boundaries of the existence conditions for the larger and the smaller nontrivial solutions are, respectively, the same as the lower and upper boundaries of the instability conditions for the zero solution. That is, the instability interval of the zero solution is

$$-2|k_1| < \mu < 2|k_1|. \tag{56}$$

The amplitudes, the existence boundaries, and the stability of steady-state responses, will be numerically represented for the principal parametric resonance of the first two modes. Effects of the material parameters, the axial-speed-fluctuation amplitude, and the initial tension on the response will be demonstrated. In all figures, the solid lines and the dash–dot lines represent the amplitude of stable and unstable responses. In the computation, the fixed parameters are selected as $E_1 = 3 \times 10^{10}(\mathrm{N/m^2})$, $E_2 = 3 \times 10^{10}(\mathrm{N/m^2})$, $T/A = 7.5 \times 10^7(\mathrm{N/m^2})$, $\eta = 3 \times 10^8(\mathrm{Ns/m^2})$, $\rho = 7.68 \times 10^3(\mathrm{kg/m^3})$, $L = 1.0(\mathrm{m})$, $c = 0.2$ and $c_1 = 0.1$, if no other values are assigned. Equations 17 and 24 yield the first two linear natural frequencies as $\omega_1 = 3.016$ and $\omega_2 = 6.032$.

To understand the effects of the material parameters on the steady-state responses, consider the stress-relaxation function defined by the standard linear solid model (2)

$$G(t) = \frac{E_1 E_2}{E_1 + E_2} + \frac{E_1^2}{E_1 + E_2} \mathrm{e}^{-\frac{(E_1+E_2)t}{\eta}}. \tag{57}$$

The properties of the stress-relaxation function include the initial value of the stress-relaxation function $G(0) = E_1$, the long-time equilibrium value of the stress-relaxation function $G(\infty) = E_1 E_2/(E_1 + E_2)$ and the characteristic relaxation time $t_0 = \eta/(E_1 + E_2)$. Figures 1–3 show the effects of the material parameters on the steady-state responses in the first two principal parametric resonances. The upper boundaries of the existence conditions for both steady-state responses increase with the material parameters, and the lower boundaries of the existence conditions are independent of those parameters. Figure 1 shows that the amplitude of the steady-state responses increases with $G(0)$. Figure 2 shows that the amplitude decreases with $G(\infty)$. Figure 3 displays that the amplitude decreases slightly with t_0.

Figure 4 depicts the effects of axial speed-fluctuation amplitude δ on the steady-state responses in the first two principal parametric resonances. With the increasing axial speed-fluctuation amplitude δ, the lower boundaries of the existence conditions for the larger steady-state response decrease, and the lower boundaries of the existence conditions for the larger periodic response increase. Therefore, the instability interval of the straight equilibrium increases with axial-speed-fluctuation amplitude δ. With increasing axial-speed-fluctuation amplitude δ, the upper boundaries of the existence conditions increase dramatically, and the amplitudes of stable (unstable) steady-state responses increase (decrease) slightly.

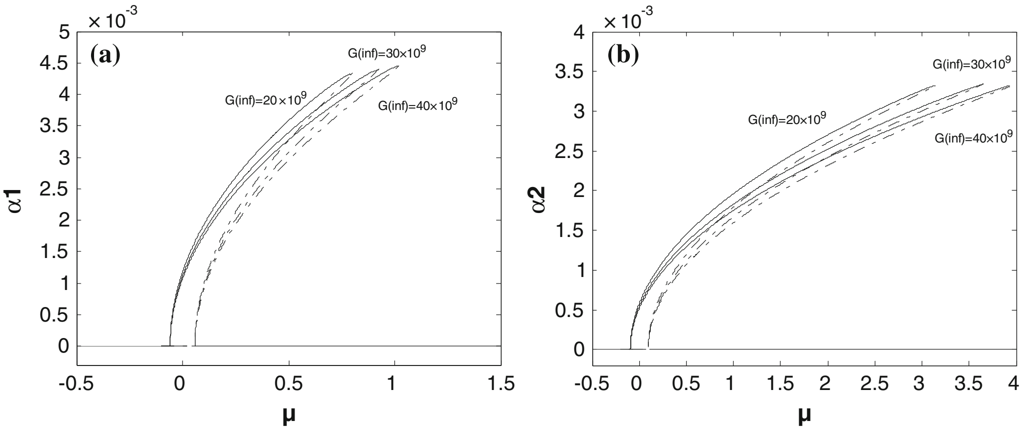

Fig. 2 Effect of long time stress relaxation function equilibrium value $G(\infty)$ **a** the first principal parametric resonance; **b** the second principal parametric resonance

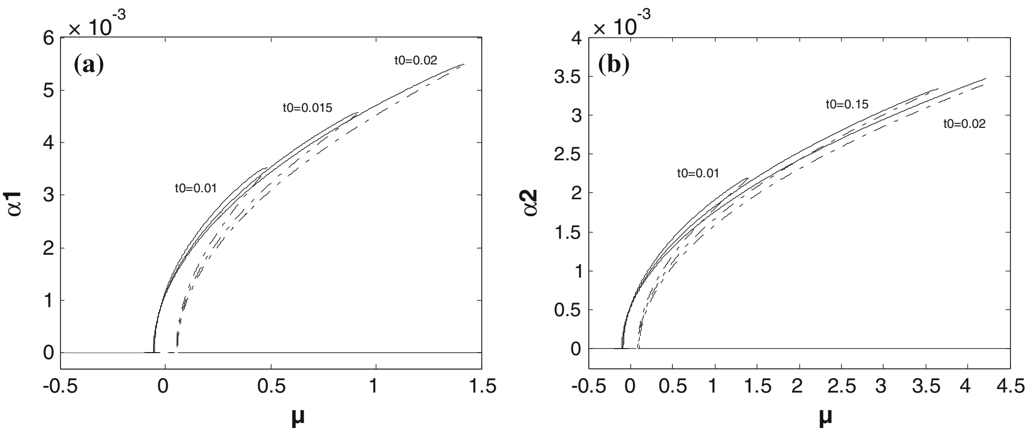

Fig. 3 Effect of characteristic relaxation time t_0 **a** the first principal parametric resonance; **b** the second principal parametric resonance

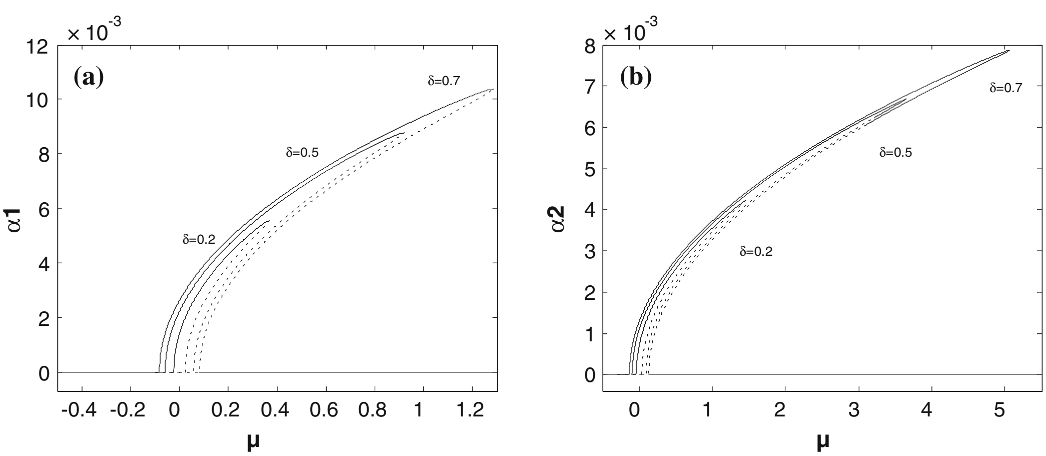

Fig. 4 Effect of axial speed fluctuation amplitude δ **a** the first principal parametric resonance; **b** the second principal parametric resonance

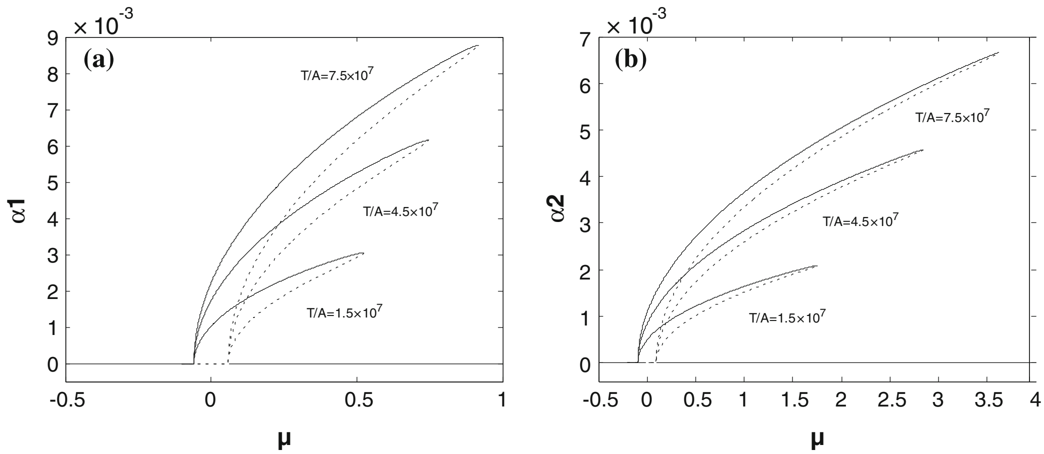

Fig. 5 Effect of initial stress T/A **a** the first principal parametric resonance; **b** the second principal parametric resonance

Figure 5 reveals the effects of initial tension T/A on the steady-state responses in the first two principal parametric resonances. The lower boundaries of the existence conditions of steady-state responses are independent of the initial tension, and hence the initial tension does not change the instability interval of the straight equilibrium. The upper boundaries of the existence conditions and the amplitude of the steady-state responses increase with initial tension T/A.

5 Steady-state responses based on Mote's model

The asymptotic approach developed for Kirchhoff's model in Sect. 3 can also be used for Mote's model. Actually, the solvability condition can still be expressed in Eq. 45, while coefficient k_2 is defined by the following equation rather than Eq. 46

$$k_2 = -\frac{1}{2\mathrm{i}j\pi}\int_0^1 \left[g_2(x)\bar{\phi}_j''(x) + g_2'(x)\bar{\phi}_j'(x) + g_1(x)\phi_j''(x) + g_1'(x)\phi_j'(x)\right]\bar{\phi}_j(x)\mathrm{d}x. \tag{58}$$

Therefore the amplitude and the existence conditions are still given by (54) and (55).

Effects of the material parameters, the axial speed fluctuation amplitude, and the initial tension on the amplitude and the existence condition of the steady response will be numerically demonstrated for the principal parametric resonance of first two modes. If no other values are assigned, the parameters are chosen as $E_1 = 3 \times 10^{10}$(N/m^2), $E_2 = 3 \times 10^{10}$(N/m^2), $T/A = 7.5 \times 10^7$(N/m^2), $\eta = 3 \times 10^8$(Ns/m^2), $\rho = 7.68 \times 10^3$(kg/m^3), $L = 1.0$(m), $c = 0.2$ and $c_1 = 0.1$. Figures 6–10 show the effects of the initial-stress-relaxation function value $G(0)$, long-time stress-relaxation-function equilibrium value $G(\infty)$, characteristic relaxation time t_0, axial speed-fluctuation amplitude δ, and initial stress T/A. In these figures, the solid lines and the dash–dot lines represent the amplitude of stable and unstable responses. For all parameters, Mote's model and Kirchhoff's model yield the same changing tendencies. Besides, based on both models, the amplitudes of the steady-state responses in the first principal parametric resonance are larger than those in the second principal parametric resonance.

Although Mote's model and Kirchhoff's model qualitatively yield the same results, two modes are quantitatively different. Figure 11 contrasts the results based on two the models. The Kirchhoff model predicts the larger amplitudes and the larger upper existence boundaries of steady-state responses.

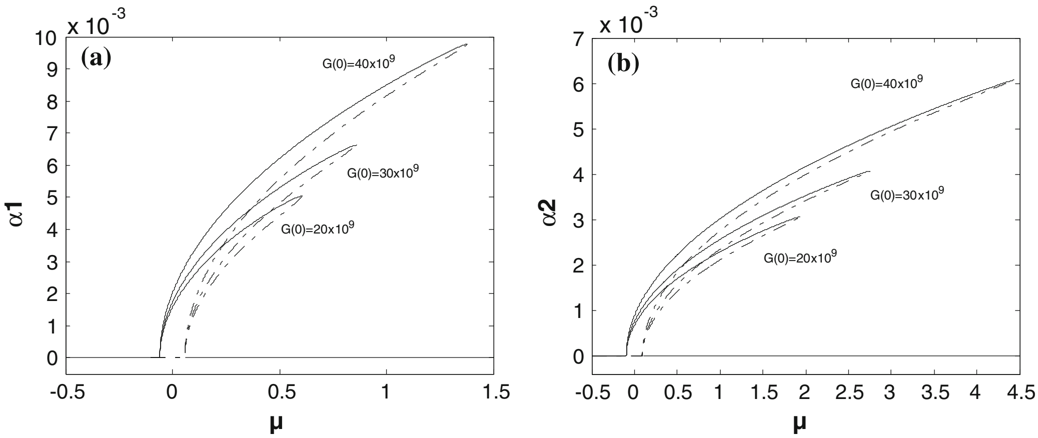

Fig. 6 Effect of initial stress relaxation function value $G(0)$ based on Mote's model **a** the first principal parametric resonance; **b** the second principal parametric resonance

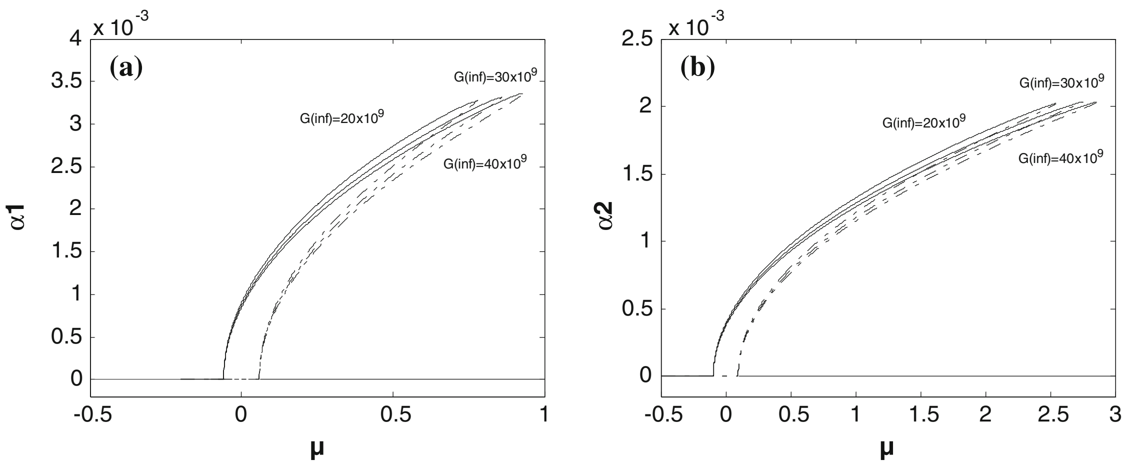

Fig. 7 Effect of long-time stress-relaxation-function equilibrium value $G(\infty)$ based on Mote's model **a** the first principal parametric resonance; **b** the second principal parametric resonance

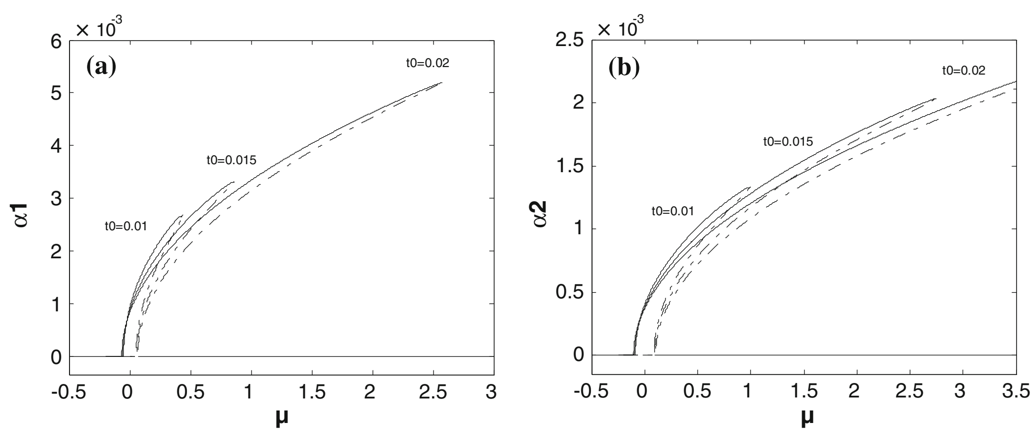

Fig. 8 Effect of characteristic relaxation time t_0 based on Mote's model **a** the first principal parametric resonance; **b** the second principal parametric resonance

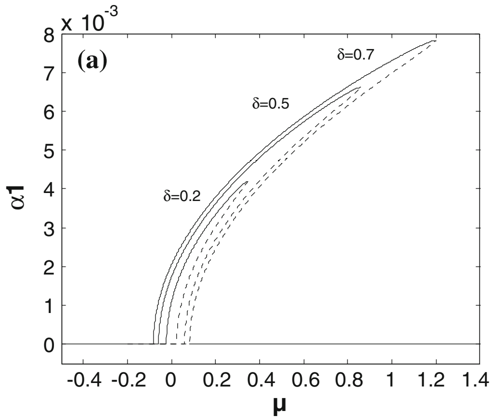

Fig. 9 Effect of axial speed-fluctuation amplitude δ based on Mote's model **a** the first principal parametric resonance; **b** the second principal parametric resonance

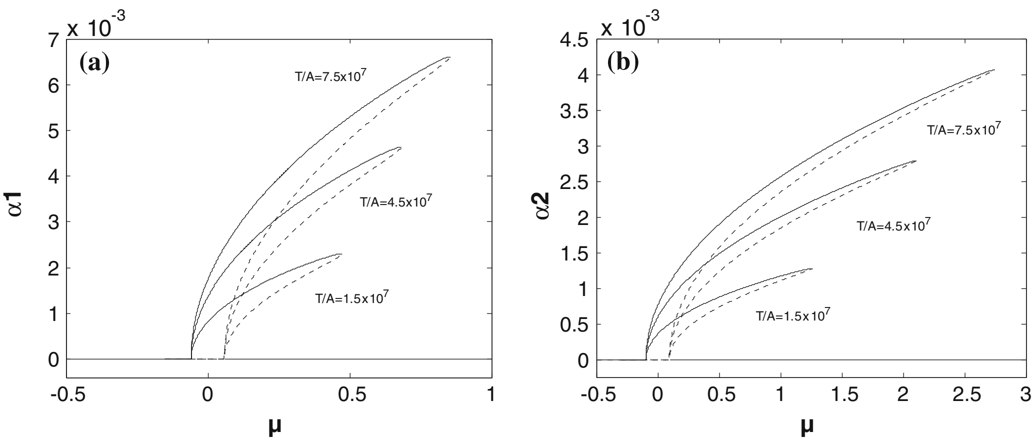

Fig. 10 Effect of initial stress T/A based on Mote's model **a** the first principal parametric resonance; **b** the second principal parametric resonance

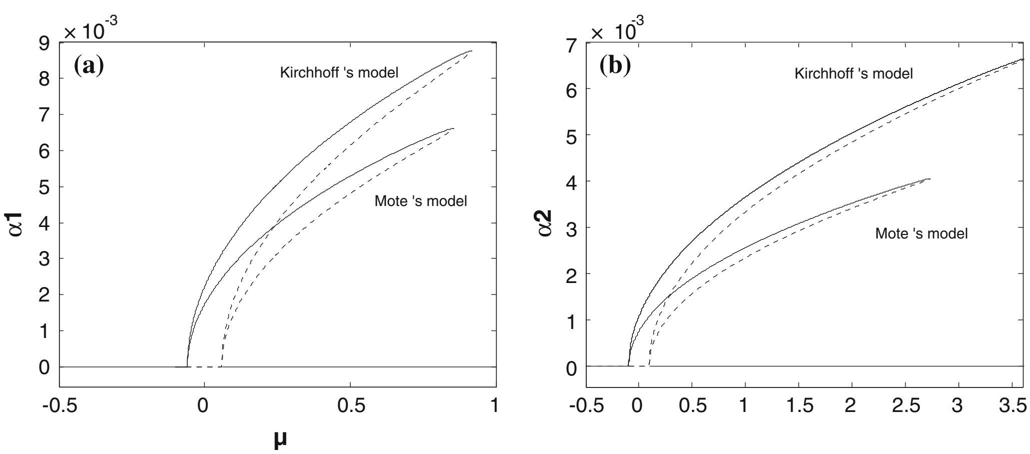

Fig. 11 Comparison of two model (18) **a** the first principal parametric resonance; **b** the second principal parametric resonance

6 Conclusions

Principal parametric resonance of axially accelerating viscoelastic strings has been investigated via an asymptotic perturbation approach. The constitutive relation of the string is the standard linear solid model using the material time derivative. Two nonlinear models of transverse motion were derived based on a coordinate transformation. An asymptotic perturbation approach has been proposed to study principal resonance based on the two models. The amplitude and the existence conditions steady-state responses are presented by locating the nonzero fixed points in the modulation equations resulting from the solvability condition. The investigation yields the following conclusions: (1) if the axial speed-fluctuation frequency is close enough to double a natural frequency, there are two steady-state responses in addition to the straight equilibrium. The larger (smaller) steady-state response is always stable (unstable); (2) the amplitudes of the steady-state responses in the first principal parametric resonance are larger than those in the second principal parametric resonance; (3) the amplitude of steady-state response increases with the initial stress relaxation function value, the initial stress, and the axial-speed-fluctuation amplitude, while it decreases with the long-time stress-relaxation-function equilibrium value and the characteristic relaxation time; (4) the lower boundary of the existence condition for the stable (unstable) steady-state response decreases (increases) with the axial-speed-fluctuation amplitude, and does not depend on the material parameters and the initial stress. The upper boundary of the existence condition for both steady-state responses increases with the material parameters, the initial stress, and the axial-speed-fluctuation amplitude; (5) qualitatively, Kirchhoff's model and Mote's model predict the same changing tendencies of the amplitudes and the existence intervals of steady-state responses. Quantitatively, both models yield the same existence intervals of steady-state responses, while Kirchhoff's model predicts the larger periodic response amplitude.

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