On some Hard-to-Solve versions of the Assignment Problem

Eduard Kh. Gimadi *

* Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, 4 Acad. Koptyuga ave., 630090 Novosibirsk, Russia
gimadi@math.nsc.ru

Abstract. Some hard-to-solve versions of the Assignment Problem are studied. Results of scientific research of approximation algorithms for the multi-index Assignment Problem are presented in axial and planar cases. There are established performance guarantees and conditions of asymptotic exactness for approximate solutions obtained.

Keywords: multi-index Assignment Problem, axiality; planarity; m-layer Assignment Problem; Peripatetic Salesman Problem; single-cyclic permutation; approximation algorithm; performance guarantee; asymptotic exactness

1. Introduction

The assignment problem (AP) is well studied in the classical two-index version. Several properties are known respective to existence of integer optimal solution for its relaxation, mathematical expectation of optimal value on random instances, etc. It is known that the classical AP with a random $n \times n$-matrix can be solved by an asymptotically optimal (approximation) algorithm whose running time is linear in the problem size [7].

A natural generalization of the AP is the multi-index AP (MAP), that has a lot of applications in communication, logistic, manufacturing, economics. For the insight with multi-index assignment problems we refer the reader to productive surveys of Burkard, at al. Studia Informatica Universalis.
[RE99, RMS09], and Spieksma [F.C00], where variants of the MAP are considered under different conditions (axiality, planarity, type of permutation, at al.). We describe results for some hard-to-solve versions of the MAP whose presentation is absent (or almost absent) in [RE99, RMS09, F.C00]. These results were obtained by the author and his colleagues in the Sobolev Institute of Mathematics substantially in last ten years. The author hopes that the contribution presented will be useful further to above mentioned reviews.

Let us formulate the general MAP with parameters \((m, n, r)\) [VMM81].

\[
\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i_1 i_2 \cdots i_m} x_{i_1 i_2 \cdots i_m} \rightarrow \min_{x_{i_1 i_2 \cdots i_m} \in \{0,1\}}
\]

where \(c_{i_1 i_2 \cdots i_m}\) are given numbers s.t.

\[
\sum_{i_{k_1}=1}^{n} \sum_{i_{k_2}=1}^{n} \cdots \sum_{i_{k_r}=1}^{n} x_{i_{k_1} i_{k_2} \cdots i_{k_r}} = 1, \ i_s = 1, n, \ s = 1, m, \ s \neq k_1, \cdots, k_r.
\]

For \(r = m - 1\) we have multi-index axial AP (MAAP).

In the case of \(r = 1\) the problem called multi-index planar AP (MPAP).

The case of \(m = 2\) corresponds to classic (two-index) AP, that is solved in running time \(O(n^3)\) by exact algorithm in [EM69].

Contrary to the classical AP, the multi-index AP is unlikely to be efficiently solvable. For the number of indexes at least three both problems (axial and planar) are NP-hard [R.M72, A.M83]. This stimulates a construction of approximation algorithms with performance guarantees for solutions of MAP. Therefore, we pay attention to designing and analyzing approximation algorithms which can solve the MAP in a polynomial time.
2. On asymptotically exact approach with regard to solving the MAP on random inputs

For solving the MAP on random input matrixes we use the approach suggested in [ENV75] to solve discrete optimization problems on random inputs using fast (effective, polynomial time) approximation algorithms for the purpose of justifying conditions of asymptotic exactness for solutions obtained.

Let $C_{mn}$ be a class of the MAP, where elements of $n \times \cdots \times n$-matrix $(c_{i1i2\cdots im})$ are random independent variables in the segment $(a_n, b_n)$, $a_n > 0$, with common distribution function $P(x) = Pr\{\xi < x\}$ of normalized random variable

$$\xi = \frac{(c_{i1i2\cdots im} - a_n)}{(b_n - a_n)}, \quad 0 \leq \xi \leq 1.$$

Let as denote $OPT(I)$ as the optimal value of an objective, and $f_A(I)$ as a value of the objective obtained by an algorithm $A$ with the input $I$. We say that the algorithm $A$ has performance estimations $(\varepsilon_A, \delta_A)$ in the class of considered problems if the following inequalities hold:

$$Pr\{f_A(I) > (1 + \varepsilon_A) OPT(I)\} \leq \delta_A,$$

where $\varepsilon_A$ is the relative error of the solution obtained by the algorithm $A$, and $\delta_A$ is the fault probability of the algorithm $A$ (i.e. a part of occurrences when the algorithm $A$ does not guarantee declared exactness in the supposed polynomial time).

One of the first algorithm with performance guarantees for the classic AP on random instances was presented in the paper of Borovkov [A.A62].

Algorithm $A_1$ for the classic AP:
1. See successively rows \( n \times n \)-matrix \((c_{ij})\). In the next row \( i = 1, n \) there is noted the number of the column \( j \), where the smallest element locates (among earlier non marked elements), and put \( \pi(i) = j \).

2. As a result we have the approximate solution with the objective value \( \tilde{f}_{A1} = \sum_{i=1}^{n} c_{i,\pi(i)} \) and variables \( \tilde{x}_{ij} = \begin{cases} 1, & j = \pi(i), \\ 0, & \text{else.} \end{cases} \)

Obviously algorithm \( A1 \) has the time complexity \( O(n^2) \) and that on random instances \( \xi_{2,n} \) the algorithm \( A1 \) is asymptotically exact under conditions

\[
\frac{b_n}{a_n} = o\left(\frac{n}{\max\{n\gamma_n, J_n\}}\right), \quad J_n = \int_{\gamma_n}^{1} dx \frac{1}{P(x)} \rightarrow \infty,
\]

where \( \gamma_n \) is a root of the equation \( P(x) = \frac{1}{n} \).

On inputs \( \xi_{2,n} \) with uniformly distributed elements of the matrix the condition of asymptotic optimality for the problem on minimum looks simpler: \( b_n/a_n = o(n/\log n) \) (for distribution on the continuous segment \((a_n, b_n)\) and \( r_n = o(n/\log n) \) (in the case of the integer segment \([1, r_n]\)). At that for the problem on maximum asymptotic exactness there is without additional requirements.

3. Multi-index axial AP (MAAP)

The MAAP (or \( m \)-AAP) is formulated as follows:

\[
\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i_1i_2\ldots i_m} x_{i_1i_2\ldots i_m} \rightarrow \min_{x_{i_1i_2\ldots i_m} \in \{0,1\}}
\]

subject to axial sums

\[
\sum_{i_1=1}^{n} \cdots \sum_{i_{s-1}=1}^{n} \sum_{i_{s+1}=1}^{n} \cdots \sum_{i_m=1}^{n} x_{i_1i_2\ldots i_m} = 1 \text{ for all } i_s = \overline{1, n}; \quad s = \overline{1, m},
\]

where \( c_{i_1i_2\ldots i_m} \) are given real numbers.

The MAAP can also be stated in a purely algebraic way: Let \( S_n \) be the set of all permutations on the set \( \{1, 2, \ldots, n\} \). Given the
Assignment Problem

\[ \text{find the permutations } \pi_1, \ldots, \pi_{m-1} \text{ in } S_n \text{ which minimize:} \]

\[ \sum_{i=1}^{n} C_{i, \pi_1(i), \sigma_3 \pi_1(i) \ldots, \sigma_m \pi_1(i)}, \]  

where

\[ \sigma_{kk'} \in S_n \text{ for } 1 \leq k' < k \leq m, \]  

and \( \sigma_{kk'} \) is the permutation \( \pi_{k-1} \pi_{k-2} \cdots \pi_{k'+1} \pi_{k'}, 1 \leq k' < k \leq m. \)

Among the MAAP, the tree-index AAP has a vivid geometrical interpretation and the simplest mathematical formulation:

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} C_{ijk} X_{ijk} \rightarrow \min \]  

subject to axial sums

\[ \sum_{j=1}^{n} \sum_{k=1}^{n} X_{ijk} = 1, \ i = 1, n; \]

\[ \sum_{i=1}^{n} \sum_{k=1}^{n} X_{ijk} = 1, \ j = 1, n; \]

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ijk} = 1, \ k = 1, n. \]

The problem consists in such choice of elements of cubic matrix \((c_{ijk})\) of size \(n\) (one by one in each section of the matrix), in order to minimize the sum of chosen elements. (The section of the matrix is its subset that consist in \(n^2\) elements where one of the three indexes \((i, j \text{ or } k, 1 \leq i, j, k \leq n)\) is fixed.) The problem has the following algebraic form:

\[ \sum_{i=1}^{n} C_{i, \pi(i), \sigma \pi(i)} \rightarrow \min_{\pi, \sigma \in S_n}, \]

where \(S_n\) is the set of permutations from symmetric group of size \(n\).

Let us describe an algorithm \(A2\) to solve the 3-index AAP [EA99]:
1. Form $n \times n$-matrix: $d_{ij} = c_{ij}$, for every $1 \leq i, j \leq n$.

2. Use the algorithm $A1$ to the matrix $(d_{ij})$.

3. There is an approximate solution of the problem: the objective value

$$\tilde{F}_{A2} = \sum_{i=1}^{n} d_{i,\pi(i)}$$

and variables

$$\tilde{x}_{ijk} = \begin{cases} 1, & \text{if } j = k = \pi(i), \\ 0, & \text{else}. \end{cases}$$

It is notable that the algorithm $A2$ for the 3-index AAP has the properties like the algorithm $A1$ for solving the classical two-index AP.

The following Statement is true under the assumption $P(x) \geq x$, $0 \leq x \leq 1$ [EN04]:

Claim. If $b_n/a_n = o\left(n/\ln n\right)$, then Algorithm $A2$ yields an asymptotically optimal solution of the 3-index AAP on class of problems $C_{3n}$ in $O(n^2)$-time.

Note, that we can repeat Algorithm $A2$ n-folding time for n different matrices $d'_{ij} = c_{i,j,k(j,r)}$, $r = 1, \ldots, n$, where $k(j,r) = j + r$ in the case $j + r \leq n$ and $k(j,r) = j + r - n$ in the case $j + r > n$. We choose a solution with the minimal value $f$. As a result we decrease essentially the probability-of-failure bound to value $\left(\delta_{A2}\right)^n \to 0$ as $n \to \infty$. Certainly the time complexity of modified Algorithm $A2$ is increased in $n$ times.

The statements formulated above for the 3-index AAP can be also established in the case of the $m$-index AAP, $m > 3$. As elements of a $n \times n$-matrix $(d_{ij})$ used in Algorithm $A2$ the numbers $d_{ij} = c_{i_1,\ldots,i_m}$ for $i_1 = i$ and $i_2 = \ldots = i_m = j$, $1 \leq i, j \leq n$, can be taken. It is of interest that sublinear-time Algorithm $A2$ applied to the 3-index AAP has sublinear running time in the case of the $m$-index AAP, $m > 3$, as well. Moreover, in this case the running time of Algorithm $A2$ normalized to the problem size decreases as the function $n^{2-m}$ when $m$ grows.

The similar propositions are true for the $m$-index AAP, $m > 3$. In this case numbers $d_{ij} = c_{i_1,\ldots,i_m}$ if $i_1 = i$ and $i_2 = \ldots = i_m = j$, $i, j = 1, n$ appear as elements of $n \times n$-matrix that formed for accordingly modified algorithms like $A1$. 
Interestingly enough the bounds of time complexity, relative error and to-fault-probability of algorithms (modified for the number indexes at least three) does not become worse with increase of $m$.

4. On solvability of the multi-index AAP on single-cyclic permutations

The essential distinction of the multi-index AP on single-cyclic permutations from problem (1)-(2) lies in the fact that the condition (2) is replaced by condition

$$\sigma_{kk'} \in P_n \text{ for } 1 \leq k' < k \leq m,$$

where $P_n$ is the set of all single-cyclic permutations in $S_n$.

The multi-index AP is NP-hard if the number of indexes exceeds 2. For the two-index version the problem is polynomially solvable. However the two-index AP problem on single-cyclic permutations is not polynomially solvable, because it coincides with the classic traveling salesman problem (TSP). It follows that the problem considered is MAX SNP-hard.

More over, the solution of the multi-index AP on single-cyclic permutations can be failed for any input data. The criterion of solvability for the 3-index AP on single-cyclic permutation was proved: the problem is solvable if and only if $n$ is odd [EA99]. Later the criterion of solvability of the $m$-index AP on one-cyclic permutations for $3 \leq m \leq 7$ is established also: there exists a number $n_m$, such that for every $n \geq n_m$ the problem is solvable if and only if $n$ is odd [KNA99].

Recently this ten years old result was extended to the 8-index AAP [O.Y11].

5. The 3-index planar AP (3-PAP) and the $m$-layer 3-PAP

The classical the 3-index PAP consist in the choice of $n^2$ elements of the input $n \times n \times n$-matrix s.t. there are exactly $n$ elements in each parallel layer in order to minimize the sum of chosen elements.
Kravtsov and Krachkovsky [MA73] designed a polynomial-time approximation algorithm for the 3-PAP and claimed it to be asymptotically optimal. However, this claim is not correct as it is shown in [IEM01].

A modification of the classical 3-PAP is the $m$-layer three-index planar AP (the $m$-layer 3-PAP): given an $n \times n \times m$ cost array $C = (c_{ijk})$, we ask for $m$ mutually distinct permutations $\pi_1, \pi_2, \ldots, \pi_m$ such that $
 \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij, \pi(i)(j)}$ is a minimum. (The 3-PAP is obtained for $m = n$.) The $m$-layer 3-PAP can also be viewed as a special case of the 3-PAP with coefficients $c_{ijk} = 0$ for all $i \geq m$. The problem is NP-hard for $m \geq 2$ [D.G97].

In [EN04] for the $m$-layer 3-PAP it is presented the following simple polynomial-time approximation algorithm which consist of the following Stages $i = 1, m$:

Consecutively for $j = 1, \ldots, n-2(i-1)$ do:
1. Find $k_j = \arg\min\{c_{ijk} | k = 1, n\}$.
2. Set $\pi'(j) = k_j$.
3. Forbid (set to infinity) elements $c_{ij'k_j}$ for $j' = j + 1, \ldots, n$ and $c_{i'jk_j}$ for $i' = i + 1, \ldots, m$.

Use Hopcroft-Karp algorithm [JR73] to solve (classical) AP for the matrix formed by the rest $2(i-1)$ lines ($j = n - 2(i-1) + 1, \ldots, n$) of $i$'th layer of matrix $C$ and construct the rest part of permutations $\pi_i$ according to this solution.

So the algorithm on each Stage $i = 1, m$ at first works by principle "to select a minimal permissible element in the next line" and constructs a partial assignment. Then a special procedure is used to complete this partial assignment to a total assignment. The algorithm in [EN04] is correct for $m < n$ and its running time is $O(mn^2 + m^{7/2})$.

In [EN04] for solving the $m$-layer 3-PAP on random inputs an asymptotical optimality was proved for the case when $m \sim \ln n$ and $c_{ijk}$ has a uniform distribution. Later this algorithm is exposed to more thorough analysis and its asymptotical optimality is proved for the extended region $m \sim n^\theta$ ($\theta = const$) and $c_{ijk}$ has a minorized type distribution function [EY06]. Also performance estimates of the algorithm are improved.
6. *m*-layer 3-PAP on single-cyclic permutation (*m*-layer 3-PAPS)

The *m*-layer 3-PAPS is varied from the *m*-layer 3-PAP by the single-cyclicity of each permutation $\pi_i$, $i = 1, m$. The problem is known also as the *m*-Peripatetic Salesman Problem (*m*-PSP) that at first was formulated by Krarup J. [J.75]. This problem can be viewed as a problem of finding *m* edge-disjoint Hamiltonian circuits in complete weighted graph with the extremal total weight of chosen edges.

De Kort [J.B93] shows that the 2-PSP is NP-hard by constructing a polynomial-time reduction from the Hamiltonian Path Problem. By a similar argument one can prove that the *m*-PSP is NP-hard for each $m > 2$.

Recently there are designed some polynomial approximation algorithms with performance guarantees for solving the 2-PSP when weight functions of both Hamiltonian circuits are common, and different also.

Assume that algorithm A solving a problem on minimum (maximum) for input $I$ computes a solution with the objective $f(I)$. Approximation ratio of A is defined from the equation: $\rho_A = \min_{(I)} \frac{f(I)}{Opt(I)}$ (respectively, $\max_{(I)} \frac{f(I)}{Opt(I)}$), where $Opt(I)$ is equal to minimum (maximum) value of objective for given input $I$. The algorithm with performance ratio $\rho$ we call as $\rho$-approximation algorithm.

The following special cases are considered: the symmetric case when $c_{ij} = c_{ji}$ for all pairs $(i, j)$; the metric case when $c_{ij} = c_{ji}$ for all pairs $(i, j)$ and $c_{ij} + c_{jk} \geq c_{ik}$ for all triples $(i, j, k)$; Euclidean case when vertexes in graph correspond to points in Euclidean space $\mathbb{R}^k$, and edge weights equal to lengths of relative intervals in $\mathbb{R}^k$.

The following statement was useful for design and analysis algorithms: in *n*-vertex 4-regular graph a pair of edge-disjoint partial tours with total number of edges at least $8n/5$ can be found in quadratic time complexity [EYA09].

The case of common weight function on Hamiltonian circuits.
For the symmetric MAX $2$-PSP there is $3/4$-approximation algorithm with time complexity $O(n^3)$ [AAE07]. This bound slightly improved recently to performance ratio $7/9$ by Glebov and Zambalaeva.

For the metric MIN $2$-PSP there are designed $9/4$-approximation algorithm with time complexity $O(n^3)$ [YKM04] and 2-approximation algorithm with time complexity $O(n^2 \log n)$ [AP07].

For the MIN $2$-PSP with weights of edges valued 1 and 2 there are known $5/4$-approximation [ADE’09] and $6/5$-approximation [EYA09] algorithms with time complexity $O(n^3)$.

The MAX $2$-PSP with weights valued in the segment $(1, q)$ can be solved in time $O(n^3)$ with performance ratio $(3q + 2)/(4q + 1)$ (that means the bound $8/9$ for $q = 2$) [EE11].

The case of different weight functions of Hamiltonian circuits.

For the MIN $2$-PSP with edges valued 1 and 2 there are known $11/7$-approximation algorithms with time complexity $O(n^3)$ and $4/3$-approximation algorithm with time complexity $O(n^3)$ [AD11].

$\frac{11\rho - 8}{18\rho - 18}$ - approximation algorithm with time complexity $O(n^3)$ is constructed in [EE11] for the MAX $2$-PSP with edges valued 1 and 2. At this point $\rho$ is an approximation ratio, known for the problem on minimum. So for the bound $\rho = 4/3$ from [AD11] a performance guarantee for the problem is equal to $20/27$.

The case of graphs in the multidimensional Euclidean space $\mathbb{R}^k$.

For the MAX $2$-PSP with distances in Euclidean space $\mathbb{R}^k$ asymptotically exact algorithm with time complexity $O(n^3)$ designed [E.K08].

For finding $m$ edge-disjoint Hamiltonian circuits on the graph in Euclidean space $\mathbb{R}^k$ polynomial algorithm was constructed in [YK11] where the following statement was proved.

Let $m = m(n) = o\left(n^{(k-1)/(k+1)}\right)$. Then an asymptotically optimal solution for the Euclidean $m$-PSP can be found in $O(n^3)$-running time with performance ratio at least \(1 - \beta_k n^{-\frac{k+1}{k+1}}\) → 1 as $n \to \infty$. 

7. Conclusion

We considered some versions of the multi-index Assignment Problem which are hard to solve in general case. Approximation polynomial algorithms are constructed in some axial and planar cases on random and deterministic inputs. There are established performance guarantees of the algorithms. Conditions of asymptotic exactness for approximate solutions for the axial and the $m$-layer planar MAP on random inputs and for the $m$-peripatetic salesmen problem in multi-dimensional Euclidean space are presented. It will be interest to research of versions of the MAP with different assumptions on elements of input multi-index matrix.

Acknowledgements

This research was supported by the Russian Foundation for Basic Research (grants 12-01-00093 and 10-07-00195), target program No 2 of RAS Presidium (project No 227), target programs SB RAS (integration projects No 44 and No 30), and Federal target grant (government contract No. 14.740.11.0362)

References


