Calculating Outage Probability of Block Fading Channels Based on Moment Generating Functions

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Abstract—Block fading channels present a realistic model in many communication scenarios. We propose a method by which the outage probability of block fading channels can be evaluated in a straightforward and computationally efficient manner by utilizing the moment generating function of a single block’s mutual information. The flexibility of moment generating functions helps obtain outage probabilities in various cases such as wideband noise jamming and multi-access interference. The effect of discrete input alphabets on outage is also investigated and turns out to be vital especially under interference.

Index Terms—Block fading, outage, moment generating function, jamming, multi-access interference, receive diversity, finite constellation size.

I. INTRODUCTION

The block fading channel model is used widely in the design and analysis of wireless communication systems. In the block fading channel model, the transmission of a piece of information, i.e., a codeword, is over a limited number of blocks where the channel is constant within blocks and change usually independently from block to block. This is a very good model for many delay sensitive systems employing frequency hopping (FH), OFDM, time diversity techniques in time selective channels etc. since delay sensitivity necessitates a limited transmission time. In addition, interference, especially in the form of noise jamming, multi-access interference in time-slotted systems etc., may also change across blocks. Hence, a combination of all these phenomena may affect the system analysis and design.

We concentrate on the scenarios with no channel state information at the transmitter (CSIT) hence no resource allocation among the blocks is possible. The availability of perfect channel state information at the receiver is assumed possibly by a pilot insertion in each block. The blocks undergo independent fading. Achievable performance of coded systems in block fading channels without CSIT has been studied by many authors. One of the pioneering works in this topic is [1] where it was shown that error performance is dominated by information outage probability. The information outage probability becomes a lower bound to error performance of coded modulation schemes and using this fact coded modulation schemes that can perform quite close to the corresponding information outage probabilities can be designed [1]–[4].

As the information outage probability provides a lower bound to achievable error performance that many practical coded modulation schemes can perform very close to, methods to calculate the information outage probability for block fading channels has recently received attention. In [5], a method based on probability density function (PDF) of mutual information is proposed where the outage probability is found by convolutions and no closed form formula is provided. Using a few approximations, another PDF based technique is introduced in [6] for the purpose of evaluating performance of hybrid-ARQ in block fading channels.

Our aim in this study is to develop analytical expressions that facilitates a simple calculation of outage probability under block fading in various scenarios. Moment generating function (MGF) based error rate evaluations are well known in the literature [7] where MGFs are used for the error terms usually in the form of an exponential. A recent survey on the use of MGF for calculating capacity can be found in [8] where the authors also propose a method for calculating the ergodic capacity of fading channels based on an integral involving the exponential integral function. To the best of authors’ knowledge, however, MGF based techniques have not been utilized in the literature for the purpose of outage probability calculation of block fading channels although it is very natural to do so.

The main contribution is the application of an MGF-based method to the calculation of outage probability for block fading channels. The method’s simple structure enables consideration of issues such as jamming and multi-access interference in a very straightforward manner unlike other methods. What is needed in the method is the MGF of a single block’s mutual information. We also study the effect of discrete input alphabets on the outage probability by writing the MGF with an upper bound to a block’s mutual information.

II. PRELIMINARIES

The approach of this study in finding outage probabilities is to utilize moment generating functions which can be evaluated easily in many cases. The invertibility of the Laplace transform entails a simple numerical procedure enabling the evaluation of the cumulative distribution function through the random variable’s MGF [7]. The MGF of a positive random variable $X$ with PDF $f_X(x)$ is defined as $\theta_X(s) = E[e^{sX}] = \int_0^\infty f_X(x)e^{sx}dx$. It is given in [7] that for a positive random variable $X$ with MGF $\theta_X(s)$

$$P(X \leq x) = \frac{2^{-K}e^{A/2}}{x} \times$$

$$\sum_{k=0}^{K} \left(\frac{K}{k}\right) \sum_{n=0}^{N+k} \frac{(-1)^n}{\alpha_n} \frac{g_X}{2x} \left(\frac{A + 2\pi jn}{2x}\right) + E(A, K, N) \tag{1}$$
where \( g_X(a) = \Re\{\theta_X(-a)/a\} \) and \( E(A, K, N) \) is an error term that can be bounded by
\[
E(A, K, N) \leq \frac{e^{-A}}{1 + e^{-A}} + \frac{2 - 2K e^{A/2}}{x} \times \left[ \sum_{k=0}^{K} \binom{K}{k} (-1)^{k+N+1} g_X\left(\frac{A + 2\pi j(k + N + 1)}{2x}\right) \right],
\]
for parameters \( A, K, N \) with \( \alpha_n = \delta[n] + u[n] \), \( \delta[] \) is the discrete Dirac delta function, and \( u[\cdot] \) is the discrete unit step function.

We will refer to the first term in the RHS of (1) as the calculated probability and the RHS of (2) as the accuracy since accuracy can be set through the error term’s bound in (2). In our studies we adopt the following approach. The default values for the parameters used throughout the paper are \( A = 30, K = 10, N = 10 \). The calculated probability and the corresponding accuracy are both evaluated. If the ratio of the accuracy to the calculated probability is not below \( 10^{-2} \), the parameters should be changed. Although increasing the values of \( K \) or \( N \) is usually sufficient to enhance the accuracy, the value of \( A \) may sometimes be more important in setting the accuracy due to the \( e^{-A}/(1 + e^{-A}) \) term in the RHS of (2). We use the algorithm given below to determine the values of \( A, K, N \) online.

**Algorithm 1** Pseudocode for computing (1). (The values used in the pseudocode are arbitrary and representative of what is usually sufficient in practice.)

\[
A \leftarrow 30, \quad K \leftarrow 10, \quad N \leftarrow 10 \\
DONE \leftarrow 0 \\
\text{while } DONE = 0 \text{ do} \\
\quad \text{Calculate } p, \text{ the RHS of (1).} \\
\quad \text{Calculate } a, \text{ the RHS of (2).} \\
\quad \text{if } e^{-A}/(1 + e^{-A})/p > 10^{-2} \text{ then} \\
\quad \quad A \leftarrow A + 5 \\
\quad \text{else if } a/p > 10^{-2} \text{ then} \\
\quad \quad K \leftarrow K + 5, \quad N \leftarrow N + 5 \\
\quad \text{else} \\
\quad \quad DONE \leftarrow 1 \\
\text{end if} \\
\text{end while}
\]

The signal model follows from the block fading channel model
\[
y_n = h_n s_n + w_n,
\]
\( n = 1, \ldots, M \), where \( y_n, s_n, w_n \) are the received, transmitted, and noise signals of the \( n^{th} \) block respectively. No indices for the signals within a block are used since our interest is toward outage probability which is directly linked with mutual information per complex symbol (or channel use). The unit of the mutual information per channel use will be either nats per use (npu) or bits per use (bpu) in this script. The average received power \( E[|s_n|^2] \) is set to \( E_s \). The noise term \( w_n \) is a complex circularly symmetric white Gaussian random variable with zero mean and variance \( N_0 \). The fading coefficient \( h_n \) is constant for the duration of the block. There are a total of \( M \) blocks through the whole transmission and the fading coefficient’s power at block \( n \) will be denoted as \( X_n = |h_n|^2 \).

The mutual information per channel use for the block fading channel averaged over \( M \) blocks is defined as
\[
R = \frac{1}{M} \sum_{n=1}^{M} C_n, \tag{4}
\]
where \( C_n \) is the mutual information per channel use in the \( n^{th} \) block [1]. The information outage probability is the probability that \( R \) is below a certain transmission rate: \( P_{\text{out}}(R_0) = P(R < R_0) \).

When the fading coefficients \( h_n \) are independent across blocks, as is the case in the block fading channel model here, \( C_n \)'s are independent. Hence, the MGF of \( R \) can be found by multiplying the MGF of the mutual information \( C_n \) per channel use of a single block since
\[
\theta_R(s) = E[e^{sR}] = E[e^{sM \sum_{n=1}^{M} C_n}] = \prod_{n=1}^{M} \theta_{C_n}(s/M). \tag{5}
\]
The corresponding outage probability is obtained when the moment generating function of \( R \) is used in (1).

### III. Outage Probability for Some Channels and Input Alphabets

We present some examples of fading distributions of utmost importance in practice for which we were able to obtain \( \theta_{C_n}(s) \) in terms of well-known functions. It is obvious that this method can be applied to other fading distributions and multi-antenna scenarios as long as closed-form MGF of a single block’s mutual information can be found as given for some in [9]. In the derivations, the mutual information per channel use will be in units of npu although we will present the results in the more familiar unit of bpu.

#### A. Nakagami-m Fading Channel

A fading model that is widely used in the literature is the Nakagami-m fading since its PDF can be fitted to the PDFs of many other fading models such as Rayleigh and Rician by varying \( m \). When a Gaussian input alphabet is assumed along with the average power constraint \( E[|s_n|^2] \leq E_s \), the mutual information per channel use for the \( n^{th} \) block is given by
\[
C_n = \ln (1 + \gamma X_n) \text{ (npu)} \quad \text{with } \gamma = E_s/N_0 \text{ is the average signal-to-noise power ratio and}
\]
\[
f_{X_n}(x) = \frac{m^m}{\Gamma(m)} x^{m-1} e^{-mx}, \quad x > 0
\]
where \( m \geq 0.5 \) is the fading parameter [10]. This leads to the following MGF
\[
\theta_{C_n}(s, \gamma) = \int_0^\infty (1 + \gamma x)^s \frac{m^m}{\Gamma(m)} x^{m-1} e^{-mx} dx \tag{7}
\]
\[
= \left( \frac{m}{\gamma} \right)^{m} U(m, 1 + m + s, m/\gamma) \tag{8}
\]
where $U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1}(1 + t)^{b-a-1} dt$ is the (Tricomi) confluent hypergeometric function that exists for $\Re(z) > 0$ which is the case here.

Plotted in Fig. 1 are the calculated outage probabilities for $m = 0.5$ and $m = 1.5$. The curves are obtained through simulations. At least 50 outage events are recorded for each simulation point throughout this study. We observe that the proposed method perfectly predicts the outage probabilities. Although the method has been explained with the rate in bpu, we will write the transmission rates in all the figures in units of bits per channel use (bpu) since it is usually more natural to do so.

We like to mention here that a very similar expression is obtained for receive diversity schemes. Let $X_n = \sum_{i=1}^k X_{n,i}$ be the resultant fading coefficient after maximal-ratio combining of $k$ branches all with Rayleigh fading of unity power. The PDF of $X_n$ is well known and equals $f_{X_n}(x) = x^{k-1} e^{-x}/(k - 1)!$. The MGF corresponding to the mutual information is then evaluated as $k \gamma^{-k} U(k, 1 + k + s, 1/\gamma)$.

B. Rayleigh Fading Channel with Discrete Input Alphabets

The use of Gaussian alphabets is not practical and discrete input alphabets are used in reality. The mutual information corresponding to discrete alphabets are obviously strictly smaller than the channel capacity and level off to the logarithm of the constellation sizes at high SNR [12].

Introduced in [5] is an upper bound to mutual information with discrete inputs of size $2^L$

$\tilde{C}_n = \left\{ \begin{array}{ll} \ln (1 + \gamma X_n) , & X_n \leq \tau_X \\ L \cdot \ln 2 , & X_n > \tau_X \end{array} \right.$ (9)

where $\tau_X$ is the point where the mutual information for the Gaussian alphabet equals $L$ bits so that $\tau_X = (2^L - 1)/\gamma$.

This formulation naturally does not carry the exact information of the discrete input but only of its size. It is shown in [5] that this upper bound to mutual information provides a tight lower bound to outage probability. As we also show later in the figures, this upper bound to the mutual information will closely characterize the discrete alphabet effect.

The MGF of $\tilde{C}_n$ is evaluated as

$\theta_{\tilde{C}_n}(s) = P(X_n > \tau_X) 2^{sL} + \int_0^{\tau_X} (1 + \gamma x)^s f_{X_n}(x) dx$

$= e^{-\tau_X} 2^{sL} + \gamma^s e^{\frac{1}{\gamma}} \left( \Gamma(s + \frac{1}{\gamma}) - \Gamma(s + 1, \frac{1}{\gamma} + \tau_X) \right)$ (11)

where $\Gamma(\cdot, \cdot)$ is the (upper) incomplete Gamma function. One can now utilize $\theta_{\tilde{C}_n}(s)$ instead of $\theta_{C_n}(s)$ in the overall MGF $\theta_B(s)$ of the rate. We will provide some numerical results with regard to this subsection in Section IV.

IV. JAMMING AND MULTI-ACCESS INTERFERENCE SCENARIOS

The MGF based approach to obtain outage probabilities can be applied in a simple way to many cases which are of significant interest to communications. Random jamming and multi-access interference will be investigated under Rayleigh fading in this section. Although (5) allows the use of non-identical fading distributions in blocks, identical distributions will be considered for clear illustration of the main ideas. We will choose the parameters used in the simulations arbitrarily to show the applicability of the method in arbitrary settings.

A. Jamming

When a jammer does not have prior knowledge about the time/frequency location of the blocks transmitted for communication, as in fast FH systems, it interferes with the blocks in a random fashion. This gives way to a block fading/interference model where each block undergoes fading and jamming where both events are independent of each other. As is the case in many communication under jamming studies (see [13], [14] among many others), we consider a partial band noise jammer generating white Gaussian noise with a constant PSD $J_0$ in a block. A block is hit by the jammer with probability $P_{hit}$ independently of other hits.

The received SNR in each block is $\gamma = E_s/N_0$ with probability $(1 - P_{hit})$ and $\gamma_j = E_s/(N_0 + J_0)$ with probability $P_{hit}$. In this case, the mutual information of the $n^{th}$ block per channel use is denoted by $C_n^j$ and its moment generating function is evaluated as

$\theta_{C_n^j}(s) = (1 - P_{hit}) \theta_{C_n}(s, \gamma) + P_{hit} \theta_{C_n}(s, \gamma_j)$ (12)

by standard probability arguments.

Outage probabilities under jamming and Rayleigh fading are drawn in Fig. 2 in a scenario of $M = 6$ and a signal-to-jammer power ratio (SJR) $E_s/J_0 = 0$ dB. Two jammer hit probabilities are inspected and the calculations from the analytical expressions are in perfect agreement with the simulation results. An error floor may be observed at high values.
of hit probability as seen for $P_{\text{hit}} = 0.3$. An error floor forms since there is a nonnegligible likelihood that all blocks are under jamming and in that case the outage probability is solely determined by SJR (recall that the jammer power increases as the signal power increases when SJR is fixed) and not SNR.

In Fig. 3, the difference between Gaussian alphabets and discrete input alphabets is observed. The parameter $A$ in (1) is set to 40 to satisfy the accuracy requirement. The dashed curve for $P_{\text{hit}} = 0.1$ corresponds to the outage probability for 8-PSK $(L = 3)$ where mutual information for blocks are determined by the simulation based method in [12]. When it is compared to the Gaussian alphabet outage probability for $P_{\text{hit}} = 0.1$, one can conclude that the alphabet size significantly affects the performance. This parallels the rate-diversity tradeoff argument in [1]. It is clear from these results that upper bounding the mutual information as in (9) is quite effective in capturing the characteristics of block fading channels under jamming with discrete alphabets. As the close match between simulations and the calculations from the analytical expressions are well-established at this point, we will show only results derived through the analytical expressions in the remainder of the script.

B. Multi-access Interference

A very interesting scenario with regard to block fading channels forms when there are many users accessing the same channel simultaneously. For instance, many frequency hopping systems such as Bluetooth are working in the same physical location though within different networks. These networks usually operate without any cooperation with the other networks and this causes multiple access interference in the channel. We will base the idea mainly on frequency hopping communication although the same set of assumptions and procedures may be applied in other cases.

In the multiple access scenario considered here, there are $P$ other users operating in the same region along with the intended transmitter and receiver pair. Each user has a likelihood $P_{\text{int}}$ of causing interference in any block. The users independently and identically cause interference. The interferers also operate under Rayleigh fading where the fading values are all independent. The channel power gain for an interferer $i$ in block $n$ is denoted by $X_{n,i}$. The received signals from the interferers and the transmitter have the same average SNR $\gamma$. No multiple access demodulation/decoding scheme is employed and the interference signals are regarded as additive white Gaussian noise. The blocks of the intended signal and interference have significant overlap so that they will be taken as time-synchronized.

When there are $k$ interferers in a block $n$ and $E[X_{n,i}] = 1$, let us define a random variable $Z = 2 \sum_{i=1}^{k} X_{n,i}$ which has chi-square distribution of $2k$ degrees of freedom with PDF

$$f_Z(z) = \frac{z^{k-1}e^{-z/2}}{2^k(k-1)!}, z \geq 0. \quad (13)$$

Based on the overall received signal-to-noise and interference power ratio (SINR), the mutual information for block $n$ is written as

$$C^\text{int}_n = \ln \left(1 + \frac{\gamma X_n}{\gamma Z/2 + 1} \right). \quad (14)$$

The problem now at hand is to come up with the MGF of $C^\text{int}_n$.

The random variable $B^k = \frac{\gamma X_n}{Z/2 + 1}$ corresponds to the SINR and will be instrumental in finding $\theta_{C^\text{int}_n}$. The cumulative distribution function of $B^k$ can be evaluated as

$$F_{B^k}(b) = P(B^k \leq b) = P(X_n \leq b(Z/2 + 1/\gamma)) \quad (15)$$

$$= E \left[1 - e^{-b(Z/2 + 1/\gamma)}\right] \quad (16)$$

$$= 1 - e^{-b/\gamma} \int_{0}^{\infty} \frac{k-1 \cdot e^{-(b+1)z/2}}{2^k(k-1)!} \, dz \quad (17)$$

$$= 1 - e^{-b/\gamma} \left(\frac{b}{(b+1)^k}\right) \quad (18)$$

for $b > 0$ through which $f_{B^k}(b)$ can be directly evaluated.

When there are $k$ interfering nodes in a block, the moment generating function for the mutual information in that block
Outage probabilities can be evaluated using the distribution of SINR derived above as follows:

\[ \Theta_{\psi_{\text{int}}} (s, k, \gamma) = \int_{0}^{\infty} (1 + b)^s \times \left( \frac{1}{\gamma(b + 1)^k} + \frac{k}{(b + 1)^k + 1} \right) e^{-b/\gamma} dB \]  

where \( \gamma \) is observed when interference over a block. Hence, nearly the same performance is achieved against fading and interferences in comparison to [5], the randomness of numbers of interferers in any block is easily incorporated into the MGF of the overall mutual information as in (20).

The expression for SINR derived in a simple way above can be seen as a special case of the general results for SINR in [15].

As a major advantage of the MGF based evaluation of outage probabilities in comparison to [5], the randomness of the number of interferers in any block is easily incorporated into the MGF of the overall mutual information as in

\[ \theta_{\psi_{\text{int}}} (s, \gamma) = \sum_{P=0}^{P} \binom{P}{k} P_{\text{hit}}^k (1 - P_{\text{hit}})^{P-k} \theta_{\psi_{\text{int}}} (s, k, \gamma). \]  

Outage probabilities can be evaluated by using the MGF \( \theta_{\psi_{\text{int}}} (s, \gamma) \) obtained above and (1).

Outage probabilities for various cases are depicted in Fig. 4. A doubling of the interference probability \( P_{\text{hit}} \) results in a dramatic degradation in performance. Hence, one should decrease the probability of interference as much as possible in system design. What is important is the overall probability of interference over a block. Hence, nearly the same performance is observed when \( P \times P_{\text{hit}} \) is kept constant.

The discrete input alphabet consideration should also be accounted for in the multiple access case due to the reasons explained before. An upper bound to mutual information is defined for \( k \) interferers present in a block as

\[ \tilde{C}_{\text{int}} = \begin{cases} \ln (1 + B^k) & , B^k \leq \tau_B \\ L \cdot \ln 2 & , B^k > \tau_B \end{cases} \]  

where \( \tau_B = 2^L - 1 \). One can evaluate the MGF of \( \tilde{C}_{\text{int}} \) as seen in (23).

\[ R_0=2 \text{ bpu}, M=20 \] 

\[ R_0=0.5 \text{ bpu}, M=8, P=2, E/E_{0}=0 \text{ dB} \]
\[ \theta_{\text{CM}}(s, k, \gamma) = (1 - F_{B^s}(\tau_B))2^{sL} + \int_0^{\tau_B} (1 + b)^s \left( \frac{1}{\gamma(b + 1)^k} + \frac{k}{(b + 1)^{k+1}} \right) e^{-\frac{\tau}{\gamma}db} \]
\[ = \frac{e^{-\frac{\tau}{\gamma}2^{sL}} + \gamma^{s-k}e^{-\frac{\tau}{\gamma}}}{{(1 + \tau_B)^k} + \gamma^{s-k}e^{-\frac{\tau}{\gamma}}} \left( \Gamma(s - k + 1, \frac{1}{\gamma}) - \Gamma(s - k + 1, \frac{1 + \tau_B}{\gamma}) \right) + k \left( \Gamma(s - k, \frac{1}{\gamma}) - \Gamma(s - k, \frac{1 + \tau_B}{\gamma}) \right) \]

Fig. 6. Minimum number of FH hops to meet \( P_{\text{out}} = 10^{-3} \) vs message length.

VI. CONCLUSION

We developed a method based on MGF of a single block’s mutual information that facilitates an efficient evaluation of outage probability for block fading channels. We showed the flexibility of the method by applying it to various fading distributions and including the effects of jamming, multi-access interference, and discrete input alphabets in a straightforward and simple manner. The method works perfectly in all the scenarios considered here and can be extended to other distributions and MIMO transmission as long as a closed form moment generating function can be written for a single block’s mutual information.

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REFERENCES