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CHARACTER SHEAVES AND GENERALIZED SPRINGER CORRESPONDENCE

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Abstract. Let G be a connected reductive algebraic group over an algebraic closure of a finite field of characteristic p. Under the assumption that p is good for G, we prove that for any character sheaf A on G, there exists a unipotent class C_A canonically attached to A, such that A has non-zero restriction on C_A, and any unipotent class C in G such that A has non-zero restriction on C has dimension strictly smaller than that of C_A. When G is of classical type, and the family of representations of the Weyl group of G associated with A is cuspidal, we prove that C is contained in the Zariski closure of C_A.

1. Introduction

Let $\overline{\mathbb{F}}_q$ be an algebraic closure of a finite field $\mathbb{F}_q$ of q elements. Let G be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$. Let T be a maximal torus in G, T^* be a maximal torus in the Langlands dual G^* of G which is dual to T, and let W denote the Weyl group of G with respect to T, that we identify with the Weyl group of G^* with respect to T^*. Let $s \in T^*$. Lusztig has defined a canonical surjective map from the set $\hat{G}$ of characters sheaves on G to the set of W–orbits on T^*. We will denote by $\hat{G}_s$ the set of character sheaves in the fiber over (s) of this map. We set

$$W_s := \{w \in W \mid w(s) = s\}. \tag{1.1}$$

Lusztig has also defined a map form $\hat{G}_s$ to the set of two-sided cells of W_s. For $\mathbf{c}$ a given two-sided cell of W_s, we will denote by $\hat{G}_{s,\mathbf{c}}$ the set of character sheaves in the fiber over $\mathbf{c}$ of this map. Lusztig has described a canonical construction, that we will recall in (1.2), by which we can associate a well-defined unipotent class $C_{s,\mathbf{c}}$ with each pair $(s, \mathbf{c})$, where $s \in T^*$ and $\mathbf{c}$ is a two-sided cell in W_s, and has proved the following theorem, under the assumption that the characteristic p of $\mathbb{F}_q$ is large enough. One of our purposes is to replace that assumption by the weaker assumption "p is good for G".

Theorem 1.1. *For any $s \in T^*$ and any two-sided cell $\mathbf{c}$ in W_s, the following hold:*

(a) *If $A \in \hat{G}_{s,\mathbf{c}}$ and if C is a unipotent class in G such that the restriction of A to the G–conjugacy class of u is non-zero, for some $u \in C$, then*

$$\dim C \leq \dim C_{s,\mathbf{c}}, \quad \textit{with equality only for } C = C_{s,\mathbf{c}}; \tag{1.2}$$

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(b) *There exists some* $A \in \hat{G}_{s,\mathbf{c}}$ *and some* $u \in C_{s,\mathbf{c}}$ *such that the restriction of* A *to the* G*–conjugacy class of* u *is non-zero.*

When G is of exceptional type, and, in one case of type C, we will derive our result from Lusztig's one. For the other cases, we will prove it, under the assumption that the element s is isolated in G^*, using direct computations in the spirit of [15].

We need first to recall some facts about the Generalized Springer Correspondence. Let $\mathcal{N}_G$ be the set of pairs $(C, \mathcal{E})$ such that C is a unipotent conjugacy class in G, and $\mathcal{E}$ is a G–equivariant irreducible local system on C, given up to isomorphism. Let C be a unipotent class in G, and fix $u \in C$. Then the isomorphism classes of irreducible G–equivariant $\overline{\mathbb{Q}}_\ell$–local systems on C are in bijection with the isomorphism classes of irreducible representations of the group

$$A_G(u) := A(u) := \mathrm{C}_G(u)/\mathrm{C}^0_G(u) \tag{1.3}$$

of components of the centralizer $\mathrm{C}_G(u)$ in G of u. Thus, we can identify a pair $(C, \mathcal{E}) \in \mathcal{N}_G$ with the G–conjugacy class $[u, \rho]_G$ of a pair (u, ρ) where u is a unipotent element in G and ρ is an irreducible representation of the group $A_G(u)$. The Springer Correspondence is an injective map from the set of isomorphism classes of irreducible representations of W into $\mathcal{N}_G$. Not all pairs $(C, \mathcal{E})$ in $\mathcal{N}_G$ occur in that correspondence. The Generalized Springer Correspondence, due to Lusztig, produces the missing pairs.

Let P be a parabolic subgroup of G, with unipotent radical U, and let L be a Levi subgroup of P. Let u be a unipotent element in G and u_L be a unipotent element in L. The group $\mathrm{C}_G(u) \times \mathrm{C}_L(u_L)U$ acts on the variety

$$\mathcal{Y}_{u,u_L} := \{x \in G \,:\, x^{-1}ux \in u_L U\} \tag{1.4}$$

by $(g, l) \cdot x := gxl^{-1}$, ($g \in \mathrm{C}_G(u)$, $l \in \mathrm{C}_L(u_L)U$, $x \in \mathcal{Y}_{u,u_L}$). Let

$$d_{u,u_L} := \frac{1}{2}\left(\dim \mathrm{C}_G(u) + \dim(\mathrm{C}_L(u_L))\right) + \dim U. \tag{1.5}$$

It is known (see [13, §1]) that $\dim \mathcal{Y}_{u,u_L} \le d_{u,u_L}$. The group $A_G(u) \times A_L(u_L)$ acts on the set of all irreducible components of $\mathcal{Y}_{u,u_L}$ of dimension d_{u,u_L}; the corresponding permutation representation will be denoted by ε_{u,u_L}. A pair (u, ρ) (or the corresponding pair $(C, \mathcal{E}) \in \mathcal{N}_G$), where ρ is an irreducible representation of the group $A_G(u)$, is said to be *cuspidal* in G if it does not appear in the permutation representation ε_{u,u_L} for any P, u_L as above, with $P \neq G$. Very few pairs (u, ρ) are cuspidal (for the list of cuspidal representations, see [13, Introduction and §15]). We will say that G is *cuspidal* if there exist a cuspidal pair (u, ρ) in G. There exist characters sheaves on G with non-zero restriction of the variety G_{unip} of unipotent elements in G if and only if G is cuspidal.

Let $\mathfrak{B}(G)$ denote the set of G–conjugacy classes of triples (L, u_L, ρ_L) where L is the Levi subgroup of some parabolic subgroup of G, u_L is a unipotent element in L and ρ_L is an irreducible representation of the group $A_L(u_L)$ such that the pair (u_L, ρ_L) is cuspidal in L. We denote by $\mathfrak{b} = [L, u_L, \rho_L]_G$ the G–conjugacy class of (L, u_L, ρ_L). Given $[u, \rho]_G \in \mathcal{N}_G$, there exists $\mathfrak{b} = [L, u_L, \rho_L]_G \in \mathfrak{B}(G)$ such that

$$\langle \rho \otimes \tilde{\rho}_L, \varepsilon_{u,u_L} \rangle_{A_G(u) \times A_L(u_L)} \neq 0. \tag{1.6}$$

(Here $\tilde{\rho}_L$ denotes the dual of ρ_L.) We will denote by $\mathcal{N}^{\mathfrak{b}}_G$ the set of $[u, \rho]_G \in \mathcal{N}_G$ which satisfied (1.6) for a given $\mathfrak{b} = [L, u_L, \rho_L]_G \in \mathfrak{B}(G)$. The sets $\mathcal{N}^{\mathfrak{b}}_G$ are called *blocks.* There is a bijective map

$$b^G\colon \mathcal{N}_G \longrightarrow \bigsqcup_{\mathfrak{b}\in\mathfrak{B}(G)} \mathcal{N}_G^{\mathfrak{b}}. \tag{1.7}$$

We set $W_L^G := \mathrm{N}_G(L)/L$, and we denote by $\mathrm{Irr}\left(W_L^G\right)$ the set of all the isomorphism classes of irreducible representations of the group W_L^G. For a fixed $\mathfrak{b} = [L, u_L, \rho_L]_G \in \mathfrak{B}(G)$, there is a bijection

$$\tau_{\mathfrak{b}}^G\colon \mathcal{N}_G^{\mathfrak{b}} \longrightarrow \mathrm{Irr}\left(W_L^G\right). \tag{1.8}$$

The *Generalized Springer Correspondence* is the bijection

$$\tau^G : \mathcal{N}_G \longrightarrow \bigsqcup_{\mathfrak{b}\in\mathfrak{B}(G)} \mathrm{Irr}\left(W_L^G\right), \tag{1.9}$$

obtained by composing the maps b^G and $\tau_{\mathfrak{b}}^G$. Let ν^G denote the inverse map of τ^G, and let $\nu_{\mathfrak{b}}^G$ denote the inverse map of $\tau_{\mathfrak{b}}^G$.

If $L^* \supset T^*$ is a dual pair to $L \supset T$, we will set

$$W_{L,s}^G = W_{L,s} := \mathrm{N}_{\mathrm{C}_{G^*(s)}}(\mathrm{C}^0_{L^*}(s))/\mathrm{C}^0_{L^*}(s). \tag{1.10}$$

For groups of classical type, Theorem 1.1 will be a consequence of the following result:

Theorem 1.2. *Let G be one of the groups PSp_{2n}, SO_{2n+1}, SO_{2n}. We assume that p is odd.*

Then there exists a natural order $\underset{\mathcal{N}_G}{\leq}$ on the set $\mathcal{N}_G$ such that the following hold:

(a) *if $(C', \mathcal{E}') \underset{\mathcal{N}_G}{\leq} (C, \mathcal{E})$, for $(C', \mathcal{E}')$, $(C, \mathcal{E})$ in $\mathcal{N}_G$, then the class C' is contained in the Zariski closure of the class C;*

(b) *For any cuspidal Levi subgroup $L \supset T$ of a parabolic subgroup of G (such that $L \neq T$ if $G = \mathrm{PSp}_{2n}$), for any an isolated element $s \in T^*$, and any irreducible representation E' of the group $W_{L,s}^G$, there exists an irreductible representation E of the group W_L^G occuring with multiplicity one in the induced representation $\mathrm{Ind}_{W_{L,s}^G}^{W_L^G}(E')$, such that $\nu^G(E_1) \underset{\mathcal{N}_G}{\leq} \nu^G(E)$, for any irreducible representation E_1 of W_L^G which occurs in $\mathrm{Ind}_{W_{L,s}^G}^{W_L^G}(E')$.*

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1.1. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of a finite field $\mathbb{F}_q$ of q elements. Let G be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$. Let W_s^0 be the Weyl group of the connected component $\mathrm{C}^0_{G^*}(s)$ of the centralizer $\mathrm{C}_{G^*}(s)$ of s in G^*. It is a normal subgroup of the group W_s defined in (1.1). Let Φ_s denote the root system of W_s^0, and let Φ_s^+ be the set of positive roots of Φ_s. Let Π_s be the set of simple roots in Φ_s. We have

$$W_s = W_s^0 \rtimes A_s, \quad \text{where } A_s := \left\{ w \in W_s \mid w(\Phi_s^+) \subset \Phi_s^+ \right\}. \tag{1.11}$$

Let $l\colon W_s \to \mathbb{N}$ be the function defined by

$$l(w) := \left| \left\{ \alpha \in \Phi_s^+ \mid w(\alpha) \in \Phi_s^+ \right\} \right|. \tag{1.12}$$

Then l extends the length function of the Coxeter group W_s^0.

Let V be the rational vector space spanned the simple roots in Φ_s. Lusztig has associated (see [8], [12, (4.1.2)]) with any any irreducible representation E' of W_s^0 an integer $b_{E'} \geq 0$ by the requirement that $b_{E'}$ is the smallest integer i such that E' occurs in the i–th symmetric power of V.

Let $\overline{\mathbb{F}}_{q'}$ be an algebraic closure of a finite field $\mathbb{F}_{q'}$ with q' elements and let $G_{s,\mathrm{ad}}$ be the adjoint Chevalley group over with root system Φ_s. Let B_s be a Borel subgroup of $G_{s,\mathrm{ad}}$ defined over $\mathbb{F}_{q'}$. Let $G_{s,\mathrm{ad}}(q')$ and $B_s(q')$ denote the groups of $\mathbb{F}_{q'}$–rational points of $G_{s,\mathrm{ad}}$ and B_s respectively.

There is a one-to-one correspondence $E' \overset{h_{q'}}{\longleftrightarrow} E'_{q'}$ between the set of isomorphism classes of irreducible representations of the group W_s^0 and the set of isomorphism classes of irreducible representations of the group $G_{s,\mathrm{ad}}$ occuring in the induced representation $\mathrm{Ind}_{B_s(q')}^{G_{s,\mathrm{ad}}(q')}(1)$. The dimension of $E'_{q'}$ is independent of the choice of the correspondence $h_{q'}$, and equals $D_{E'}(q)$ where $D_{E'}(X)$ is a well-defined polynomial with rational coefficients, independent of q'. The polynomial $D_{E'}(X)$ is called the "generic degree" of the representation E'. Lusztig has associated in [8], [12, (4.1.1)], with any irreducible representation E' of W_s^0, an integer $a_{E'} \geq 0$ and an integer $f_{E'} > 0$ by the requirement that

$$D_{E'}(X) = f_{E'}^{-1} X^{a_{E'}} + \text{higher powers of X}. \tag{1.13}$$

As observed in [8], we have always $a_{E'} \leq b_{E'}$. This justifies the following definition (see [8]): an irreducible representation E' of W_s^0 is said to be *special* if $a_{E'} = b_{E'}$.

Let X be an indeterminate, and let $S_s := \{w_\alpha \mid \alpha \in \Pi_s\}$ denote the set of simple reflexions in W_s^0 (defined by Φ_s^+). Let $\mathcal{H}(W_s, X)$ denote the Hecke algebra (over $\mathcal{A} := \mathbb{Z}[X^{\frac12}, X^{-\frac12}]$) corresponding to W_s; it is a free $\mathcal{A}$–module with basis $(T_w)_{w \in W_s}$. The multiplication is characterized by

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{si } l(ww') = l(w) + l(w'), \\ (T_{w_\alpha} + 1)(T_{w_\alpha} - X) = 0, & \text{if } w_\alpha \in S_s. \end{cases} \tag{1.14}$$

Let $\leq$ denote the usual partial order on W_s^0. Kazhdan and Lusztig [5] have shown that, for every $w \in W_s^0$, and every $y \leq w$ in W_s^0, there exists a well-defined polynomial $P_{y,w} \in \mathbb{Z}[X]$ such that

(a) $\deg P_{y,w} \leq \frac12 (l(w) - l(y) - 1)$, if $y < w$;
(b) $P_{w,w} = 1$, for any $w \in W_s^0$;

(c) $\sum_{y\leq w}(-1)^{l(y)}\,X^{l(w)-l(y)}\,P_{y,w}(X^{-1})\,T_y \;=\; \sum_{y\leq w}(-1)^{l(y)}\,X^{l(y)}\,P_{y,w}(X)\,T_{y^{-1}}^{-1}$ in $\mathcal{H}(W_s,X)$.

For any y, w in W_s^0, we will write $y \prec w$ if the following conditions hold:

(a) $y < w$;
(b) $l(w) - l(y)$ is odd;
(c) $P_{y,w} = m(y,w)\,X^{\frac{1}{2}(l(w)-l(y)-1)} +$ lower powers of X where $m(y,w)$ is a non-zero integer.

Two elements w, w' in W_s^0 are said to be *linked* if $w \prec w'$ or $w' \prec w$. For any $w \in W_s^0$, we set

$$\mathcal{L}(w) := \{w_\alpha \mid \alpha \in \Pi_s, w_\alpha w < w\}\,, \quad \mathcal{R}(w) := \{w_\alpha \mid \alpha \in \Pi_s, w w_\alpha < w\}\,. \tag{1.15}$$

Preorders $\leq_L$, $\leq_R$ et $\leq_{LR}$ are defined on W_s^0 as follows: for $w \in W_s^0$, $w' \in W_s^0$, we write $w \leq_L w'$ if there exists a sequence $w = w_0, \ldots, w_n = w'$ of elements of W_s^0, such that, for any $i \in \{1,2,\ldots,n\}$, the elements w_{i-1} and w_i are linked and $\mathcal{L}(w_{i-1}) \not\subset \mathcal{L}(w_i)$ (then [5, 2.4] $\mathcal{R}(w) \subset \mathcal{R}(w')$); we will write $w \leq_R w'$ if $w^{-1} \leq_L w'^{-1}$, and we will write $w \leq_{LR} w'$ if there exists a sequence $w = w_0$, $\ldots$, $w_m = w'$ of elements of W_s^0, such that, for any $i \in \{1,2,\ldots,m\}$, we have $w_{i-1} \leq_L w_i$ or $w_{i-1} \leq_R w_i$. Let $\simeq_{LR}$ denote the equivalence relation associated to the preorder $\leq_{LR}$; the corresponding equivalence classes are called *two-sided cells* of W_s^0.

The notion of two-sided cell has been extended to the group W_s in [14, (16.3)]: a subset of W_s is said to be a two-sided cell if it is of the form $A_s\mathbf{c}A_s$ for some two-sided cell $\mathbf{c}$ in W_s^0.

Let ℓ be a prime number, distinct of the characteristic p of $\mathbb{F}_q$. Let $\mathcal{H}_s$ be the free $\mathcal{A}$–module with basis $e_{\mathbf{w}}$ indexed by the sequences $\mathbf{w} := (w_{\alpha_1}, w_{\alpha_2}, \ldots, w_{\alpha_n})$ in $S_s \cup \{1\}$ $(n \geq 1)$, such that $w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_n} \in W_s$. To each isomorphism class of irreducible $\overline{\mathbb{Q}}_\ell$–representations E' of W_s one can associate canonically an $\mathcal{H}_s \otimes_{\mathcal{A}} \overline{\mathbb{Q}}_\ell[X^{\frac{1}{2}}, X^{-\frac{1}{2}}]$–module E'_X as in [10, 1.1, 1.2], [12, 3.3]. The corresponding modules $E'_X \otimes \overline{\mathbb{Q}}_\ell(X^{\frac{1}{2}})$ form a complete set of irreducible representations of $\mathcal{H}_s \otimes_{\mathcal{A}} \overline{\mathbb{Q}}_\ell(X^{\frac{1}{2}})$. We have $\mathrm{Trace}(T_w, E'_X) \in \zeta \cdot \mathbb{Z}[X^{\frac{1}{2}}]$, where ζ is a root of unity of order dividing $|A_s|$, which depends only on E' and on the W_s–coset of $w \in W_s$, see [14, (12.9.3)]. For any irreducible representation E' of W_s, any $h \in \mathcal{H}_s$ and any integer i, we define $\mathrm{Trace}\,(h, E'_X; i) \in \overline{\mathbb{Q}}_\ell$ by

$$\mathrm{Trace}\,(h, E'_X) = \sum_i \mathrm{Trace}\,(h, E'_X; i)\; X^{\frac{i}{2}}. \tag{1.16}$$

Under the specialization $\mathcal{A} \to \overline{\mathbb{Q}}_\ell$, $X^{\frac{1}{2}} \mapsto q^{\frac{1}{2}}$, where q is a fixed square root of q in $\overline{\mathbb{Q}}_\ell$, E'_X becomes an $\mathcal{H}_s \otimes_{\mathcal{A}} \overline{\mathbb{Q}}_\ell$–module E'_q and the E'_q form again a complete set of irreducible representations of $\mathcal{H}_s \otimes_{\mathcal{A}} \overline{\mathbb{Q}}_\ell$. (Note that E'_q coincide with the previous definition in case $W_s = W_s^0$.) For any irreducible representation E' of W_s and any element w of W_s, we define $c_{w,E'} \in \overline{\mathbb{Q}}_\ell$, as in [14, (16.2.1], by setting:

$$c_{w,E'} := (-1)^{l(w)}\,\mathrm{Trace}\left(X^{-\frac{l(w)}{2}}T_w, E'_X; -a_{E'}\right). \tag{1.17}$$

Hence it is an integer times a root of unity. It is known that, given E', there is a unique two-sided cell $\mathbf{c}$ of W_s such that $c_{w,E'} \neq 0$ for some $w \in \mathbf{c}$. We then say that E' *belongs* to $\mathbf{c}$. Given $\mathbf{c}$, there is a unique irreducible representation of W_s

(up to isomorphism) belonging to $\mathbf{c}$ such that $c_{w,E'}$ is an integer ≥ 0 for $w \in \mathbf{c}$ and it is > 0 for some $w \in \mathbf{c}$. We denote it as $E'(\mathbf{c})$. The representation $E'(\mathbf{c})$ is called the *special representation* corresponding to $\mathbf{c}$, see [17, (10.4)].

The intersection $\mathbf{c} \cap W_s^0$ is a union of two-sided cells of W_s^0, which form an orbit under the action of W_s^0, and the restriction of $E'(\mathbf{c})$ to W_s^0 is a direct sum of the special representations of W_s^0 corresponding to these two-sided cells of W_s^0.

Let E' be an irreducible representation of W_s, and let $\tilde{E}'$ be an irreducible representation of W_s^0 which occurs in the restriction of E' to W_s^0. We will set $a_{E'} := a_{\tilde{E}'}$. It does not depend on the choice of the representation $\tilde{E}'$, see also [14, (16.5)]. The function $E' \mapsto a_{E'}$ is constant on the set of irreducible representations E' of W_s which belong to $\mathbf{c}$, see [12, (4.14.1) and (5.25)]. For any such representation E', we will set $a_{s,\mathbf{c}} := a_{E'}$.

1.2. For each unipotent element u of G, we will denote by $\mathcal{B}_u^G = \mathcal{B}_u$ the variety of Borel subgroups of G containing u. The group $A(u)$ acts naturally by permutation on the set of irreducible components of the variety $\mathcal{B}_u$. (According to Spaltenstein, all irreducible components of $\mathcal{B}_u$ have the same dimension, say $d(u)$.) Hence $A(u)$ acts naturally on $\mathrm{H}^{2d(u)}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$. Under the assumption that p is large enough, Springer [24] has defined a natural representation of W on $\mathrm{H}^{2d(u)}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$ which commutes with the action of $A(u)$. A definition valid in arbitrary characteristic has been given by Lusztig in [9]. For any irreducible representation (ρ, V_ρ) of $A(u)$, the vector space

$$E_{u,\rho}^G \;=\; E_{u,\rho} \;:=\; \mathrm{Hom}_{A(u)}(V_\rho, \mathrm{H}^{2d(u)}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)) \qquad (1.18)$$

is a W–module.

The main properties of Springer representation are the following:

(1) $E_{u,\rho}$ is either $\{0\}$ or an irreducible W–module;
(2) $E_{u,\rho} \simeq E_{u',\rho'}$ if and only if (u,ρ), (u',ρ') are conjugate under G;
(3) all irreducible representations of W arise in this way;
(4) $E_{u,1}$ is always non-zero. (Hence it is an irreducible W–module.)

We will denote by ν_T^G the injective map from $\mathrm{Irr}(W)$ to $\mathcal{N}_G$, given by the *Springer Correspondence*, that is the map which associates to the isomorphism class of $E_{u,\rho}$ the pair $\mathfrak{n} = [u,\rho]_G$. The map ν_T^G is well-defined because of the above properties.

We set $G_s := \mathrm{C}^0_{G^*}(s)$. Any special irreducible representation of W_s^0 is of the form $E_{u,1}^{G_s}$ for a well-defined unipotent element u (up to conjugacy) in G_s. A unipotent element u (or its class in G_s) is said to be *special* if $E_{u,1}^{G_s}$ is a special representation of W_s^0, or equivalently (see [12, (13.11)]), if $a_{E_{u,1}^{G_s}} = \dim \mathcal{B}_u^{G_s}$.

Let $\mathbf{c}$ be a two-sided cell in W_s. Recall that $E'(\mathbf{c})$ denotes the unique special representation of W_s which belongs to $\mathbf{c}$. Let $\tilde{E}'(\mathbf{c})$ be an irreducible component of the restriction of $E'(\mathbf{c})$ to W_s^0; this is a special representation of W_s^0. Hence the induced representation $\mathrm{Ind}_{W_s^0}^W(\tilde{E}'(\mathbf{c}))$ contains a unique irreducible W–submodule $E(s,\mathbf{c})$ such that $b_{E(s,\mathbf{c})} = b_{\tilde{E}'(\mathbf{c})}$, see [12, (13.3)], [8], [11]. It is easy to check that $E(s,\mathbf{c})$ is independent of the choice of $\tilde{E}'(\mathbf{c})$, see also [17, (10.5)].

Then there is a well-defined unipotent element $\hat{u}$ in G (up to conjugacy) such that $E(s,\mathbf{c})$ is the Springer representation $E_{\hat{u},1}^G$, [12, (13.3)]. Let $C_{s,\mathbf{c}}$ denote the G–conjugacy class of $\hat{u}$. We set $d(C_{s,\mathbf{c}}) := d(\hat{u})$ (the dimension of the variety of

Borel subgroups of G containing $\hat{u}$). The element $\hat{u}$ is special, see [12, (13.3)]. Hence we have

$$a_{s,\mathbf{c}} = d(C_{s,\mathbf{c}}). \tag{1.19}$$

2. Character sheaves

Let $\mathcal{D}G$ be the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on G, and let $\mathcal{M}G$ be the full subcategory of $\mathcal{D}G$ consisting of perverse sheaves. We denote by $\mathcal{S}(T)$ the set of isomorphism classes of tame local systems $\mathcal{L}$ on T such that $\operatorname{rank}\mathcal{L} = 1$. Take a local system $\mathcal{L}$ in $\mathcal{S}(T)$ such that $w^*\mathcal{L} \simeq \mathcal{L}$ for some $w \in W$. Then one can construct a complex $K_w^{\mathcal{L}} \in \mathcal{D}G$ as in [14, 12.1]. For each $\mathcal{L} \in \mathcal{S}(T)$, we denote by $\hat{G}_{\mathcal{L}}$ the set of isomorphism classes of irreducible perverse sheaves A on G such that A is a constituent of the i-th cohomology perverse sheaf ${}^pH^i(K_w^{\mathcal{L}})$ of $K_w^{\mathcal{L}}$ for some w and some $i \in \mathbb{Z}$. The set $\hat{G}$ of character sheaves of G is defined as the union of $\hat{G}_{\mathcal{L}}$, where $\mathcal{L}$ runs over all the elements in $\mathcal{S}(T)$.

Assume chosen, once and for all, an isomorphism of abstract groups from the set of isomorphisms classes of tame local systems of rank one on $\overline{\mathbb{F}}_q^{\times}$ onto $\overline{\mathbb{F}}_q^{\times}$, this gives rise to an isomorphism $\mathcal{S}(T)\tilde{\to}T^*$ (see [16, Section 1.6]). Hence any tame local system $\mathcal{L}$ of rank one can be interpreted as a semisimple element s of G^*. If $\mathcal{L}$ and s correspond to each other, we will set $\hat{G}_s := \hat{G}_{\mathcal{L}}$. The subset $\hat{G}_s$ of $\hat{G}$ will be called the *series of character sheaves defined by s*.

We will assume from now that p of is good for G. We denote by G_{unip} the unipotent variety of G. By [14, Corollary 11.4], there is a canonical surjective map from the set $\hat{G}$ to the W–orbits on T^*. Let $s \in T^*$, then $\hat{G}_s$ is the set of character sheaves in the fiber over $(s)_{G^*}$ of this map.

By [14, Corollary 16.7], there exists a well-defined surjective map from $\hat{G}_s$ to the set of two-sided cells in W_s. For $\mathbf{c}$ a two-sided cell in W_s, we will denote by $\hat{G}_{s,\mathbf{c}}$ the set of character sheaves in the fiber over $\mathbf{c}$ of this map.

Proposition 2.1. *Let $s \in T^*$ and let $\mathbf{c}$ be a two-sided cell in W_s, such that, for any $A \in \hat{G}_{s,\mathbf{c}}$, and any $u \in G_{\mathrm{unip}}$ satisfying $A_{|\{u\}} \neq 0$, the element u lies in $C_{s,\mathbf{c}}$ or in a unipotent class of dimension strictly smaller than that of $C_{s,\mathbf{c}}$.*

Then there exists some $A \in \hat{G}_{s,\mathbf{c}}$ and some $u \in C_{s,\mathbf{c}}$ such that $A_{|\{u\}} \neq 0$.

Proof. Let us assume that the restriction of each character sheaf $A \in \hat{G}_{s,\mathbf{c}}$ to $C_{s,\mathbf{c}}$ is zero. We choose a Frobenius map $F\colon G \to G$ as in (2.2). Then we can similarly associate to the pair $(s,\mathbf{c})$ a set of irreducible representations of G^F, as in [12]. Let π such a representation. By [12] and [14], we can express the character of π as a linear combination of characteristic functions of character sheaves in $\hat{G}_{s,\mathbf{c}}$. The assumption that these character sheaves have zero restriction to $C_{s,\mathbf{c}}$ implies that the restriction of the character χ_π of π is zero too. Using that expression of the character χ_π and the fact that if $A_{|\{u\}} \neq 0$ for some $A \in \hat{G}_{s,\mathbf{c}}$, some unipotent class C in G, and some $u \in C$, then $\dim C \leq \dim C_{s,\mathbf{c}}$, we obtain that χ_π is zero on all unipotent classes of dimension $\geq \dim C_{s,\mathbf{c}}$. But we know, by [4], that there exists a unique F–stable unipotent class C_π in G of maximal dimension such that

$$\sum_{i=1}^{r}[G^F : \mathrm{C}_G(u_i)^F]\,\chi_\pi(u_i) \,\neq\, 0,$$

where $u_1, u_2, \ldots, u_r \in G^F$ are representatives for the G^F–conjugacy classes contained in C^F and $\mathrm{C}_G(u_i)$ denotes the centralizer in G of u_i. It follows that $\dim C_\pi < \dim C_{s,\mathbf{c}}$. By [4, Th. 1.4], the p–part of the dimension of π equals $q^{d(C_\pi)}$, where $d(C_\pi)$ denotes the dimension of the variety of Borel subgroups which contain a fixed element in the class C_π. Using the equality $\dim C_{s,\mathbf{c}} = \dim G - \mathrm{rank}\,(G) - 2d(C_{s,\mathbf{c}})$ (see [23]), we get $d(C_\pi) > d(C_{s,\mathbf{c}})$. On the other hand, the p–part of the dimension of π also equals $q^{a_{s,\mathbf{c}}}$, see [12, (4.26.3)]. It follows that $a_{s,\mathbf{c}} > d(C_{s,\mathbf{c}})$, which contradicts the equality (1.19). Hence there exists a character sheaf A in $\hat{G}_{s,\mathbf{c}}$ such that $A_{|\{u\}} \neq 0$ for some $u \in C_{s,\mathbf{c}}$. □

2.1. Lusztig has proved in [17, (10.9)] that, if p is large enough, then the following property always holds.

Property 2.2. *For any $s \in T^*$, any two-sided cell $\mathbf{c}$ in W_s, and any $A \in \hat{G}_{s,\mathbf{c}}$, if $C \neq C_{s,\mathbf{c}}$ is a unipotent class in G such that $A_{|\{u\}} \neq 0$ for some $u \in C$, then $\dim C < \dim C_{s,\mathbf{c}}$.*

Lemma 2.3. *Let p be a prime number. Assume that all reductive connected algebraic groups defined over a finite field of characteristic p, which have connected center and are simple modulo their center, satisfy Property 2.2. Then all reductive connected algebraic groups defined over a finite field of characteristic p satisfy Property 2.2.*

Proof. Let G be any reductive connected algebraic group defined over $\mathbb{F}_q$, where q is a power of p. There exists (see [3]) a connected reductive group $\tilde{G}$ defined over $\mathbb{F}_q$ satisfying the following properties:

(a) G is a subgroup of $\tilde{G}$;
(b) G contains the derived group of $\tilde{G}$;
(c) the center $\tilde{Z}$ of $\tilde{G}$ is connected.

We will denote by $\iota\colon G \hookrightarrow \tilde{G}$ the canonical injection. Such a map is called a regular embedding. It follows from these properties that $\mathrm{Z} = \tilde{Z} \cap G$, where Z denote the center of G, and that $\tilde{G} = G \cdot \tilde{Z}$. The map ι gives a bijection between the set of unipotent classes in G and the set of unipotent classes in $\tilde{G}$, and the truth of Property 2.2 for G is equivalent with its truth for $\tilde{G}$.

Hence we may assume that the center of G is connected. We can find a reductive group G' over $\mathbb{F}_q$ and a surjective homorphism $\phi\colon G' \to G$ of algebraic groups over $\mathbb{F}_q$ such that the center of G' is connected and the derived subgroup of G' is semisimple and simply-connected. The map ϕ induces a bijection between the unipotent classes of G' and G, and the truth of Property 2.2 for G is equivalent with its truth for G'.

Hence we may now assume that the derived group G_{der} of G is simply-connected. Let us write $G_{\mathrm{der}} = \mathrm{R}_{t_1}(G_1) \times \mathrm{R}_{t_2}(G_2) \times \cdots \times \mathrm{R}_{t_n}(G_n)$, where each G_i is a closed simply-connected subgroup and R_t denotes the restriction of scalars from $\mathbb{F}_{q^t}$ to $\mathbb{F}_q$ (for some $t \geq 1$). We can embedd each G_i regularly (over $\mathbb{F}_{q^{t_i}}$) into a connected reductive group $\tilde{G}_i$ with connected center and which is simple modulo its center. Let $G'' := \mathrm{R}_{t_1}(\tilde{G}_1) \times \mathrm{R}_{t_2}(\tilde{G}_2) \times \cdots \times \mathrm{R}_{t_n}(\tilde{G}_n)$. Then we also have a regular embedding $G_{\mathrm{der}} \hookrightarrow G''$ (over $\mathbb{F}_q$). Finally, there exists a connected reductive group G''' with connected center and defined over $\mathbb{F}_q$ and there exits regular embeddings $G \hookrightarrow G'''$, $G'' \hookrightarrow G'''$ (over $\mathbb{F}_q$) which are compatible with the regular embedding $G_{\mathrm{der}} \hookrightarrow G''$.

Then the truth of Property 2.2 for G is equivalent with its truth for G''', which itself is equivalent with its truth for G''. Now G'' has a decomposition into a direct product of various factors of the form $\mathrm{R}_{t_i}(\tilde{G}_i)$, and truth of Property 2.2 for each factor $\tilde{G}_i$ implie its truth for G''. □

2.2. We will assume from now that the center of G is connected. We choose an $\mathbb{F}_q$–rational structure on G, with corresponding Frobenius map $F\colon G \to G$, such that T is F–stable and split. Then G^* is also defined over $\mathbb{F}_q$, and T^* is F–stable. Moreover, replacing q by q^t for some integer $t \geq 1$ if necessary, we may assume that $s \in T^{*F}$ and that F acts trivially on W_s^*. Then the set $\hat{G}_s$ is F–stable. Since it is finite, we can choose t in such a way that the permutation of $\hat{G}_s$ induced by F is trivial. Thus, all the character sheaves $A \in \hat{G}_s$ are F–stable (see [14, (13.8.0)]). For each irreducible representation E' of W_s^* let $R_s(E')$ denote the corresponding almost character, defined in [12, (3.7)] as a certain rational linear combination of Deligne-Lusztig virtual characters of G^F. (Note that it only depends on E' and not on an extension of E' since F acts trivially on W_s.) The restriction of a Deligne-Lusztig virtual character to the set G^F_{unip} of unipotent elements in G^F is given in terms of the corresponding Green function. For $w \in W$, let $T_w \subset G$ denote an F–stable maximal torus obtained by twisting the torus T with w, and let $Q^G_{T_w}\colon G^F_{\mathrm{unip}} \to \mathbb{Z}$ be the corresponding Green function, defined by $Q^G_{T_w} := R^G_{T_w}(\theta)(1)$. For any irreducible representation E of W, we set

$$Q_E := \frac{1}{|W|} \sum_{w \in W} \mathrm{Trace}(w, E)\, Q^G_{T_w}. \tag{2.1}$$

Let $\mathrm{Irr}(W)$ denote the set of isomorphism classes of irreducible representations of W. Then, using the definition of almost characters given in [12, (3.7)], we obtain the following relation:

$$R_s(E')_{|G^F_{\mathrm{unip}}} = \sum_{E \in \mathrm{Irr}(W)} m(E, E')\, Q_E, \tag{2.2}$$

where $m(E, E')$ denotes the multiplicity of E in the representation $\mathrm{Ind}^W_{W_s}(E')$. An element u in G^F_{unip} is said to be *split* if each irreducible component of $\mathcal{B}_u$ is stabilized by F. Since p is good, for each F–stable unipotent class C, there exist a unique split element u up to G^F–conjugacy, except in the following case: G is of type E_8, $q \equiv -1 \pmod 3$ and u is of type $D_8(a_3)$. In this exceptional case, split elements do not exist in C^F. If u is split, it is verified that F acts trivially on $A_G(u)$. We will assume tht $q \equiv 1 \pmod 3$ in case G is of type E_8. Let C_u be a unipotent class with u split. Then the G^F–conjugacy classes in G^F correspond bijectively to the conjugacy classes in $A_G(u)$ as follows. For each $a \in A_G(u)$, there exits $g \in G$ such that $g^{-1}F(g) = \dot{a}$, where $\dot{a}$ denotes a representative of a in $\mathrm{C}_G(u)$. Then the $u_a := gug^{-1}$ ($a \in A_G(u)$, up to conjugate) give representative of the G^F–conjugacy classes in C^F. Let E be an irreducible representation of W, and let (u, ρ) be its image under the map ν^G_T. Then (see [19, (5.2)]), the following holds:

$$\begin{cases} Q_E(u_a) = q^{d(u)}\, \rho(a), & \text{for } a \in A_G(u); \\ Q_E(g) = 0 & \text{if } g \notin \overline{C}^F_u. \end{cases} \tag{2.3}$$

2.3. Let $L \supset T$ be a Levi subgroup of a parabolic subgroup P of G and let $\hat{L}$ be the set of character sheaves on L. In [14, 4.1], Lusztig introduced the notion of induction ind_L^G of character sheaves. In particular, for each $A_L \in \hat{L}_s$, $\mathrm{ind}_L^G(A_L)$ is a semisimple perverse sheaf on G, and each irreducible direct summand is a character sheaf which belongs to $\hat{G}_s$ (see [14, Proposition 4.8 (b)]).

A character sheaf on G is said to be *cuspidal* if it is not contained in $\mathrm{ind}_L^G(A_L)$ for any Levi subgroup L of a proper parabolic subgroup of G and any $A_L \in \hat{L}$. Let $\Sigma \subset G$ be a $G \times \mathrm{Z}_G$–orbit, and let $\mathcal{E}$ be an irreducible $G \times \mathrm{Z}_G$–equivariant local system on Σ. Since p is good (almost good would be sufficient here), it is known by [14, Prop. 3.12, (7.1.2)] that, for any cuspidal such pair $(\Sigma, \mathcal{E})$, the (shift of) intersection cohomology complex $\mathrm{IC}(\bar{\Sigma}, \mathcal{E})\,[\dim \Sigma]$, extended to the whole of G by zero on $G - \bar{\Sigma}$, is a cuspidal character sheaf on G. All the cuspidal character sheaves on G are obtained in this way. Let $\hat{G}^0$ denote the set of cuspidal character sheaves on G.

Remark 2.4. Assume that G is simple of adjoint type (with $\mathrm{rank}\, G \neq 0$). Then $\hat{G}_s$ contains at most one cuspidal character sheaf with non-zero restriction to G_{unip} and such a character sheaf exists exactly in the following cases (see [14, Section 23], [1, (2.3), Appendix A]):

Type of G	Condition on n	Type of W_s
B_n	$n = 2t(t+1)$	$C_{t(t+1)} \times C_{t(t+1)}$
C_n	$n = 2t(4t \pm 1)$	$D_{4t^2} \times B_{4t^2 \pm 2t}$
D_n	$n = 8t^2$	$D_{4t^2} \times D_{4t^2}$
G_2		G_2
F_4		F_4
E_8		E_8

where $t \geq 1$.

Any character sheaf A of G is obtained as a direct summand of $\mathrm{ind}_L^G(A_L)$ for the Levi complement L of some parabolic subgroup P of G and a cuspidal character sheaf A_L on L. The pair (L, A_L) is unique up to conjugacy, an we will set

$$\mathcal{I}(A) := \left\{(gLg^{-1}, A_L^g) \mid g \in G\right\}. \tag{2.4}$$

Assume that $\hat{L}^0$ is non-emtpy. Then by [13, Theorem 9.2 (a)], the group $W_L^G = \mathrm{N}_G(L)/L$ is a finite Coxeter group. For $s \in T^*$, we we set $\hat{L}_s^0 := \hat{L}_s \cap \hat{L}^0$. Now let $s \in T^*$ such that $\hat{L}_s^0$ is not empty, and let $A_L \in \hat{L}_s^0$. It implies that s is isolated in L^*, that is, $\mathrm{C}_{G^*}(s)$ has the same semisimple rank as G^* (see [14, (17.12)]).

Shoji proves in [20, (5.16.1) and II, proof of (4.21)] (see also [12, (8.5)]) that the stabilizer of the cuspidal character sheaf A_L in W_L^G is the image of the canonical map

$$(\mathrm{N}_{G^*}(L^*) \cap \mathrm{C}_{G^*}(s))/\mathrm{C}_{L^*}(s) \hookrightarrow \mathrm{N}_{G^*}(L^*)/L^*, \tag{2.5}$$

where $L^* \subseteq G^*$ denotes the standard Levi subgroup dual to L and $\mathrm{C}_{G^*}(s)$ (resp. $\mathrm{C}_{L^*}(s)$) denotes the centralizer of s in G^* (resp. L^*). We see that this stabilizer only depends on G, s and L, and we shall therefore denote it by $W_{L,s}^G = W_{L,s}$. Note that we have $W_{T,s} = W_s$.

Let $\hat{L}_{\mathrm{unip}}^0$ be the subset of all $A_L \in \hat{L}^0$ such that A_L has non-zero restriction to L_{unip}. Assume that s isolated in G^* and that the set $\hat{L}_s^0 \cap \hat{L}_{\mathrm{unip}}^0$ is non-empty.

Type of G	Type of W_L^G	Type of $W_{L,s}^G$
B_n	$C_{n-2t(t+1)}$	$C_a \times C_b$ where $a+b=n-2t(t+1)$
C_n	$B_{n-2t(4t\pm 1)}$	$\begin{cases} B_a \times B_b \text{ where } a+b=n-2t(4t\pm 1), & \text{if } t \geq 1 \\ D_a \times B_b \text{ where } a+b=n, & \text{if } t=0 \end{cases}$
D_n	D_{n-8t^2}	$D_a \times D_b$ where $a+b=n-8t^2$

Let $A_L \in \hat{L}^0_{\mathrm{unip}}$. We set $K := \mathrm{ind}_L^G(A_L)$. Now let $\mathcal{H}(G, A_L) := \mathrm{End}_{\mathcal{M}G}(K)$ be the endomorphism algebra of K in $\mathcal{M}G$. It is known by [15, (2.4) (a), (2.5) (b)] (see also [13, (3.4)]), that $\mathcal{H}(G, A_L)$ is isomorphic to the group algebra $\overline{\mathbb{Q}}_\ell[W^G_{L,s}]$. Hence we have a decomposition

$$\mathrm{ind}_L^G(A_L) = \sum_{E' \in \mathrm{Irr}(W^G_{L,s})} E' \otimes A^s_{E'}, \quad \text{where } A^s_{E'} \in \hat{G}_s. \tag{2.6}$$

Let $\hat{G}_{\mathrm{unip}}$ be the set of isomorphism classes of characters sheaves $A \in \hat{G}$ such that the restriction of A to G_{unip} is non-zero. We then get a bijective map from the set $\tilde{\mathcal{T}}(G)$:

$$\left\{ \tilde{\mathfrak{t}} := [L, s, E']_G \ : \ \begin{array}{l} L \text{ Levi subgroup of some parabolic subgroup of } G \\ s \in T^* \text{ such that } \hat{L}^0_s \cap \hat{L}^0_{\mathrm{unip}} \text{ is non empty} \\ E' \text{ isomorphism class of irreducible representations of} \\ W^G_{L,s} \end{array} \right\}$$

to $\hat{G}_{\mathrm{unip}}$, by sending (L, s, E) to $A^s_E \in \hat{G}_{\mathrm{unip}}$. Then $[L, s]_G$ will be called the *inertial support* of A. The character sheaves which have non-zero restriction to the unipotent variety of G and have a given inertial support are in bijection with the irreducible characters of the group $W^G_{L,s}$. Let L_{ad} be the adjoint group of L and let $\pi \colon L \to L_{\mathrm{ad}}$ be the canonical map. Let L_{der} be the derived subgroup of L. By [14, (17.10)], we can write any $A \in \hat{L}^0$ in the form $A = \pi^*(\bar{A}) \otimes \mathcal{L}$ where $\bar{A} \in \hat{L}^0_{\mathrm{ad}}$ and $\mathcal{L}$ is a tame local system on L which is the inverse image of a local system on L/L_{der} under the canonical map $L \to L/L_{\mathrm{der}}$.

We have a corresponding embedding of dual groups $L^*_{\mathrm{der}} \subseteq L^*$. If $\bar{A}_L$ lies in the series of L_{ad} defined by $\overline{s} \in T^* \cap L^*_{\mathrm{der}}$ and $\mathcal{L}$ corresponds to the central element z of L^*, then A_L lies in the series of L defined by $s := \overline{s}z$. Clearly, we have $\mathrm{C}_{L^*}(s) = \mathrm{C}_{L^*}(\overline{s})$. Note that if $L = T$ then $\overline{s} = 1$.

Assume that L_{ad} has a cuspidal character sheaf $\bar{A}_L$ such that $\bar{A}_L$ has non-zero restriction to the unipotent variety of L_{ad}. Then $\bar{A}_L$ is uniquely determined. Let $\overline{s} \in T^* \cap L^*_{\mathrm{der}} \subseteq L^*$ such that $\bar{A}_L$ lies in the series defined by $\overline{s}$. By [13, Theorem 9.2 (b)], we have $W^G_{L,\overline{s}} = W^G_L$. We consider the decomposition

$$\mathrm{ind}_L^G(\pi^*(\bar{A}_L)) = \sum_{E \in \mathrm{Irr}(W^G_L)} E \otimes A^{\overline{s}}_E, \quad \text{where } A^{\overline{s}}_E \in \hat{G}_{\overline{s}}. \tag{2.7}$$

Let $A_L \in \hat{L}$ be any cuspidal character sheaf. If the restriction of A_L to L_{unip} is zero then the restrictions of all components of $\mathrm{ind}_L^G(A_L)$ to G_{unip} will also be zero (see [15, (2.9)]). Assume now that the restriction of A_L to L_{unip} is non-zero. We can write $A_L = \pi^*(\bar{A}_L) \otimes \mathcal{L}$ where $\bar{A}_L$ lies in the series of L_{ad} defined by $\overline{s} \in T^* \cap L^*_{\mathrm{der}}$ and $\mathcal{L}$ is pulled back from a local system on L/L_{der}. Let A_L, $\bar{A}_L$ lie in the series defined by $s, \overline{s} \in T^*$, respectively.

It is clear that A_L and $\pi^*(\bar{A}_L)$ have the same restriction to L_{unip}, [15, (2.6) (c)]. Moreover, the restriction of the decomposition to G_{unip} is related to the restriction of that in (a) by the following formula, see [15, (2.6) (e)]:

$$A^s_{E'} = \bigoplus_E m(E,E')\, A^{\overline{s}}_E \ \text{ on } G_{\mathrm{unip}}, \tag{2.8}$$

where $E' \in \mathrm{Irr}(W^G_{L,s})$, $E \in \mathrm{Irr}(W^G_L)$ and $m(E,E')$ denotes the multiplicity of E in the induced representation $\mathrm{Ind}_{W^G_{L,s}}^{W^G_L}(E')$.

Proposition 2.5. *Let E' be an irreducible representation of W_s, and let $\mathbf{c}$ be the two-sided cell in W_s to which E' belongs. If C is a unipotent class in G such that $A^s_{E'|\{u\}} \neq 0$ for some $u \in C$, then $\dim C \le \dim C_{s,\mathbf{c}}$, with equality only for $C = C_{s,\mathbf{c}}$.*

Proof. Assume first that $s = 1$. Then $W_s = W$. We can choose a Frobenius map $F\colon G \to G$ as in (2.2) so that the characteristic function of $A^1_{E'}$ coincides (for a suitable normalization) with the almost character $R^1_{E'}$ of G^F, see, for instance, [6, Cor. 2.3.2]. The restriction of $R^1_{E'}$ to G^F_{unip} is given in terms of Green functions, as recalled in (2.2). We have seen there that, via the Springer Correspondence, the representation E' corresponds to a pair$(C', \mathcal{E}')$, where C' is a unipotent class in G and $\mathcal{E}'$ is a certain G–invariant irreducible local system on C', and that the restriction of the almost character $R^1_{E'}$ to G^F_{unip} has non-zero values only in $\overline{C}'^F$ (see (2.3)). The unique special character belonging to $\mathbf{c}$ corresponds to the pair $(C_{s,\mathbf{c}}, \overline{\mathbb{Q}}_\ell)$ via Springer Correspondence, and we have seen that $a_{s,\mathbf{c}} = d(C_{s,\mathbf{c}})$. By [17, Cor. 10.9], we have $\dim C' \le \dim C_{s,\mathbf{c}}$, with equality only for $C' = C_{s,\mathbf{c}}$. Note that this statement is true whenever p is good for G, since the Springer Correspondence as well the dimensions of varieties of Borel subgroups are independent ot the characteristic as long as the characteristic is good (see the tables in [2], for instance). Now, if C is a unipotent class such that $A^s_{E'|\{u\}} \neq 0$ for some $u \in C$, then we may assume, as in (2.2), that $F(u) = u$ and that $R^1_{E'}(u) \neq 0$, and the the above remarks on the values of Green functions imply that the class C lies in the Zariski closure of C'. Hence we have $\dim C \le \dim C' \le \dim C_{s,\mathbf{c}}$, with equality only for $C = C' = C_{s,\mathbf{c}}$. $\dim C \le \dim C_{s,\mathbf{c}}$, with equality only for $C = C_{s,\mathbf{c}}$.

When s is not equal to 1, the result follows from [17, Cor. 10.9], the formula (2.8), and the case $s = 1$. □

We are now able to prove Theorem 1.1 for groups of exceptional type.

Proposition 2.6. *Assume that G is of exceptional type. Then Theorem 1.1 holds for G when p is good.*

Proof. By using Lemma 2.3, we can assume that G is simple modulo its center, which can be assumed to be connected. Then let $A \in \hat{G}$, and let $(L, A_L) \in \mathcal{I}(A)$. Since G is of exceptional type, the only possibilities for L are either $L = T$ or

$L = G$, see [14, §18]. When $L = T$, the result follows from Proposition 2.5. When $L = G$, we have $A = \mathrm{IC}(\overline{C} \cdot \mathrm{Z}_G, \mathcal{E} \boxtimes \mathcal{L})[\dim(\overline{C} \cdot \mathrm{Z}_G)]$, where $(C, \mathcal{E})$ is a cuspidal pair, and $\mathcal{L}$ is a tame local system on L. If G is of type E_6, E_7, there are no cuspidal pairs for $L = G$ which are suported on a unipotent class, see [1, Appendix].

Hence we can assume that G is of type G_2, F_4, E_8. Let C' a unipotent class in G such that $A_{|\{u'\}} \neq 0$ for some $u' \in C'$. It implies that C' is contained in the Zariski closure of C. We have seen in (1.2) that the unique special representation E of W belonging to $\mathbf{c}$ correspond to the pair $(C_{s,\mathbf{c}}, \overline{\mathbb{Q}}_\ell)$ via the Springer Correspondence. By [14, (20.6) and §21], we know that E belongs to the unique family $\mathcal{F}$ of representations of W with 4, 7, 17 elements, for G of type G_2, F_4, E_8, respectively. But, by [12, (13.1.3)], the order of $A(u)$ for $u \in C_{s,\mathbf{c}}$ equals the order of the finite group $\mathcal{G}_{\mathcal{F}}$ associated to the family $\mathcal{F}$. Hence the order of $A(u)$ is 6, 24, 120, for G of type G_2, F_4, E_8, respectively. But, for G of type G_2, F_4, E_8, there is exactly one cuspidal pair for $L = G$ supported on a unipotent class, say C, and the class C is uniquely determined by the condition that the group $A_G(u)$, for $u \in C$, has order 6, 24, 120, respectively. Hence $C = C_{s,\mathbf{c}}$. The assertion (a) of Theorem 1.1 follows. Then the assertion (b) follows from Proposition 2.1. □

Remark 2.7. The case of groups of classical type is more difficult, because it involves the group $W_{L,s}$, with $L \neq T$, and not only the group W_s. Because of that we will need to use the Generalized Springer Correspondence instead of the ordinary one. In particular, we will prove a generalization of [17, Cor. 10.9] in that case.

3. The classical groups case

In this section, we will prove Theorem 1.1 for classical groups under the assumption that p is odd and that s is isolated in G^*. First note that, by Proposition 2.1, it is sufficient to prove the assertion (a) of Theorem 1.1. Note also that if $L = T$, it follows from Proposition 2.5. Hence we can, if necessary, assume that $L \neq T$ (we will assume this when G is of type C). Finally, by Proposition 2.3, we can assume that the center of G is connected and that G is simple modulo its center.

3.1. The strategy of the proof.

3.1.1. *The restriction of character sheaves to the unipotent variety.* Let $A \in \hat{G}$. Then by [15, (2.6)(e)] and [13, (6.5)], the restriction of a character sheaf $A \in \hat{G}$ to G_{unip} can be expressed uniquely as follows

$$A_{|G_{\mathrm{unip}}} = \sum_{\mathfrak{n} \in \mathcal{N}_G} m_{\mathfrak{n}}(A)\, A_{\mathfrak{n}}, \quad \text{where } A_{\mathfrak{n}} = \mathrm{IC}(\overline{C}, \mathcal{E})[d_{\mathfrak{n}}], \text{ with } \mathfrak{n} = (C, \mathcal{E}), \tag{3.1}$$

where the $m_{\mathfrak{n}}^G(A) = m_{\mathfrak{n}}(A)$ are certain non-negative integers, and

$$d_{\mathfrak{n}} := \dim C + \dim \mathrm{Z}_L. \tag{3.2}$$

If the restriction of A to G_{unip} is equal to zero then $m_{\mathfrak{n}}(A) = 0$ for all $\mathfrak{n} \in \mathcal{N}_G$. We will now assume that A has non-zero restriction to G_{unip}. Then A is a component of an induced complex $\mathrm{ind}_L^G(A_L)$ where $L \supset T$ is a Levi subgroup of a parabolic subgroup P of G and A_L is a cuspidal character sheaf on L, with non-zero restriction on L_{unip}. We have

$$A_L = \mathrm{IC}(\overline{C}_L \mathrm{Z}_L, \mathcal{E}_L \boxtimes \mathcal{L})[\dim C_L + \dim Z_L], \tag{3.3}$$

where $\mathfrak{n}_L := (C_L, \mathcal{E}_L) \in \mathcal{N}_L$ is a cuspidal pair and $\mathcal{L}$ is an irreducible $\overline{\mathbb{Q}}_\ell$–local system of Z_L.

Thus we have

$$m^L_{\mathfrak{n}_L}(A_L) = 1 \quad \text{and} \quad m^L_{\mathfrak{n}'_L}(A_L) = 0 \quad \text{for } \mathfrak{n}'_L \in \mathcal{N}_L \text{ such that } \mathfrak{n}'_L \neq \mathfrak{n}_L.$$

Moreover, A_L is *clean* (see [14, (23.1)]). So the restriction of A_L to L_{unip} is zero on $L_{\mathrm{unip}} \backslash C_L$ and is equal to $\mathcal{E}_L$ on C_L (up to shift). Conversely, since, if $A'_L \in \hat{L}$ is a component of a complex induced from a Levi subgroup L_1 of a proper parabolic subgroup of L, then $m^L_{\mathfrak{n}'_L}(A'_L) = 0$ unless $\mathfrak{n}'_L$ lies in a block of $\mathcal{N}_L$ with associated Levi subgroup conjugate to L_1 (see [15, (2.6)] and [13, (6.5)]), we get

$$m^L_{\mathfrak{n}_L}(A'_L) = 0 \quad \text{for all non-cuspidal } A'_L \in \hat{L}. \tag{3.4}$$

A complex $K \in \mathcal{D}G$ is said to be F-stable if $F^*K \simeq K$. For an F-stable complex K, with a given isomorphism $\varphi : F^*K \tilde{\to} K$, following [14, (8.4)], we define a characteristic function $\chi_{K,\varphi} : G^F \to \overline{\mathbb{Q}}_\ell$ by

$$\chi_{K,\varphi}(x) = \sum_i (-1)^i \operatorname{Trace}(\varphi, \mathcal{H}^i_x(K)), \tag{3.5}$$

where $\mathcal{H}^i_x(K)$ denotes the stalk at $x \in G^F$ of i-th cohomology sheaf $\mathcal{H}^i(K)$ of K, and φ is the induced linear map on $\mathcal{H}^i_x(K)$. If K is a G-equivariant perverse sheaf, $\chi_{K,\varphi}$ gives rise to a class function on G^F.

Let $\mathfrak{n} = (C, \mathcal{E}) \in \mathcal{N}_G$. We choose the natural mixed structure on $\mathcal{E}$, that is, the one (see [15, §3.2–3.4]) which has the property that $F^*\mathcal{E} \tilde{\to} \mathcal{E}$ induces on the stalk over a split element u of C the identity map times $q^{\frac{1}{2}(\dim G - d_{\mathfrak{n}})}$. (This property characterizes the mixed structure on $\mathcal{E}$.) This mixed structure on A_L exenteding the natural mixed structure on $\mathcal{E}_L$ is compatible with the isomorphism $\mathcal{H}(G, A_L) \simeq \overline{\mathbb{Q}}_\ell[W_{L,s}]$. The induced complex $\operatorname{ind}^G_L(A_L)$ inherits a natural mixed structure, and we will denote by $\chi_{A^s_{E'}}$ the corresponding characteristic function of $A^s_{E'}$.

On the other hand, we associate with any $(C, \mathcal{E}) \in \mathcal{N}_G$ the class function $Y_{(C,\mathcal{E})} : G^F \to \overline{\mathbb{Q}}_\ell$ defined as follows:

$$Y_{(C,\mathcal{E})}(g) := \chi_{\mathrm{IC}(\overline{C},\mathcal{E})}(g), \quad g \in \overline{C}^F, \tag{3.6}$$

extended by zero on $G^F - \overline{C}^F$ (where the mixed structure on $\mathrm{IC}(\overline{C}, \mathcal{E})$ is that extending the natural one on $\mathcal{E}$).

We assume that G is a split classical group of adjoint type and that p is odd. Let $s \in T^*$ and $E' \in \operatorname{Irr}(W_{L,s})$. Let C be an F–stable unipotent class in G and let $u \in C^F$ be a split element. If s is isolated in G^* and $\mathrm{C}_{L^*}(s)$ is cuspidal, then (see [1, Th. 3.2]) we have the following formula (which gives a much more precise expression than (3.1))

$$\chi_{A^s_{E'}}(u) = \sum_{E \in \operatorname{Irr}(W^G_L)} m(E, E')(-1)^{\operatorname{rank} G} q^{\dim G - d(\nu^G(E))} Y_{\nu^G(E)}(u), \tag{3.7}$$

where $m(E, E')$ denotes the multiplicity of E in the representation $\operatorname{Ind}^{W^G_L}_{W^G_{s,L}}(E')$. The formula 3.7 shows that to finish the proof of Theorem 1.1, it is sufficient to prove Theorem 1.2. The rest of this section will be devoted to the proof of Theorem 1.2. We will compute the multiplicites $m(E, E')$, and, in order to to that, we will first describe the groups $W^G_{s,L}$ and W^G_L.

3.1.2. *Type of the Levi subgroup.* First, we note that by the explicit classification of cuspidal character sheaves recalled in (2.4) the Levi subgroup L above must be of type isomorphic to

(B_n): $\mathrm{SO}_{(2t+1)^2} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(C_n): $\mathrm{PSp}_{16t^2 \pm 4t} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(D_n): $\mathrm{PSO}_{16t^2} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,

for some integer $t \geq 0$. We denote by $\tilde{n}$ the number of factors GL_1 in each type of Levi above, that is:

(B_n): $\tilde{n} = n - 2t^2 - 2t$,
(C_n): $\tilde{n} = n - (8t^2 \pm 2t)$,
(D_n): $\tilde{n} = n - 8t^2$.

3.1.3. *Description of the ramification subgroup of L.* We denote by W_n the group of all permutations of the set $\{1, 2, \ldots, n, n^*, \ldots, 2^*, 1^*\}$ which commute with the involution $i \mapsto i^*$, $i^* \mapsto i$, $(1 \leq i \leq n)$. We set $W_0 = \{1\}$. For each j, $1 \leq j \leq n-1$, let $s_j \in W_n$ be the permutation which interchanges j with $j+1$ and also j^* with j^*+1 and leaves the other elements unchanged. Let $\sigma_a \in W_n$ $(1 \leq a \leq n)$ be the permutation which interchanges a with a^* and leaves the other elements unchanged. Let $S = \{s_1, s_2, \ldots, s_{n-1}, \sigma_n\}$. Then (W_n, S) is a Coxeter group of type $B_n = C_n$.

Let $\varphi_n \colon W_n \to \{-1, 1\}$ be the homorphism defined by the condition $\varphi_n(s_j) = 1$ $(1 \leq j \leq n-1)$, $\varphi_n(\sigma_n) = -1$. It is straightforward to check that $\sigma_a = s_a s_{a+1} \cdots s_{n-1} \sigma_n s_{n-1} \cdots s_{a+1} s_a$ $(1 \leq a \leq n)$. It follows that $\varphi_n(\sigma_a) = -1$, hence the restriction of φ_n to W_a is φ_a. We shall denote φ_n simply by φ. Let W'_n be the kernel of φ, see (3.2.2). It is a group of type D_n.

Then we can identify the group W_L^G with the group $W_{\tilde{n}}$ if if G is of type B_n or C_n, or if $t \geq 1$ and G is of type D_n as follows. Consider a basis $e_1, \ldots, e_n, e_n^*, \ldots, e_1^*$ of V such that $(e_i, e_i^*) = 1$, $(e_i^*, e_i) = 1$ (resp. $(e_i^*, e_i) = -1$) if $(\ ,\ )$ is orthogonal (resp. symplectic) and all other scalar products equal to zero. We assume that L is the set of $g \in G$ which map each of the vectors $e_1, \ldots, e_{\tilde{n}}, e_{\tilde{n}}^*, \ldots, e_1^*$ into a scalar multiple of itself. Then each element of $\mathrm{N}_G(L)/L$ defines a permutation of the set of lines $\langle e_1 \rangle, \ldots, \langle e_{\tilde{n}} \rangle, \langle e_{\tilde{n}}^* \rangle, \ldots, \langle e_1^* \rangle$ and this gives the wanted isomorphism. If G is of type D_n and $t = 0$ then $W_L^G = W'_{\tilde{n}} = W'_n$.

3.1.4. *Dual side.* The group L^* is of type

(B_n): $\mathrm{Sp}_{4t^2+4t} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(C_n): $\mathrm{Spin}_{16t^2 \pm 4t+1} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(D_n): $\mathrm{Spin}_{16t^2} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$.

Since the cuspidal character sheaf A of L is supported by the closure of a unipotent class, the group $\mathrm{C}_{L^*}(s)$ is of type

(B_n): $\mathrm{Sp}_{2t^2+2t} \times \mathrm{Sp}_{2t^2+2t} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(C_n): $\mathrm{Spin}_{8t^2} \times \mathrm{Spin}_{8t^2 \pm 4t+1} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$,
(D_n): $\mathrm{Spin}_{8t^2} \times \mathrm{Spin}_{8t^2} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$.

3.1.5. *The ramification subgroup of the cuspidal pair.* We can identify in a standard way the group $W_{L,s}^G$ with

- $W_{\tilde{n}'} \times W_{\tilde{n}''}$ if G is of type B_n or if $t \geq 1$ and G is of type C_n or D_n,
- $W'_{\tilde{n}'} \times W_{\tilde{n}''}$ if G is of type C_n and $t = 0$,
- $W'_{\tilde{n}'} \times W'_{\tilde{n}''}$ if G is of type D_n and $t = 0$

where $\tilde n' + \tilde n'' = \tilde n$; the isomorphism is canonical up to conjugation with inner automorphism of $W_{\tilde n'} \times W_{\tilde n''}$ (resp. $W'_{\tilde n'} \times W_{\tilde n''}$, $W'_{\tilde n'} \times W'_{\tilde n''}$). Hence an irreducible representation of $W^G_{L,s}$ may be identified with the corresponding representation of $W_{\tilde n'} \times W_{\tilde n''}$ (resp. $W'_{\tilde n'} \times W_{\tilde n''}$, $W'_{\tilde n'} \times W'_{\tilde n''}$).

3.2. **On induced representations of Weyl groups of classical type.**

3.2.1. *Type A.* Let $\mathfrak{S}_{\tilde n}$ be the permutation group of the set $\{1, 2, \ldots, \tilde n\}$. We call "partitions" of $\tilde n$ the non-decreasing sequences of non-negative integers

$$\alpha = (0 \le \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m) \text{ such that } \sum_{i=0}^{m} \alpha_i = \tilde n;$$

two sequences, differing only by the number of zeros, are identified. We shall denote by $v(\alpha)$ the lowest i such that $\alpha_i \neq 0$, and we shall write $\alpha \dashv \tilde n$ to mean that α is a partition of the integer n.

Each $E \in \mathrm{Irr}(\mathfrak{S}_{\tilde n})$ is parametrized by a partition of $\tilde n$; we shall denote by E_α the irreducible representation of $\mathfrak{S}_{\tilde n}$ corresponding to the partition α.

Let $\tilde n'$, $\tilde n''$ be two non-negative integers such that $\tilde n' + \tilde n'' = \tilde n$. Let $\alpha' = (\alpha'_0 \le \alpha'_1 \le \ldots \le \alpha'_{m'})$ be a partition of $\tilde n'$, let $\alpha'' = (\alpha''_0 \le \alpha''_1 \le \ldots \le \alpha''_{m''})$ be a partition of $\tilde n''$. By allowing 0 on the entries of α', α'', we may assume that $m' = m'' = m$.

Since $\tilde n' + \tilde n'' = \tilde n$, the group $\mathfrak{S}_{\tilde n'} \times \mathfrak{S}_{\tilde n''}$ is naturally embedded in $\mathfrak{S}_{\tilde n}$ as the stabilizer of $\{1, 2 \ldots, \tilde n'\} \subset \{1, 2, \ldots, \tilde n\}$. It allows us to construct from representations $E_{\alpha'}$ of $\mathfrak{S}_{\tilde n'}$ and $E_{\alpha''}$ of $\mathfrak{S}_{\tilde n''}$ the representation

$$I(\alpha', \alpha'') := \mathrm{Ind}_{\mathfrak{S}_{\tilde n'} \times \mathfrak{S}_{\tilde n''}}^{\mathfrak{S}_{\tilde n}} (E_{\alpha'} \otimes E_{\alpha''}) \tag{3.8}$$

of $\mathfrak{S}_{\tilde n}$ induced from the tensor product representation $E_{\alpha'} \otimes E_{\alpha''}$ of the subgroup $\mathfrak{S}_{\tilde n'} \times \mathfrak{S}_{\tilde n''}$ of $\mathfrak{S}_{\tilde n}$.

We first recall in a convenient way for our purpose the Littlewood-Richardson rule which describes the representation $I(\alpha', \alpha'')$. Let us introduce some combinatorial terminology.

Let α be a partition of $\tilde n$. Let

$$D(\alpha) := \{(i, j) \in \mathbb{N}^* \times \mathbb{N}^* \ : \ j \le \alpha_{m-i+1}\} \tag{3.9}$$

be the Young diagram associated with α (here $\mathbb{N}$ and $\mathbb{N}^* = \mathbb{Z}_0$ stand respectively for the set of non-negative integers and the set of positive integers). When we show diagrams graphically, we assume the i–axis to go downwards while the j–axis goes to the right. Let i_0 be a fixed integer. Then the set of points $(i_0, j) \in D(\alpha)$ will be called the i_0–th row of $D(\alpha)$. Define the partial order $\underset{\mathcal{P}}{\le}$ on $\mathbb{N}^* \times \mathbb{N}^*$ by

$$a = (i, j) \underset{\mathcal{P}}{\le} a' = (i', j') \quad \text{if} \quad i \le i', j \le j'. \tag{3.10}$$

Let α' be a partition of $\tilde n'$. If $D(\alpha) \supset D(\alpha')$ we shall denote by $D(\alpha \backslash \alpha')$ the difference $D(\alpha) \backslash D(\alpha')$. By a "numbering" of $D(\alpha \backslash \alpha')$ we mean any map $f \colon D(\alpha \backslash \alpha') \to \mathbb{N}$ which is a morphism of partially ordered sets where $D(\alpha \backslash \alpha')$ is ordered by $\underset{\mathcal{P}}{\le}$ and $\mathbb{N}$ as usual.

Geometrically, a numbering will be shown as an array of integers obtained by replacing each point x of $D(\alpha) \backslash D(\alpha')$ by the number $f(x)$. Such an array represents a numbering if and only if its numbers are non-decreasing along the rows and down the columns of $D(\alpha) \backslash D(\alpha')$; it has "$\alpha''$–type" where $\alpha'' = (\alpha''_0, \alpha''_1, \ldots, \alpha''_{m''})$ is a partition of $\tilde n''$ if any number $k \in \mathbb{N}$ occurs α''_{m-k+1} times.

We shall say that the numbering is "column-strict" if the numbers increase down the columns of $D(\alpha)\backslash D(\alpha')$. We shall say that it is "well-readable" if it satisfies the following condition: if n_1, n_2, …, $n_{m-m'}$ is the sequence of the numbers $f(x)$ read from right to left along the first row of $D(\alpha)\backslash D(\alpha')$, next right to left along the second row, etc., then for any $h \in \{1, 2, \dots, m-m'\}$ and $l \in \mathbb{N}$ the number of l's among n_1, n_2, …, n_h is not less than the number of $(l+1)$'s among n_1, n_2, …, n_h.

The Littlewood-Richardson rule states (see for instance [26, Proposition 4.18]) that

$$I(\alpha', \alpha'') = \sum_{\alpha \dashv n} m(\alpha'', \alpha\backslash\alpha')\, E_\alpha, \tag{3.11}$$

where $m(\alpha'', \alpha\backslash\alpha')$ equals the number of well-readable column-strict numberings of $D(\alpha)\backslash D(\alpha')$ which have α''–type.

Let $\leq$ be the natural partial order on partitions defined as follows. Let $\alpha = (0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m)$ and $\beta = (0 \leq \beta_0 \leq \beta_1 \leq \cdots \leq \beta_m)$ be two partitions of $\tilde{n}$. Then

$$\beta \leq \alpha \quad \text{if} \quad \begin{cases} \beta_m \leq \alpha_m \\ \beta_{m-1} + \beta_m \leq \alpha_{m-1} + \alpha_m \\ \beta_{m-2} + \beta_{m-1} + \beta_m \leq \alpha_{m-2} + \alpha_{m-1} + \alpha_m \\ \vdots \\ \beta_1 + \beta_2 + \cdots + \beta_{m-1} + \beta_m \leq \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} + \alpha_m. \end{cases} \tag{3.12}$$

For any partition α of $\tilde{n}$, and any integer $i \geq 1$, we will denote by $c_i(\alpha)$ the numbers of integers j such that $\alpha_j = i$. Let $\tilde{n}'$, $\tilde{n}''$ two integers such that $\tilde{n}' + \tilde{n}'' = \tilde{n}$, and let α', α'' be partitions of $\tilde{n}'$, $\tilde{n}''$ respectively. We will denote by $\alpha' \cup \alpha''$ the unique partition of $\tilde{n}$ such that

$$c_i(\alpha' \cup \alpha'') = c_i(\alpha') + c_i(\alpha''),$$

for any integer $i \geq 1$, and by $\alpha' + \alpha''$ the partition of $\tilde{n}$ defined by

$$\alpha' + \alpha'' := (0 \leq \alpha'_0 + \alpha''_0 \leq \alpha'_1 + \alpha''_1 \leq \ldots \leq \alpha'_m + \alpha''_m), \tag{3.13}$$

Proposition 3.1. *Let $\tilde{n}'$, $\tilde{n}''$ be such that $\tilde{n}' + \tilde{n}'' = \tilde{n}$. Let α, α', α'' be partitions of $\tilde{n}$, $\tilde{n}'$, $\tilde{n}''$ respectively. Then the following hold.*

(a) *Both the representations $E_{\alpha'\cup\alpha''}$ and $E_{\alpha'+\alpha''}$ occur with multiplicity one in $I(\alpha', \alpha'')$.*
(b) *If E_α occurs in $I(\alpha', \alpha'')$, then $\alpha' \cup \alpha'' \leq \alpha \leq \alpha' + \alpha''$.*

Proof. For the facts that $E_{\alpha'\cup\alpha''}$ occurs with multiplicity one in $I(\alpha', \alpha'')$, and that $\alpha' \cup \alpha'' \leq \alpha$, for any E_α occurring in $I(\alpha', \alpha'')$, see [25, VIII.2.(5)].

Now, for the other assertions, let f be a well-readable numbering of $D(\alpha' + \alpha'')\backslash D(\alpha')$. We define j_1 by $j \leq j_1$ for all j such that $(1, j) \in D(\alpha' + \alpha''\backslash\alpha')$ (geometrically, the point $(1, j_1)$ is the last point on the first row of $D(\alpha' + \alpha''\backslash\alpha')$), the fact that f is well-readable implies that $f(1, j_1) = 1$. We set $I_1 := \{\alpha'_m + 1, \alpha'_m + 2, \dots, j_1\}$. We denote by $|I_1|$ the cardinality $j_1 - \alpha'_m$ of I_1. Now since f is a numbering (hence the numbers are non-decreasing along the rows), we see that $f(1, j) = 1$ for all $j \in I_1$. Assume moreover that f has α''–type. Then it follows that $|I_1| = \alpha''_m$. Then the number 1 is used up, that is we have $f(i, j) \neq 1$ if $i \neq 1$. Now let $i \geq 2$. By induction on i, we see that for all $(i, j) \in D(\alpha' + \alpha'')\backslash D(\alpha')$

we have $f(i,j) = i$ and $f(k,j) \neq i$ if $k \neq i$. It follows that there is exactly one well-readable numbering of $D(\alpha' + \alpha'')\backslash D(\alpha')$ which has α''–type: the numbering f which is defined by $f(i,j) = i$ for all $(i,j) \in D(\alpha' + \alpha'')\backslash D(\alpha')$. Hence (a) is proved.

We shall prove the second part of (b). We first note that if E_α occurs in $I(\alpha', \alpha'')$ then the Young diagram $D(\alpha)$ is obtained from $D(\alpha')$ by adding n'' points. Then because of the condition "column-strict" in the Littlewood-Richardson rule, we see that if $(i,j) \in D(\alpha)$ then $j \leq 2m - v(\alpha') - v(\alpha'') + 2$. By choosing $m \geq v(\alpha') + v(\alpha'')$, we may assume that $\alpha = (0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_m)$, $\alpha' = (0 \leq \alpha'_0 \leq \alpha'_1 \leq \ldots \leq \alpha'_m)$, $\alpha'' = (0 \leq \alpha''_0 \leq \alpha''_1 \leq \ldots \leq \alpha''_m)$.

We shall denote by $\alpha^{r_1, r_2, \cdots, r_{n''}}$ the partition of $\tilde{n}$ such that $D(\alpha^{r_1, r_2, \cdots, r_{n''}})$ is obtained from $D(\alpha')$ by adding a point to the $(m - r_l + 1)$–th row, for l running over $\{1, 2, \ldots, n''\}$. We do not assume the numbers r_l to be distinct. Now if E_α occurs in $I(\alpha', \alpha'')$ there exists a sequence $(r_1, r_2, \ldots, r_{\tilde{n}''})$ such that $\alpha = \alpha^{r_1, r_2, \cdots, r_{\tilde{n}''}}$. Note that we have already shown that $\alpha' + \alpha'' = \alpha^{r_1^{\alpha'+\alpha''}, r_2^{\alpha'+\alpha''}, \cdots, r_{\tilde{n}''}^{\alpha'+\alpha''}}$ where

$$(r_1^{\alpha'+\alpha''}, r_2^{\alpha'+\alpha''}, \ldots, r_{\tilde{n}''}^{\alpha'+\alpha''}) = (\underbrace{m, \ldots, m}_{\alpha''_m}, \underbrace{m-1, \ldots, m-1}_{\alpha''_{m-1}}, \ldots, \underbrace{1, \ldots, 1}_{\alpha''_{m-v(\alpha'')+1}}). \tag{3.14}$$

We now fix a $\tilde{n}''$–tuple $(r_1, r_2, \ldots, r_{\tilde{n}''})$. Let $r \in \{r_1, r_2, \ldots, r_{\tilde{n}''}\}$. We denote by J^r the set of $l \in \{1, 2, \ldots, \tilde{n}''\}$ such that $r_l = r$. Let $|J^r|$ be the cardinality of J^r. Let f a well-readable column-strict numbering of $D(\alpha^{r_1, r_2, \cdots, r_{\tilde{n}''}})\backslash D(\alpha')$ which have α''–type. We regard f as an array of integers. Now on the first row only the number 1 is allowed. Indeed that the condition of being well-readable implies that the sequence 1, 1, …, 1, j, with $j \neq 1$ is not allowed, the condition of being no decreasing along the row implies that j, 1 is neither not allowed. Thus, on the first row we at most α''_m added squares, *i.e.,* $|J^m| \leq \alpha''_m$.

Next, on the second row, only numbers 1 and 2 are allowed. Indeed the the condition of being non-decreasing along the row implies that we cannot have $j \neq 1, 2$ on the left of 1 or on the left of 2, the condition of being well-readable implies that we cannot have $j \neq 1, 2$ on the right.

Fix now $j_0 \leq m$. Let us state the following hypothesis

> (*) For any $i < j_0$, on the i–th row of f, only numbers 1, 2, …, i are allowed.

We shall prove by induction on j_0 that (*) holds true. Let k be the first number we meet on the j_0–th row of f when we are reading this row from right to left. Assume that $k \notin \{1, 2, \ldots, j_0\}$. Then using the induction hypothesis we see that k does not appear on the rows 1, 2, …, $j_0 - 1$. Thus the numbering f is not well-readable. We get a contradiction. Hence $k \in \{1, 2, \ldots, j_0\}$. Next, because the numbering is non decreasing along the rows, all the numbers belonging to the j_0–th row which are on the left of k (that is all of them) are lower than k, hence they all belong to $\{1, 2, \ldots, j_0\}$. Then, on the j_0–th rows we only have numbers 1, 2, …, j_0. It implies that the number of added squares which appear on these rows is bounded by the total number of 1, 2, …, j_0. Since that number equals $\alpha''_m + \alpha''_{m-1} + \cdots + \alpha''_{m-j_0+1}$, we get

$$\alpha_m + \alpha_{m-1} + \cdots + \alpha_{m-j_0+1} < (\alpha'_m + + \cdots + \alpha'_{m-j_0+1}) + (\alpha''_m + \cdots + \alpha''_{m-j_0+1}),$$

for any $j_0 \in \{1, 2, \ldots, m\}$, *i.e.,* $\alpha \leq \alpha' + \alpha''$. □

3.2.2. *Types B, C and D.* A permutation in $W_{\tilde{n}}$ defines a permutation of the $\tilde{n}$ element set consisting of the unordered pairs $(1,1^*)$, $(2,2^*)$, $\ldots$, $(\tilde{n},\tilde{n}^*)$. Thus we have a natural homomorphism of $W_{\tilde{n}}$ onto $\mathfrak{S}_{\tilde{n}}$. Let a, b be two non-negative integers such that $a+b=\tilde{n}$. The subgroup $W_{a,b}$ of $W_{\tilde{n}}$ consisting of all permutations which map $\{1,2,\ldots,a,a^*,\ldots,2^*,1^*\}$ into itself and hence also map $\{a+1,\ldots,\tilde{n},\tilde{n}^*,\ldots,(a+1)^*\}$ into itself may be regarded in a natural way as a product $W_a\times W_b$. Hence $W_{a,b}$ has a natural map (as above) onto the product of symmetric groups $\mathfrak{S}_a\times\mathfrak{S}_b$. Let E be an irreducible representation of $\mathfrak{S}_a$ and let E^* be an irreducible representation of $\mathfrak{S}_b$. We can regard $E\otimes E^*$ as a representation of $W_{a,b}$ via the projection $W_{a,b}\to\mathfrak{S}_a\times\mathfrak{S}_b$. We induce the representation $E\otimes\varphi E^*$ from $W_{a,b}$ to $W_{\tilde{n}}$. We obtain thus an irreducible representation of $W_{\tilde{n}}$. In this way we get a bijection

$$\begin{aligned} H:\ \bigcup_{\substack{a,b\\ a+b=\tilde{n}}} \operatorname{Irr}(\mathfrak{S}_a\times\mathfrak{S}_b) &\longrightarrow \operatorname{Irr}(W_{\tilde{n}})\\ ((a,b)\,;E\otimes E^*) &\mapsto \operatorname{Ind}_{W_{a,b}}^{W_{\tilde{n}}}(E\otimes\varphi E^*). \end{aligned}$$

Let $\mathcal{R}(\mathfrak{S}_{\tilde{n}})$ be the Grothendieck group of the category of finite dimensional complex representations of $\mathfrak{S}_{\tilde{n}}$. The tensor product gives an isomorphism

$$\mathcal{R}(\mathfrak{S}_a\times\mathfrak{S}_b)\xrightarrow{\sim}\mathcal{R}(\mathfrak{S}_a)\otimes\mathcal{R}(\mathfrak{S}_b)$$

and the operations of induction and restriction give rise to the $\mathbb{Z}$–linear maps

$$i_{a,b}:\ \mathcal{R}(\mathfrak{S}_a)\otimes\mathcal{R}(\mathfrak{S}_b)\to\mathcal{R}(\mathfrak{S}_{\tilde{n}})\ \text{ and }\ r_{a,b}:\ \mathcal{R}(\mathfrak{S}_{\tilde{n}})\to\mathcal{R}(\mathfrak{S}_a)\otimes\mathcal{R}(\mathfrak{S}_b).$$

It is convenient to consider all these maps together. Consider the graded group $\mathcal{R}(\mathfrak{S}):=\bigoplus_{\tilde{n}\geq 0}\mathcal{R}(\mathfrak{S}_{\tilde{n}})$ (here $\mathfrak{S}_0=\{1\}$). Define graded group morphisms

$$m^{\mathfrak{S}}:\ \mathcal{R}(\mathfrak{S})\otimes\mathcal{R}(\mathfrak{S})\to\mathcal{R}(\mathfrak{S})\ \text{ and }\ {}^*m^{\mathfrak{S}}:\ \mathcal{R}(\mathfrak{S})\to\mathcal{R}(\mathfrak{S})\otimes\mathcal{R}(\mathfrak{S})$$

by

$$m^{\mathfrak{S}}_{|\mathcal{R}(\mathfrak{S}_a)\otimes\mathcal{R}(\mathfrak{S}_b)}:=i_{a,b},\qquad {}^*m^{\mathfrak{S}}_{|\mathcal{R}(\mathfrak{S}_{\tilde{n}})}:=\sum_{\substack{(a,b)\\ a+b=\tilde{n}}} r_{a,b}.$$

We consider $m^{\mathfrak{S}}$ as a multiplication; it makes $\mathcal{R}(\mathfrak{S})$ into a graded algebra over $\mathbb{Z}$. Similarly, ${}^*m^{\mathfrak{S}}$ is a comultiplication making $\mathcal{R}(\mathfrak{S})$ into a coalgebra. These structure happen to be compatible and they make $\mathcal{R}(\mathfrak{S})$ into a Hopf algebra (see for instance [26, §6]).

Now let $\mathcal{R}(W_{\tilde{n}})$ be the Grothendieck group of the category of finite dimensional complex representations of $W_{\tilde{n}}$. Consider the graded group $\mathcal{R}(\mathrm{B}):=\bigoplus_{n\geq 0}\mathcal{R}(W_{\tilde{n}})$. Similarly we make $\mathcal{R}(\mathrm{B})$ into a Hopf algebra. We shall denote by m^{B} the multiplication.

The bijection H induces an isomorphism $\mathcal{R}(\mathfrak{S})\otimes\mathcal{R}(\mathfrak{S})\xrightarrow{\sim}\mathcal{R}(\mathrm{B})$ of $\mathbb{Z}$–modules. We shall set

$$X\,\dot{\boxtimes}\,X':=H(X\otimes X'),\ \text{ for } X,\ X' \text{ in } \mathcal{R}(\mathfrak{S}). \tag{3.15}$$

Then it follows from [26, §7] that $X\otimes X'\mapsto X\,\dot{\boxtimes}\,X'$ is a morphism of Hopf algebras. In particular it commutes with the multiplication in $\mathcal{R}(B)$, that is

$$m^{\mathrm{B}}\left(X\,\dot{\boxtimes}\,X',Y\,\dot{\boxtimes}\,Y'\right)=m^{\mathfrak{S}}(X,Y)\,\dot{\boxtimes}\,m^{\mathfrak{S}}(X',Y'), \tag{3.16}$$

for X, X', Y, Y' in $\mathcal{R}(\mathfrak{S})$.

3.2.3. *A property of induced representations.* First let us introduce the following notation. Let α be a partition of a and let β be a partition of b, where $a+b=\tilde{n}$. Then we shall set

$$E_{\alpha,\beta} := E_\alpha \dot{\boxtimes} E_\beta = \mathrm{Ind}_{W_{a,b}}^{W_{\tilde{n}}}(E_\alpha \otimes \varphi E_\beta). \tag{3.17}$$

We denote by $E'_{\alpha,\beta}$ the restriction of $E_{\alpha,\beta}$ to $W'_{\tilde{n}}$ when that restriction is irreducible (that is when the sets $\{\alpha_i\}$ and $\{\beta_j\}$ do not coincide); if the restriction is not irreducible we shall write $\mathrm{Res}_{W'_{\tilde{n}}}^{W_{\tilde{n}}}(E_{\alpha,\beta}) = E'_{\alpha,\beta} + E''_{\alpha,\beta}$.

Proposition 3.2. *Let $\tilde{n}'$, $\tilde{n}''$ be two integers such that $\tilde{n}'+\tilde{n}''=\tilde{n}$. Let (a',b'), (a''',b'') be two pairs of integers such that $a'+b'=\tilde{n}'$, $a''+b''=\tilde{n}''$. Let α', α'', β', β'' be partitions of a', a'', b', b'' respectively. Then*

$$\mathrm{Ind}_{W_{\tilde{n}'}\times W_{\tilde{n}''}}^{W_{\tilde{n}}}(E_{\alpha',\beta'} \otimes E_{\alpha'',\beta''}) = \sum_{\substack{(\alpha,\beta)\\ \alpha\dashv a'+a'',\ \beta\dashv b'+b''}} m(\alpha'',\alpha\backslash\alpha')\, m(\beta'',\beta\backslash\beta')\, E_{\alpha,\beta}.$$

Proof. We have

$$\begin{aligned}\mathrm{Ind}_{W_a\times W_b}^{W_{\tilde{n}}}(E_{\alpha,\beta}\otimes E_{\alpha'\otimes\beta'})) &= m^{\mathrm{B}}(E_{\alpha,\beta},E_{\alpha',\beta'})\\ &= m^{\mathrm{B}}(E_\alpha \dot{\boxtimes} E_\beta, E_{\alpha'} \dot{\boxtimes} E_{\beta'})\\ &= m^{\mathfrak{S}}(E_\alpha\otimes E_{\alpha'}) \dot{\boxtimes} m^{\mathfrak{S}}(E_\beta \otimes E_{\beta'}).\end{aligned}$$

Now $m^{\mathfrak{S}}(E_\alpha\otimes E_{\alpha'}) \dot{\boxtimes} m^{\mathfrak{S}}(E_\beta \otimes E_{\beta'})$ equals

$$\begin{aligned}&\mathrm{Ind}_{W_{a+a',b+b'}}^{W_{\tilde{n}}}\left(\mathrm{Ind}_{\mathfrak{S}_{a'}\times\mathfrak{S}_{a''}}^{\mathfrak{S}a'+a''}(E_{\alpha'}\otimes E_{\alpha''}) \otimes \varphi\,\mathrm{Ind}_{\mathfrak{S}_{b'}\times\mathfrak{S}_{b''}}^{\mathfrak{S}b'+b''}(E_{\beta'}\otimes E_{\beta''})\right)\\ &\quad = \sum_{\substack{\alpha\dashv a'+a''\\ \beta\dashv b'+b''}} m(\alpha'',\alpha\backslash\alpha')\, m(\beta'',\beta\backslash\beta')\, \mathrm{Ind}_{W_{a+a',b+b'}}^{W_{\tilde{n}}}(E_\alpha\otimes\varphi E_\beta),\end{aligned}$$

the latter by using the Littlewood-Richardson rule. □

Corollary 3.3. *We keep the notations of proposition 3.2. Then the following hold.*

(a) *Both the representations $E_{\alpha'\cup\alpha'',\beta'\cup\beta''}$ and $E_{\alpha'+\alpha'',\beta'+\beta''}$ occur with multiplicity one in the induced representation*

$$I(\alpha',\beta';\alpha'',\beta'') := \mathrm{Ind}_{W_{\tilde{n}'}\times W_{\tilde{n}''}}^{W_{\tilde{n}}}(E_{\alpha',\beta'}\otimes E_{\alpha'',\beta''}).$$

(b) *Let α, β be partitions of a, b respectively. If $E_{\alpha,\beta}$ occurs in $I(\alpha',\beta';\alpha'',\beta'')$, then we have $\alpha'\cup\alpha''\le\alpha\le\alpha'+\alpha''$ and $\beta'\cup\beta''\le\beta\le\beta'+\beta''$.*

Proof. For the facts that $E_{\alpha'\cup\alpha'',\beta'\cup\beta''}$ occurs with multiplicity one in representation $I(\alpha',\beta';\alpha'',\beta'')$, and that $\alpha'\cup\alpha''\le\alpha$ and $\beta'\cup\beta''\le\beta$ for any $E_{\alpha,\beta}$ occuring in $I(\alpha',\beta';\alpha'',\beta'')$, see [25, VIII.3.(2)].

On the other hand, it follows from Proposition 3.2 that the multiplicity of $E_{\alpha,\beta}$ in $I(\alpha',\beta';\alpha'',\beta'')$ equals $m(\alpha'',\alpha\backslash\alpha')\,m(\beta'',\beta\backslash\beta')$. By applying Proposition 3.1 (a) both to $\mathfrak{S}_{\tilde{n}'}$ and $\mathfrak{S}_{\tilde{n}''}$, we get

$$m(\alpha'',\alpha'+\alpha''\backslash\alpha') = m(\beta'',\beta'+\beta''\backslash\beta') = 1;$$

hence the multiplicity of $E_{\alpha'+\alpha'',\beta'+\beta''}$ in $I(\alpha',\beta';\alpha'',\beta'')$ equals 1.

We shall now use Proposition 3.1 (b) in order to prove (b). We assume that $E_{\alpha,\beta}$ occurs in $I(\alpha',\beta';\alpha'',\beta'')$. Then both $m(\alpha'',\alpha\backslash\alpha')\neq 0$, $m(\beta'',\beta\backslash\beta')\neq 0$. By the Littlewood-Richardson rule, it follows that E_α and E_β respectively occurs in $I(\alpha',\alpha'')$ and in $I(\beta',\beta'')$. Then Proposition 3.1 (b) implies that $\alpha\le\alpha'+\alpha''$ and $\beta\le\beta'+\beta''$. □

3.3. Symbols associated with irreducible characters of Weyl groups.

3.3.1. *Definition of symbols.* We recall now the definition of symbols, following Lusztig [7]. A symbol is an (unordered) pair $\binom{S}{T}$ of finite subsets of $\{0,1,2,\dots\}$, modulo the equivalence relation generated by the shift operation $\binom{S}{T} \sim \binom{S'}{T'}$, where $S' = \{0\} \cup (S+1)$, $T' = \{0\} \cup (T+1)$. For any real number x, we denote by $[x]$ the largest integer k such that $k \leq x$. The rank $\mathrm{rk}(\Lambda)$ of a symbol $\Lambda = \binom{S}{T}$ is defined by

$$\mathrm{rk}(\Lambda) := \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left[\left(\frac{|S| + |T| - 1}{2} \right)^2 \right]. \tag{3.18}$$

The defect $\mathrm{def}(\Lambda)$ of Λ is by definition the absolute value of $|S| - |T|$. The rank and the defect are unchanged by the shift operation. We denote by $\Phi_{\tilde{n},d}$ the set of symbols of rank $\tilde{n}$ and defect $d \geq 1$. In the case of symbols of defect zero, a symbol $\Lambda = \binom{S}{T}$ is said to be degenerate if $S = T$, and is said to be non-degenerate if $S \neq T$. We denote by $\Phi'_{\tilde{n},0}$ the set of symbols of rank $\tilde{n}$ and defect zero, where degenerate symbols are counted twice. Let $\Phi_{\tilde{n}}$ be the set of symbols of rank $\tilde{n}$ and odd defect.

Lusztig has defined in [7, Lemma 2.7], natural bijections

$$\mathrm{Irr}(W_{\tilde{n}}) \simeq \Phi_{\tilde{n},1}, \qquad \mathrm{Irr}(W'_{\tilde{n}}) \simeq \Phi'_{\tilde{n},0} \tag{3.19}$$

as follows:

$(W_{\tilde{n}})$ Each irreducible representation $E = E_{\alpha,\beta}$ of $W_{\tilde{n}}$ is parametrized by an ordered pair of partitions (α,β) with that is $|\alpha| + |\beta| = \tilde{n}$, see [2, 11.4.2]; by allowing 0 among the entries of α, β, we may assume that the partitions α, β have the following form:

$$\alpha = (0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m), \quad \beta = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m)$$

for some positive integer m such that $\sum_{i=0}^m \alpha_i + \sum_{j=1}^m \beta_j = \tilde{n}$; then we define $S := \{\lambda_0, \lambda_1, \dots, \lambda_m\}$, $T := \{\mu_1, \mu_2, \dots, \mu_m\}$ where $\lambda_i = \alpha_i + i$, $\mu_j = \beta_j + j - 1$. Then $\Lambda_1(E) = \binom{S}{T}$ belongs to $\Phi_{\tilde{n},1}$ and we define a bijection

$$\Sigma_{\tilde{n}} : \mathrm{Irr}(W_{\tilde{n}}) \to \Phi_{\tilde{n},1} \tag{3.20}$$

by setting $\Sigma_{\tilde{n}}(E) := \Lambda_1(E)$. We will set $E(\Lambda_1) := \Sigma_{\tilde{n}}^{-1}(\Lambda_1)$.

$(W'_{\tilde{n}})$ Each irreducible representation E' of $W'_{\tilde{n}}$ is parametrized by an unordered pair of partitions (α,β) with $|\alpha| + |\beta| = \tilde{n}$, which is counted twice if $\alpha = \beta$, see [2, 11.4.4]; we express α, β as

$$\alpha = (0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m), \beta = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m)$$

for some integer $m \geq 1$, and define

$$S := \{\lambda_1, \lambda_2, \dots, \lambda_m\}, \ T := \{\mu_1, \mu_2, \dots, \mu_m\}$$

by $\lambda_i = \alpha_i + i - 1$, $\mu_j = \beta_j + j - 1$. Then $\Lambda_0(E') = \binom{S}{T}$ belongs to $\Phi'_{\tilde{n},0}$ and we define a bijection

$$\Sigma'_{\tilde{n}} : \operatorname{Irr}(W'_{\tilde{n}}) \to \Phi'_{\tilde{n},0} \tag{3.21}$$

by setting $\Sigma'_{\tilde{n}}(E') := \Lambda_0(E')$. We will set $E'(\Lambda_0) := \Sigma'^{-1}_{\tilde{n}}(\Lambda_0)$.

3.3.2. *Pairs of partitions associated with symbols.*

$(W_{\tilde{n}})$ We associate with any symbol

$$\Lambda_1 := \begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \in \Phi_{\tilde{n},1}$$

two partitions α^{Λ_1}, β^{Λ_1} of the following form

$$\alpha^{\Lambda_1} = (0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m), \quad \beta^{\Lambda_1} = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m)$$

such that $\sum_{i=0}^m \alpha_i + \sum_{j=1}^m \beta_j = \tilde{n}$, by setting

$$\alpha_i := \lambda_i - i, \;\; \beta_j := \mu_j - j + 1.$$

We have $E_{\alpha^{\Lambda_1}, \beta^{\Lambda_1}} = E(\Lambda_1)$ with the notation of Section 3.3.1.

$(W'_{\tilde{n}})$ We associate with any symbol

$$\Lambda_0 := \begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \in \Phi'_{\tilde{n},0}$$

two partitions α^{Λ_0}, β^{Λ_0} of the following form

$$\alpha^{\Lambda_0} = (0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m), \quad \beta^{\Lambda_0} = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m)$$

such that $\sum_{i=1}^m \alpha_i + \sum_{j=1}^m \beta_j = \tilde{n}$, by setting

$$\alpha_i := \lambda_i - i + 1, \;\; \beta_j := \mu_j - j + 1.$$

We have $E_{\alpha^{\Lambda_0}, \beta^{\Lambda_0}} = E(\Lambda_0)$ with the notation of Section 3.3.1.

3.3.3. *Orders on symbols.* We will write:

- $\Lambda_1 \underset{\Phi_{\tilde{n},1}}{\leq} \widetilde{\Lambda}_1$, for $\Lambda_1, \widetilde{\Lambda}_1 \in \Phi_{\tilde{n},1}$, if both $\alpha^{\Lambda_1} \leq \alpha^{\widetilde{\Lambda}_1}$, and $\beta^{\Lambda_1} \leq \beta^{\widetilde{\Lambda}_1}$; hence,

$$\Lambda_1 = \begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \underset{\Phi_{\tilde{n},1}}{\leq} \begin{pmatrix} \tilde{\lambda}_0 < \tilde{\lambda}_1 < \cdots < \tilde{\lambda}_m \\ \tilde{\mu}_1 < \tilde{\mu}_2 < \cdots < \tilde{\mu}_m \end{pmatrix} = \widetilde{\Lambda}_1$$

if and only if

$$\begin{cases} \lambda_i + \lambda_{i+1} + \cdots + \lambda_m \leq \tilde{\lambda}_i + \tilde{\lambda}_{i+1} + \cdots + \tilde{\lambda}_m, & \text{for any } i \in \{0, 1, \ldots, m\}, \\ \mu_j + \mu_{j+1} + \cdots + \mu_m \leq \tilde{\mu}_j + \tilde{\mu}_{j+1} + \cdots + \tilde{\mu}_m, & \text{for any } j \in \{1, 2, \ldots, m\}. \end{cases}$$

- $\Lambda_0 \underset{\Phi'_{\tilde{n},0}}{\leq} \widetilde{\Lambda}_0$, for $\Lambda_0, \widetilde{\Lambda}_0 \in \Phi'_{\tilde{n},0}$, if both $\alpha^{\Lambda_0} \leq \alpha^{\widetilde{\Lambda}_0}$, and $\beta^{\Lambda_0} \leq \beta^{\widetilde{\Lambda}_0}$; hence,

$$\Lambda_0 = \begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \underset{\Phi'_{\tilde{n},0}}{\leq} \begin{pmatrix} \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_m \\ \tilde{\mu}_1 < \tilde{\mu}_2 < \cdots < \tilde{\mu}_m \end{pmatrix} = \widetilde{\Lambda}_0$$

if and only if

$$\begin{cases} \lambda_i + \lambda_{i+1} + \cdots + \lambda_m \leq \tilde{\lambda}_i + \tilde{\lambda}_{i+1} + \cdots + \tilde{\lambda}_m, & \text{for any } i \in \{1, 2, \ldots, m\}, \\ \mu_j + \mu_{j+1} + \cdots + \mu_m \leq \tilde{\mu}_j + \tilde{\mu}_{j+1} + \cdots + \tilde{\mu}_m, & \text{for any } j \in \{1, 2, \ldots, m\}. \end{cases}$$

3.3.4. *From defect* 1 *to defect* d *for symbols.* Recall that for the symbols with higher defects in $\Phi_{\tilde{n}}$ we have the following relation (see [7, Proposition 3.2]). Let d an integer ≥ 1. Let

$$\Lambda_1 := \begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \in \Phi_{n,1}.$$

The correspondence

$$\Lambda_1 \mapsto \begin{pmatrix} 0 < 1 < \cdots < d-2 < \lambda_0 + d - 1 < \lambda_1 + d - 1 < \cdots < \lambda_m + d - 1 \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix}$$

defines a bijection

$$\delta_{\tilde{n},d} : \Phi_{\tilde{n},1} \to \Phi_{\tilde{n}+[(\frac{d}{2})^2],d}. \tag{3.22}$$

In particular we have bijections

$$\delta_{n-t(t+1),2t+1} : \Phi_{n-t(t+1),1} \to \Phi_{n,2t+1} \quad \text{and} \quad \delta_{n-4t^2,4t} : \Phi_{n-4t^2,1} \to \Phi_{n,4t}. \tag{3.23}$$

Note that the order we have defined on the set $\Phi_{\tilde{n}}$ is compatible with the map $\delta_{\tilde{n},d}$, that is,

$$\Lambda_1 \leq \tilde{\Lambda}_1 \quad \text{if and only if} \quad \delta_{\tilde{n},d}(\Lambda_1) \leq \delta_{\tilde{n},d}(\tilde{\Lambda}_1). \tag{3.24}$$

3.4. **Symbols and induced representations of Weyl groups.**

3.4.1. *Type* B. Let $\tilde{n}'$, $\tilde{n}''$ such that $\tilde{n}' + \tilde{n}'' = \tilde{n}$, and let

$$\Lambda_1' := \begin{pmatrix} \lambda_0' < \lambda_1' < \cdots < \lambda_m' \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix} \in \Phi_{\tilde{n}',1}, \tag{3.25}$$

$$\Lambda_1'' := \begin{pmatrix} \lambda_0'' < \lambda_1'' < \cdots < \lambda_m'' \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix} \in \Phi_{\tilde{n}'',1}. \tag{3.26}$$

We set

$$I(\Lambda_1', \Lambda_1'') := \operatorname{Ind}_{W_{\tilde{n}'} \times W_{\tilde{n}''}}^{W_{\tilde{n}}} \left(E(\Lambda_1') \otimes E(\Lambda_1'') \right), \tag{3.27}$$

that is $I(\Lambda_1', \Lambda_1'') = I\left(\alpha^{\Lambda_1'}, \beta^{\Lambda_1'}; \alpha^{\Lambda_1''}, \beta^{\Lambda_1''}\right)$ in the notation of Corollary 3.3.

We then set

$$\Lambda_1' + \Lambda_1'' := \begin{pmatrix} \lambda_0' + \lambda_0'' < \lambda_1' + \lambda_1'' - 1 < \cdots < \lambda_m' + \lambda_m'' - m \\ \mu_1' + \mu_1'' < \mu_2' + \mu_2'' - 1 < \cdots < \mu_{\tilde{m}}' + \mu_{\tilde{m}}'' - m + 1 \end{pmatrix} \in \Phi_{\tilde{n},1}. \tag{3.28}$$

We have

$$\alpha_i^{\Lambda_1' + \Lambda_1''} = (\lambda_i' - i) + (\lambda_i'' - i) = \alpha_i^{\Lambda_1'} + \alpha_i^{\Lambda_1''},$$
$$\beta_j^{\Lambda_1' + \Lambda_1''} = (\mu_j' - j + 1) + (\mu_j'' - j + 1) = \beta_j^{\Lambda_1'} + \beta_j^{\Lambda_1''}.$$

The following proposition is a reformulation of Corollary 3.3 in terms on symbols instead of pairs of partitions.

Proposition 3.4. *Let* $\Lambda_1' \in \Phi_{\tilde{n}',1}$, $\Lambda_1'' \in \Phi_{\tilde{n}'',1}$, $\Lambda_1 \in \Phi_{\tilde{n},1}$. *Then the following hold.*

(a) *The representation* $E(\Lambda'_1+\Lambda''_1)$ *occurs with multiplicity one in* $I(\Lambda'_1,\Lambda''_1)$.
(b) *If* $E(\Lambda_1)$ *occurs in* $I(\Lambda'_1,\Lambda''_1)$, *then* $\Lambda_1 \leq \Lambda'_1+\Lambda''_1$.

3.4.2. *Type D*. Let $\tilde n'$, $\tilde n''$ such that $\tilde n'+\tilde n''=\tilde n$, and let

$$\Lambda'_0 := \begin{pmatrix} \lambda'_1<\lambda'_2<\cdots<\lambda'_m \\ \mu'_1<\mu'_2<\cdots<\mu'_m \end{pmatrix} \in \Phi'_{\tilde n',0}, \tag{3.29}$$

$$\Lambda''_0 := \begin{pmatrix} \lambda''_1<\lambda''_2<\cdots<\lambda''_m \\ \mu''_1<\mu''_2<\cdots<\mu''_m \end{pmatrix} \in \Phi'_{\tilde n'',0}. \tag{3.30}$$

We set

$$I'(\Lambda'_0,\Lambda''_0) := \mathrm{Ind}_{W'_{\tilde n'}\times W'_{\tilde n''}}^{W'_{\tilde n}}\left(E(\Lambda'_0)\otimes E(\Lambda''_0)\right). \tag{3.31}$$

We then set

$$\Lambda'_0+\Lambda''_0 := \begin{pmatrix} \lambda'_1+\lambda''_1<\lambda'_2+\lambda''_2-1<\cdots<\lambda'_m+\lambda''_m-m+1 \\ \mu'_1+\mu''_1<\mu'_2+\mu''_2-1<\cdots<\mu'_m+\mu''_m-m+1 \end{pmatrix} \in \Phi'_{\tilde n,0}. \tag{3.32}$$

We have

$$\alpha_i^{\Lambda'_0+\Lambda''_0} = (\lambda'_i-i+1)+(\lambda''_i-i+1) = \alpha_i^{\Lambda'_0}+\alpha_i^{\Lambda''_0},$$
$$\beta_j^{\Lambda'_0+\Lambda''_0} = (\mu'_j-j+1)+(\mu''_j-j+1) = \beta_j^{\Lambda'_0}+\beta_j^{\Lambda''_0}.$$

Proposition 3.5. *Let* $\Lambda'_0\in\Phi'_{\tilde n',0}$, $\Lambda''_0\in\Phi'_{\tilde n'',0}$, $\Lambda_0\in\Phi'_{\tilde n,0}$. *Then the following hold.*

(a) *The representation* $E(\Lambda'_0+\Lambda''_0)$ *occurs with multiplicity one in* $I'(\Lambda'_0,\Lambda''_0)$.
(b) *If* $E(\Lambda_0)$ *occurs in* $I'(\Lambda'_0,\Lambda''_0)$, *then* $\Lambda_0\leq\Lambda'_0+\Lambda''_0$.

Proof. Let $\Lambda'_0\in\Phi'_{\tilde n',0}$, $\Lambda''_0\in\Phi'_{\tilde n'',0}$. Then the irreducible representation $E(\Lambda'_0)$ of the group $W'_{\tilde n'}$ occurs in the restriction from $W_{\tilde n'}$ to $W'_{\tilde n'}$ of the representation $E(\Lambda'_1)$, where

$$\Lambda'_1 := \begin{pmatrix} 0<\lambda'_1+1<\cdots<\lambda'_m+1 \\ \mu'_1<\mu'_2<\cdots<\mu'_m \end{pmatrix} \in \Phi_{\tilde n',1}.$$

Similarly, the irreducible representation $E(\Lambda''_0)$ of the group $W'_{\tilde n''}$ occurs in the restriction from $W_{\tilde n''}$ to $W'_{\tilde n''}$ of $E(\Lambda''_1)$, where

$$\Lambda''_1 := \begin{pmatrix} 0<\lambda''_1+1<\cdots<\lambda''_m+1 \\ \mu''_1<\mu''_2<\cdots<\mu''_m \end{pmatrix} \in \Phi_{\tilde n'',1}.$$

We set $E:=E(\Lambda'_1)\otimes E(\Lambda''_1)$. We have

$$(*) \qquad \mathrm{Ind}_{W'_{\tilde n'}\times W'_{\tilde n''}}^{W'_{\tilde n}}(\mathrm{Res}_{W'_{\tilde n'}\times W'_{\tilde n''}}^{W_{\tilde n'}\times W_{\tilde n''}}(E)) \simeq \mathrm{Res}_{W'_{\tilde n}}^{W_{\tilde n}}(\mathrm{Ind}_{W_{\tilde n'}\times W_{\tilde n''}}^{W_{\tilde n}}(E)).$$

We will first prove (b). Let $\Lambda_0\in\Phi'_{\tilde n,0}$ such that $E(\Lambda_0)$ occurs in $I'(\Lambda'_0,\Lambda''_0)$. Since $E(\Lambda'_0)\otimes E(\Lambda''_0)$ occurs in the restriction from $W_{\tilde n'}\times W_{\tilde n''}$ to $W'_{\tilde n'}\times W'_{\tilde n''}$ of E, the representation $E(\Lambda_0)$ occurs in $\mathrm{Ind}_{W'_{\tilde n'}\times W'_{\tilde n''}}^{W'_{\tilde n}}(\mathrm{Res}_{W'_{\tilde n'}\times W'_{\tilde n''}}^{W_{\tilde n'}\times W_{\tilde n''}}(E))$. Using $(*)$ and Proposition 3.1, it follows that $E(\Lambda_0)$ occurs in the restriction to $W'_{\tilde n}$ of some irreducible representation $E(\Lambda_1)$ of $W_{\tilde n}$ with

$$m(\alpha^{\Lambda''_1},\alpha^{\Lambda_1}\backslash\alpha^{\Lambda'_1})\, m(\beta^{\Lambda''_1},\beta^{\Lambda_1}\backslash\beta^{\Lambda'_1}) \neq 0.$$

Now, note that $\alpha_0^{\Lambda_1'} = 0$ and $\alpha_i^{\Lambda_1'} = (\lambda_i'+1)-i = \lambda_i'-i+1 = \alpha_i^{\Lambda_0'}$, $\beta_j^{\Lambda_1'} = \mu_j'-j+1 = \beta_j^{\Lambda_0'}$, $\alpha_i^{\Lambda_1''} = \alpha_i^{\Lambda_0''}$, $\beta_j^{\Lambda_1''} = \beta_j^{\Lambda_0''}$ $(i,j \in \{1,2,\dots,m\})$. Then the result follows from Proposition 3.1 (b).

We will now prove (a). Using Proposition 3.4, we see that the restriction to $W_{\tilde{n}}'$ of the irreducible representation $E(\Lambda_1'+\Lambda_1'')$ of $W_{\tilde{n}}$ occurs in $I'(\Lambda_0',\Lambda_0'')$. We shall show that it occurs with multiplicity one. Let $E(\widetilde{\Lambda}_1)$ be an irreducible representation of $W_{\tilde{n}}$ which occurs in $I(\Lambda_1',\Lambda_1'')$ and which has the same restriction to $W_{\tilde{n}}'$ as $E(\Lambda_1'+\Lambda_1'')$. Then we have

$$E(\widetilde{\Lambda}_1) = \begin{pmatrix} \mu_1'+\mu_1'' < \mu_2'+\mu_2'' < \cdots < \mu_m'+\mu_m'' \\ 0 < \lambda_1'+\lambda_1''+2 < \cdots < \lambda_m'+\lambda_m''+2 \end{pmatrix}.$$

Since $E(\widetilde{\Lambda}_1)$ occurs in $I(\Lambda_1',\Lambda_1'')$, we get $\Lambda_1 \leq \Lambda_1'+\Lambda_1''$, by proposition 3.4, hence $\Lambda_0 \leq \Lambda_0'+\Lambda_0''$. □

3.5. u–**Symbols.** We will now recall some combinatorial objects which have been introduced by Lusztig in [13] in order to parametrize the $\mathcal{N}_G$ when G is of classical type.

3.5.1. *Definitions.* For an integer $n \geq 1$, let Ψ_{2n} be the set of all pairs $\begin{pmatrix} A \\ B \end{pmatrix}$, called *$u$–symbols*, where A is a finite subset of $\{0,1,2,\dots\}$, B a finite subset of $\{1,2,3\dots\}$ subject to the condition that

(1) A, B contain no consecutive integers;

(2)
$$\sum_{a\in A} a + \sum_{b\in B} b = n + \frac{(|A|+|B|)(|A|+|B|-1)}{2};$$

(3) $|A|+|B|$ is odd;

the pairs are taken modulo the equivalence relation generated by the shift operation $\begin{pmatrix} A \\ B \end{pmatrix} \sim \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix}$ if $\tilde{A} = \{0\} \cup (A+2)$, $\tilde{B} = \{1\} \cup (B+2)$.

Next, for any integer $N \geq 3$, let Ψ_N' be the set of (unordered) pairs $\begin{pmatrix} A' \\ B' \end{pmatrix}$, also called *$u$–symbols*, where A' and B' are finite subsets of $\{0,1,2,\dots\}$ subject to the condition that

(1′) A', B' contain no consecutive integers,

(2′)
$$\sum_{a'\in A'} a' + \sum_{b'\in B'} b' = \frac{N}{2} + \frac{(|A'|+|B'|-1)^2-1}{2}$$

(which implies that $|A'|+|B'| \equiv N \mod 2$);

the pairs are taken modulo the equivalence relation generated by the shift operation $\begin{pmatrix} A' \\ B' \end{pmatrix} \sim \begin{pmatrix} \tilde{A}' \\ \tilde{B}' \end{pmatrix}$ if $\tilde{A}' = \{0\} \cup (A'+2)$, $\tilde{B}' = \{0\} \cup (B'+2)$. Now for each $\theta = \begin{pmatrix} A \\ B \end{pmatrix} \in \Psi_N$ (resp. $\theta = \begin{pmatrix} A' \\ B' \end{pmatrix} \in \Psi_N'$), we define the defect $\mathrm{def}(\theta)$ of θ by $\mathrm{def}(\theta) = |A|-|B|$ (resp. the absolute value of $|A'|-|B'|$). An element $\binom{A'}{B'}$ of Ψ_N' of zero defect is said to be *degenerate* if $A' = B'$. We denote by $\Psi_{N,0}'$ the subset of u–symbols of zero defect in Ψ'_N, where the degenerate u–symbols are counted twice. We denote by $\Psi_{N,d}$ (resp. $\Psi'_{N,d}$, with $d \neq 0$) the subset of u–symbols in Ψ_N (resp. Ψ'_N) of defect equal to d.

3.6. If $G = \mathrm{PSp}_{2n}$, the sets there exists a bijective map $\psi^{\mathrm{PSp}_{2n}} \colon \Psi_{2n} \longrightarrow \mathcal{N}_G$, see [13, (11.6.1)]. If $G = \mathrm{SO}_N$, we have a map $\psi^{\mathrm{SO}_N} \colon \Psi'_N \longrightarrow \mathcal{N}_G$ which is bijective over the set of non-degenerate u–symbols in Ψ'_N and is such for each degenerate u–symbol in Ψ'_N its fiber has two elements, see [13, (11.7.3)].

3.6.1. *Pairs of partitions associated to u–symbols.*

$\Psi_{2\tilde{n},1}$: We associate with any u–symbol

$$\theta_1 \;:=\; \begin{pmatrix} a_0 < a_1 < \cdots < a_m \\ b_1 < b_2 < \cdots < b_m \end{pmatrix} \;\in \Psi_{2\tilde{n},1}$$

two partitions α^{θ_1}, β^{θ_1} of the following form

$$\alpha^{\theta_1} = (0 \le \alpha_0^{\theta_1} \le \alpha_1^{\theta_1} \le \cdots \le \alpha_m^{\theta_1}),\ \beta^{\theta_1} = (\beta_1^{\theta_1} \le \beta_2^{\theta_1} \le \cdots \beta_m^{\theta_1}),$$

such that

$$\sum_{i=0}^{m} \alpha_i^{\theta_1} \;+\; \sum_{j=1}^{m} \beta_j^{\theta_1} \;=\; n + m^2 + m,$$

by setting $\alpha_i^{\theta_1} := a_i - i$, $\beta_j^{\theta_1} := b_j - j + 1$.

$\Psi'_{2\tilde{n}+1,1}$: We associate with any u–symbol

$$\theta'_1 \;:=\; \begin{pmatrix} a'_0 < a'_1 < \cdots < a'_m \\ b'_1 < b'_2 < \cdots < b'_m \end{pmatrix} \;\in \Psi'_{2\tilde{n}+1,1}$$

two partitions $\alpha^{\theta'_1}$, $\beta^{\theta'_1}$ of the following form

$$\alpha^{\theta'_1} = (0 \le \alpha_0^{\theta'_1} \le \alpha_1^{\theta'_1} \le \cdots \le \alpha_m^{\theta'_1}),\ \beta^{\theta'_1} = (\beta_1^{\theta'_1} \le \beta_2^{\theta'_1} \le \cdots \beta_m^{\theta'_1}),$$

such that

$$\sum_{i=0}^{m} \alpha_i^{\theta'_1} \;+\; \sum_{j=1}^{m} \beta_j^{\theta'_1} \;=\; n + m^2,$$

by setting $\alpha_i^{\theta'_1} := a'_i - i$, $\beta_j^{\theta'_1} := b'_j - j + 1$.

$\Psi'_{2n,0}$: We associate with any u–symbol

$$\theta'_0 \;:=\; \begin{pmatrix} a'_1 < a'_2 < \cdots < a'_m \\ b'_1 < b'_2 < \cdots < b'_m \end{pmatrix} \;\in \Psi'_{2n,0}$$

two partitions $\alpha^{\theta'_0}$, $\beta^{\theta'_0}$ of the following form

$$\alpha^{\theta'_0} = (0 \le \alpha_1^{\theta'_0} \le \alpha_2^{\theta'_0} \le \cdots \le \alpha_m^{\theta'_0}),\ \beta^{\theta'_0} = (\beta_1^{\theta'_0} \le \beta_2^{\theta'_0} \le \cdots \beta_m^{\theta'_0}),$$

such that

$$\sum_{i=1}^{m} \alpha_i^{\theta'_0} \;+\; \sum_{j=1}^{m} \beta_j^{\theta'_0} \;=\; n + m^2 + m,$$

by setting $\alpha_i^{\theta'_0} := a_i - i + 1$, $\beta_j^{\theta'_0} := b_j - j + 1$.

3.6.2. *Orders on u–symbols.* We will write

- $\theta_1 \underset{\Psi_{2\tilde{n},1}}{\leq} \widetilde{\theta}_1$, for $\theta_1, \tilde{\theta}_1 \in \Psi_{2\tilde{n},1}$, if both $\alpha^{\theta_1} \leq \alpha^{\tilde{\theta}_1}$ and $\beta^{\theta_1} \leq \beta^{\tilde{\theta}_1}$; hence

$$\theta_1 = \begin{pmatrix} a_0 < a_1 < \cdots < a_m \\ b_1 < b_2 < \cdots < b_m \end{pmatrix} \underset{\Psi_{2\tilde{n},1}}{\leq} \begin{pmatrix} \tilde{a}_0 < \tilde{a}_1 < \cdots < \tilde{a}_m \\ \tilde{b}_1 < \tilde{b}_2 < \cdots < \tilde{b}_m \end{pmatrix} = \widetilde{\theta}_1$$

if and only if

$$\begin{cases} a_i + a_{i+1} + \cdots + a_m \leq \tilde{a}_i + \tilde{a}_{i+1} + \cdots + \tilde{a}_m, & \text{for any } i \in \{0, 1, \ldots, m\}, \\ b_j + b_{j+1} + \cdots + b_m \leq \tilde{b}_j + \tilde{b}_{j+1} + \cdots + \tilde{b}_m, & \text{for any } j \in \{1, 2, \ldots, m\}. \end{cases}$$

- $\theta'_1 \underset{\Psi'_{2\tilde{n}+1,1}}{\leq} \widetilde{\theta}'_1$, for $\theta'_1, \tilde{\theta}'_1 \in \Psi'_{2\tilde{n}+1,1}$, if both $\alpha^{\theta'_1} \leq \alpha^{\tilde{\theta}'_1}$ and $\beta^{\theta'_1} \leq \beta^{\tilde{\theta}'_1}$; hence

$$\theta'_1 = \begin{pmatrix} a'_0 < a'_1 < \cdots < a'_m \\ b'_1 < b'_2 < \cdots < b'_m \end{pmatrix} \underset{\Psi'_{2\tilde{n}+1,1}}{\leq} \begin{pmatrix} \tilde{a}'_0 < \tilde{a}'_1 < \cdots < \tilde{a}'_m \\ \tilde{b}'_1 < \tilde{b}'_2 < \cdots < \tilde{b}'_m \end{pmatrix} = \widetilde{\theta}'_1$$

if and only if

$$\begin{cases} a'_i + a'_{i+1} + \cdots + a'_m \leq \tilde{a}'_i + \tilde{a}'_{i+1} + \cdots + \tilde{a}'_m, & \text{for any } i \in \{0, 1, \ldots, m\}, \\ b'_j + b'_{j+1} + \cdots + b'_m \leq \tilde{b}'_j + \tilde{b}'_{j+1} + \cdots + \tilde{b}'_m, & \text{for any } j \in \{1, 2, \ldots, m\}. \end{cases}$$

- $\theta'_0 \underset{\Psi'_{2n,0}}{\leq} \widetilde{\theta}'_0$, for $\theta'_0, \tilde{\theta}'_0 \in \Psi'_{2n,0}$, if both $\alpha^{\theta'_0} \leq \alpha^{\tilde{\theta}'_0}$ and $\beta^{\theta'_0} \leq \beta^{\tilde{\theta}'_0}$; hence

$$\theta'_0 = \begin{pmatrix} a'_1 < a'_2 < \cdots < a'_m \\ b'_1 < b'_2 < \cdots < b'_m \end{pmatrix} \underset{\Psi'_{2n,0}}{\leq} \begin{pmatrix} \tilde{a}'_1 < \tilde{a}'_2 < \cdots < \tilde{a}'_m \\ \tilde{b}'_1 < \tilde{b}'_2 < \cdots < \tilde{b}'_m \end{pmatrix} = \widetilde{\theta}'_0$$

if and only if

$$\begin{cases} a'_i + a'_{i+1} + \cdots + a'_m \leq \tilde{a}'_i + \tilde{a}'_{i+1} + \cdots + \tilde{a}'_m, & \text{for any } i \in \{1, 2, \ldots, m\}, \\ b'_j + b'_{j+1} + \cdots + b'_m \leq \tilde{b}'_j + \tilde{b}'_{j+1} + \cdots + \tilde{b}'_m, & \text{for any } j \in \{1, 2, \ldots, m\}. \end{cases}$$

3.6.3. *From defect d to defect 1 for u–symbols.* Lusztig shows in [13, §12.2 and §13.2] that the structure of $\Psi_{N,d}$, $\Psi'_{N,d}$ is related to the case where $d = 1$ by the way of the following natural bijections:

$$\begin{aligned} \Delta_{N,d} : \; \Psi_{N,1} \to & \Psi_{N+d(d-1),d}, \\ \begin{pmatrix} A \\ B \end{pmatrix} \mapsto & \begin{pmatrix} \{0, 2, 4, \ldots, 2d-4\} \cup (A + 2d - 2) \\ B \end{pmatrix} \text{ if } d \geq 1, \\ \begin{pmatrix} A \\ B \end{pmatrix} \mapsto & \begin{pmatrix} A \\ \{1, 3, 5 \ldots, 1 - 2d\} \cup (B + 2 - 2d) \end{pmatrix} \text{ if } d \leq -1. \end{aligned}$$

and (assuming that $|A| > |B|$)

$$\begin{aligned} \Delta'_{N,d} : \; \Psi'_{N,1} \to & \Psi'_{N+d^2-1,d}, \\ \begin{pmatrix} A' \\ B' \end{pmatrix} \mapsto & \begin{pmatrix} \{0, 2, 4, \ldots, 2d-4\} \cup (A' + 2d - 2) \\ B' \end{pmatrix}. \end{aligned}$$

Let $\underset{\Psi_N}{\leq}$, $\underset{\Psi'_N}{\leq}$ be the orders on Ψ_N, Ψ'_N respectively induced by the orders defined on the set Ψ_N, Ψ'_N, by the maps $\Delta_{N,d}$, $\Delta'_{N,d}$. Let $\underset{\mathcal{N}_G}{\leq}$ be the order on $\mathcal{N}_G$ induced from $\underset{\Psi_N}{\leq}$, $\underset{\Psi'_N}{\leq}$ by the maps $\psi^{\mathrm{PSp}_{2n}}$, ψ^{SO_N} defined in 3.6.

3.6.4. *Combinatorial description of the Generalized Springer Correspondence.* We have partitions

$$\Psi_N = \bigcup_{\substack{d\in\mathbb{Z}\\ d \text{ odd}}} \Psi_{N,d}, \quad \Psi'_N = \bigcup_{\substack{d\geq 0\\ d\equiv N \pmod 2}} \Psi'_{N,d}. \tag{3.33}$$

By changing the variable $d' := d-1$ if $d \geq 1$, and $d' := -d$ if $d \leq -1$, we obtain a bijection

$$\Psi_N = \bigcup_{d'\geq 0} \Psi_{N-d'(d'+1),1}. \tag{3.34}$$

For the case of Ψ'_N, we have

$$\Psi'_N = \begin{cases} \displaystyle\bigcup_{\substack{d\geq 1\\ d \text{ odd}}} \Psi'_{N-d^2+1,1}, & \text{if } N \text{ odd},\\ \\ \displaystyle\Psi'^0_N \bigcup_{\substack{d\geq 2\\ d \text{ even}}} \Psi'_{N-d^2+1,1}, & \text{if } N \text{ even}. \end{cases} \tag{3.35}$$

Using the partitions above, and the bijections $\psi^{\mathrm{PSp}_{2n}}$, ψ^{SO_N} defined in 3.6, we will construct four bijections

$$\Theta_{\tilde n}\colon \mathrm{Irr}(W_{\tilde n}) \to \Psi_{2\tilde n,1}, \quad {}^t\Theta_{\tilde n}\colon \mathrm{Irr}(W_{\tilde n}) \to \Psi_{2\tilde n,1},$$
$$\Theta'_{\tilde n}\colon \mathrm{Irr}(W_{\tilde n}) \to \Psi'_{2\tilde n+1,1}, \quad \Theta'^0\colon \mathrm{Irr}(W'_n) \to \Psi'_{2n,0},$$

where in $\Psi'_{2\tilde n,0}$ the degenerate element are counted twice, such that

- $\nu^G = \psi^{\mathrm{PSp}_{2n}} \circ \Theta_{\tilde n}$, if $G = \mathrm{PSp}_{2n}$, and $\tilde n = n-(8t^2+2t)$, that is, $d = 4t+1 \geq 1$;
- $\nu^G = \psi^{\mathrm{PSp}_{2n}} \circ {}^t\Theta_{\tilde n}$, if $G = \mathrm{PSp}_{2n}$, and $\tilde n = n-(8t^2-2t)$, that is, $d = 1-4t \leq -1$;
- $\nu^G = \psi^{\mathrm{SO}_{2n+1}} \circ \Theta'_{\tilde n}$, if $G = \mathrm{SO}_{2n+1}$, and $\tilde n = n-2t^2-2t$, that is, $d = 2t+1$;
- $\nu^G = \psi^{\mathrm{SO}_{2n}} \circ \Theta'^0_n$, if $G = \mathrm{SO}_{2n}$, and $t = 0$, and $\nu^G = \psi^{\mathrm{SO}_{2n}} \circ \Theta'_{\tilde n}$, if $G = \mathrm{SO}_{2n}$, and $\tilde n = n - 8t^2$ with $t \neq 0$, that is $d = 4t \geq 2$.

$\Theta_{\tilde n}$: Let (α,β) be a pair of partitions of $\tilde n$. We denote by $E = E_{\alpha,\beta}$ the corresponding irreducible representation of $W_{\tilde n}$ and express α, β as in §3.3, that is: $\alpha = (0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m)$, $\beta = (0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m)$ for some integer $m \geq 1$. We set

$$\begin{aligned} A_\alpha &= \{\alpha_0 < \alpha_1 + 2 < \cdots < \alpha_m + 2m\},\\ B_\beta &= \{\beta_1 + 1 < \beta_2 + 3 < \cdots < \beta_m + 2m - 1\}. \end{aligned}$$

Then $\begin{pmatrix} A_\alpha \\ B_\beta \end{pmatrix} \in \Psi_{2\tilde n,1}$ and $E_{\alpha,\beta} \mapsto \begin{pmatrix} A_\alpha \\ B_\beta \end{pmatrix}$ gives the bijection $\Theta_{\tilde n}$.

Let $\sigma_{\tilde n} : \Phi_{\tilde n,1} \to \Psi_{2\tilde n,1}$ be the bijection defined by $\sigma_{\tilde n} := \Theta_{\tilde n} \circ \Sigma_{\tilde n}^{-1}$. We get

$$\sigma_{\tilde n}\begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} = \begin{pmatrix} \lambda_0 < \lambda_1 + 1 < \cdots < \lambda_m + m \\ \mu_1 + 1 < \mu_2 + 2 < \cdots < \mu_m + m \end{pmatrix}. \tag{3.36}$$

If $\theta_1 = \sigma_{\tilde{n}}(\Lambda_1)$, then we have $\alpha_i^{\theta_1} = \alpha_i^{\Lambda_1} + i$, for any $i \in \{0, 1, \dots, m\}$, $\beta_j^{\theta_1} = \beta_j^{\Lambda_1} + j$, for any $j \in \{1, 2, \dots, m\}$.

We get

$$\begin{aligned} \sum_{h=i}^{m} \alpha_h^{\theta_1} = & \sum_{h=i}^{m} (\alpha_h^{\Lambda_1} + h) = \sum_{h=i}^{m} \alpha_h^{\Lambda_1} + \left(\sum_{h=1}^{m} h - \sum_{h=1}^{m-1} h \right) \\ = & \sum_{h=i}^{m} \alpha_h^{\Lambda_1} + \frac{m(m+1)}{2} - \frac{i(i-1)}{2}, \end{aligned}$$

for any $i \in \{0, 1, \dots, m\}$,

$$\sum_{h=j}^{m} \beta_h^{\theta_1} = \sum_{h=j}^{m} \beta_h^{\Lambda_1} + \sum_{h=j}^{m} h = \sum_{h=j}^{m} \beta_h^{\Lambda_1} + \frac{m(m+1)}{2} - \frac{j(j-1)}{2},$$

for any $j \in \{1, 2, \dots, m\}$, It follows that, for any $\Lambda_1, \tilde{\Lambda}_1 \in \Phi_{\tilde{n},1}$, we have $\Lambda_1 \leq \tilde{\Lambda}_1$ if and only if $\sigma_{\tilde{n}}(\Lambda_1) \underset{\Psi_{2\tilde{n},1}}{\leq} \sigma_{\tilde{n}}(\tilde{\Lambda}_1)$.

${}^t\Theta_{\tilde{n}}$: We will also need later the following variant of $\Theta_{\tilde{n}}$. We define a bijection

$${}^t\Theta_{\tilde{n}} : \operatorname{Irr}(W_{\tilde{n}}) \to \Psi_{2\tilde{n},1}$$

by setting ${}^t\Theta_{\tilde{n}}(E_{\alpha,\beta}) := \Theta_{\tilde{n}}(E_{\alpha',\beta'})$ where $\alpha'_0 = \alpha'_1 := 0$, $\alpha'_2 := \beta_1$, $\alpha'_3 := \beta_2$, $\dots$, $\alpha'_{m+1} := \beta_m$, $\beta'_1 := \alpha_0$, $\beta'_2 := \alpha_1$, $\beta'_3 := \alpha_2$, $\dots$, $\beta'_{m+1} := \alpha_m$.
Let ${}^t\sigma_{\tilde{n}} : \Phi_{\tilde{n},1} \to \Psi_{2\tilde{n},1}$ be the bijection defined by ${}^t\sigma_{\tilde{n}} := {}^t\Theta_{\tilde{n}} \circ \Sigma_{\tilde{n}}^{-1}$. We get

$${}^t\sigma_{\tilde{n}} \begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} = \begin{pmatrix} 0 < 2 < \mu_1 + 4 < \mu_2 + 5 < \cdots < \mu_m + m + 3 \\ \lambda_0 + 1 < \lambda_1 + 2 < \cdots < \lambda_m + m + 1 \end{pmatrix}. \tag{3.37}$$

If $\theta_1 = {}^t\sigma_n(\Lambda_1)$, then we have $\alpha_0^{\theta_1} = 0$, $\alpha_1^{\theta_1} = 1$, $\alpha_i^{\theta_1} = \beta_{i-1}^{\Lambda_1} + i$ for any $i \in \{2, 3, \dots, m+1\}$, $\beta_j^{\theta_1} = \alpha_{j-1}^{\Lambda_1} + j$, for any $j \in \{1, 2, \dots, m+1\}$.

We get

$$\begin{aligned} \sum_{h=0}^{m+1} \alpha_h^{\theta_1} = & \sum_{h=1}^{m+1} \alpha_h^{\theta_1} = 1 + \sum_{h=2}^{m+1} \alpha_h^{\theta_1} = 1 + \sum_{h=2}^{m+1} (\beta_{h-1}^{\Lambda_1} + h) \\ = & \sum_{h=1}^{m} \beta_h^{\Lambda_1} + \sum_{h=1}^{m} h = \sum_{h=1}^{m} \beta_h^{\Lambda_1} + \frac{m(m+1)}{2}, \end{aligned}$$

$$\sum_{h=i}^{m+1} \alpha_h^{\theta_1} = \sum_{h=i}^{m} (\beta_h^{\Lambda_1} + h) = \sum_{h=i-1}^{m} \beta_h^{\Lambda_1} + \frac{(m+1)(m+2)}{2} - \frac{i(i-1)}{2},$$

for any $i \in \{2, 3, \dots, m\}$,

$$\sum_{h=j}^{m} \beta_h^{\theta_1} = \sum_{h=j}^{m} (\alpha_{h-1}^{\Lambda_1} + h) = \sum_{h=j-1}^{m-1} \alpha_h^{\Lambda_1} + \frac{m(m+1)}{2} - \frac{j(j-1)}{2},$$

for any $j \in \{1, 2, 3, \dots, m\}$. It follows that, for any $\Lambda_1, \tilde{\Lambda}_1 \in \Phi_{n,1}$, we have $\Lambda_1 \leq \tilde{\Lambda}_1$ if and only if ${}^t\sigma_{\tilde{n}}(\Lambda_1) \underset{\Psi_{2\tilde{n},1}}{\leq} {}^t\sigma_{\tilde{n}}(\tilde{\Lambda}_1)$.

$\Theta'_{\tilde n}$: Next we set

$$\begin{aligned} A'_\alpha &= \{\alpha_0 < \alpha_1 + 2 < \cdots < \alpha_m + 2m\}, \\ B'_\beta &= \{\beta_1 < \beta_2 + 2 < \cdots < \beta_m + 2m - 2\}. \end{aligned}$$

Then $\begin{pmatrix} A'_\alpha \\ B'_\beta \end{pmatrix} \in \Psi'_{2\tilde n+1,1}$ and $E_{\alpha,\beta} \mapsto \begin{pmatrix} A'_\alpha \\ B'_\beta \end{pmatrix}$ gives the bijection $\Theta'_{\tilde n}$.

Let $\sigma'_{\tilde n} : \Phi_{\tilde n,1} \to \Psi'_{2\tilde n+1,1}$ be the bijection defined by $\sigma'_{\tilde n} := \Theta'_{\tilde n} \circ \Sigma_{\tilde n}^{-1}$. We get

$$\sigma'_{\tilde n}\begin{pmatrix} \lambda_0 < \lambda_1 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} = \begin{pmatrix} \lambda_0 < \lambda_1 + 1 < \cdots < \lambda_m + m \\ \mu_1 < \mu_2 + 1 < \cdots < \mu_m + m - 1 \end{pmatrix}. \tag{3.38}$$

If $\theta'_1 = \sigma'_{\tilde n}(\Lambda_1)$, then we have $\alpha_i^{\theta'_1} = \alpha_i^{\Lambda_1} + i$, for any $i \in \{0, 1, \ldots, m\}$, $\beta_j^{\theta'_1} = \beta_j^{\Lambda_1} + j - 1$, for any $j \in \{1, 2, \ldots, m\}$.

We get

$$\sum_{h=i}^m \alpha_h^{\theta'_1} = \sum_{h=i}^m (\alpha_h^{\Lambda_1} + h) = \sum_{h=i}^m \alpha_h^{\Lambda_1} + \frac{m(m+1)}{2} - \frac{i(i-1)}{2},$$

for any $i \in \{0, 1, \ldots, m\}$,

$$\sum_{h=j}^m \beta_h^{\theta'_1} = \sum_{h=j}^m (\beta_h^{\Lambda_1} + h - 1) = \sum_{h=j}^m \beta_h^{\Lambda_1} + \frac{m(m-1)}{2} - \frac{(j-2)(j-1)}{2},$$

for any $j \in \{1, 2, \ldots, m\}$, It follows that, for any $\Lambda_1, \tilde\Lambda_1 \in \Phi_{\tilde n,1}$, we have $\Lambda_1 \le \tilde\Lambda_1$ if and only if $\sigma'_{\tilde n}(\Lambda_1) \underset{\Psi'_{2\tilde n+1,1}}{\le} \sigma'_{\tilde n}(\tilde\Lambda_1)$.

Θ'^0_n: We choose a pair (α, β) of partitions of n such that $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m$, $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_m$, and set

$$\begin{aligned} A'^0_\alpha &= \{\alpha_1 < \alpha_2 + 2 < \cdots < \alpha_m + 2m - 2\}, \\ B'^0_\beta &= \{\beta_1 < \beta_2 + 2 < \cdots < \beta_m + 2m - 2\}. \end{aligned}$$

Then $\begin{pmatrix} A'^0_\alpha \\ B'^0_\beta \end{pmatrix} \in \Psi'_{2n,0}$ and $E_{\alpha,\beta} \mapsto \begin{pmatrix} A'^0_\alpha \\ B'^0_\beta \end{pmatrix}$ gives the bijection Θ'^0_n.

Let $\sigma'^0_n : \Phi'_{n,0} \to \Psi'_{2n,0}$ be the bijection defined by $\sigma'^0_n := \Theta'_n \circ \Sigma'^{-1}_n$. We get

$$\sigma'^0_n\begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} = \begin{pmatrix} \lambda_1 < \lambda_2 + 1 < \cdots < \lambda_m + m - 1 \\ \mu_1 < \mu_2 + 1 < \cdots < \mu_m + m - 1 \end{pmatrix}. \tag{3.39}$$

If $\theta'_0 = \sigma'^0_n(\Lambda_0)$, the we have $\alpha_i^{\theta'_0} = \alpha_i^{\Lambda_0} + i - 1$, for any $i \in \{0, 1, \ldots, m\}$, $\beta_j^{\theta'_0} = \beta_j^{\Lambda_0} + j - 1$, for any $j \in \{1, 2, \ldots, m\}$.

We get

$$\sum_{h=i}^m \alpha_h^{\theta'_0} = \sum_{h=i}^m (\alpha_h^{\Lambda_0} + h - 1) = \sum_{h=i}^m \alpha_h^{\Lambda_0} + \frac{m(m-1)}{2} - \frac{(i-2)(i-1)}{2},$$

for any $i \in \{012, \dots, m\}$,

$$\sum_{h=j}^{m} \beta_h^{\theta'_0} = \sum_{h=j}^{m} (\beta_h^{\Lambda_0} + h - 1) = \sum_{h=j}^{m} \beta_h^{\Lambda_0} + \frac{m(m-1)}{2} - \frac{(j-2)(j-1)}{2},$$

for any $j \in \{1, 2, \dots, m\}$. It follows that, for any $\Lambda_0, \tilde{\Lambda}_0 \in \Phi'_{n,0}$, we have $\Lambda_0 \leq \tilde{\Lambda}_0$ if and only if $\sigma'^0_n(\Lambda_0) \underset{\Psi'_{2n,0}}{\leq} \sigma'^0_n(\tilde{\Lambda}_0)$.

3.7. u–**Symbols and induced representations of Weyl groups.** The following proposition is a reformulation of Proposition 3.4 (b), Proposition 3.5 (b) in terms on u–symbols, using (3.6.4).

Proposition 3.6. *Let $\tilde{n}'$, $\tilde{n}''$ such that $\tilde{n}' + \tilde{n}'' = \tilde{n}$.*

(a) *Let $\Lambda'_1 \in \Phi_{\tilde{n}',1}$, $\Lambda''_1 \in \Phi_{\tilde{n}'',1}$, $\Lambda_1 \in \Phi_{\tilde{n},1}$ such that $E(\Lambda_1)$ occurs in the induced representation $I(\Lambda'_1, \Lambda''_1)$. Then we have $\sigma_{\tilde{n}}(\Lambda_1) \underset{\Psi_{2\tilde{n},1}}{\leq} \sigma_{\tilde{n}}(\Lambda'_1 + \Lambda''_1)$, ${}^t\sigma_{\tilde{n}}(\Lambda_1) \underset{\Psi_{2\tilde{n},1}}{\leq} {}^t\sigma_{\tilde{n}}(\Lambda'_1 + \Lambda''_1)$, $\sigma'_{\tilde{n}}(\Lambda_1) \underset{\Psi'_{2\tilde{n}+1,1}}{\leq} \sigma'_{\tilde{n}}(\Lambda'_1 + \Lambda''_1)$.*

(b) *Let $\Lambda'_0 \in \Phi'_{\tilde{n}',0}$, $\Lambda''_0 \in \Phi_{\tilde{n}'',0'}$, $\Lambda_0 \in \Phi'_{\tilde{n},0}$ such that $E(\Lambda_0)$ occurs in the induced representation $I'(\Lambda'_0, \Lambda''_0)$. Then we have $\sigma'^0_{\tilde{n}}(\Lambda_0) \underset{\Psi'_{2\tilde{n},0}}{\leq} \sigma'^0_{\tilde{n}}(\Lambda'_0 + \Lambda''_0)$.*

Hence we have proved that the order $\underset{\mathcal{N}_G}{\leq}$ satifies the assertion (a) of Theorem 1.2. We will now prove that it also satisfies the assertion (b) of it.

3.8. **Similarities on symbols ans u–symbols.**

3.8.1. *Similarity on symbols.* Let Λ_1, Λ_2 be two symbols in Φ_n (resp. $\Phi'_{n,0}$). We say that Λ_1, Λ_2 are *similar* if they can be represented in the form $\binom{S_1}{T_1}$, $\binom{S_2}{T_2}$ so that $S_1 \cup T_1 = S_2 \cup T_2$, $S_1 \cap T_1 = S_2 \cap T_2$. We then write $\Lambda_1 \sim \Lambda_2$. This is an equivalence relation on Φ_n (resp. $\Phi'_{n,0}$) called *similarity*.

The set of symbols in a fixed similarity class of Φ_n (resp. $\Phi'_{n,0}$) can be organized in a natural way as an $\mathbb{F}_2$–vector space with a canonical nonsingular symplectic form (see [12, (4.5), (4.6)]).

3.8.2. *Similarity on u–symbols.* Let θ_1, θ_2 be two elements of Ψ'_{2n+1} (resp. Ψ_{2n}, Ψ'_{2n}). We say that θ_1, θ_2 are *similar* if they can be represented in the form $\binom{S_1}{T_1}$, $\binom{S_2}{T_2}$ so that $S_1 \cup T_1 = S_2 \cup T_2$, $S_1 \cap T_1 = S_2 \cap T_2$. We then write $\theta_1 \sim \theta_2$. This is an equivalence relation on Ψ'_{2n+1} (resp. Ψ_{2n}, Ψ'_{2n}) called *similarity*.

In each similarity class of Ψ'_{2n+1} (resp. Ψ_{2n}, Ψ'_{2n}) there is unique element wich can be represented by $\binom{A}{B}$ with

$$A = \{a_1 < a_2 < \dots < a_h\}, \quad B = \{b_1 < b_2 < \dots < b_l\}$$

such that the following holds:

$h = l + 1$, $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_h \leq b_h \leq a_{h+1}$, if $\binom{A}{B} \in \Psi'_{2n+1}$;
$h = l + 1$, $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_h \leq b_h \leq a_{h+1}$, if $\binom{A}{B} \in \Psi_{2n}$;
$h = l$, $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_h \leq b_h$, if $\binom{A}{B} \in \Psi'_{2n}$.

Such an element is said to be *distinguished*.

3.9. **From unipotent classes to distinguished u–symbols.**

3.9.1. *The symplectic group.* Assume that G is simple of adjoint type C_n. Let X_{2n} be the set of partitions of $2n$ such that any odd part occurs an even number of times, that is

$$X_{2n} = \left\{ \begin{matrix} \text{partition } \gamma = 0i_0 + 1i_1 + 2i_2 + 3i_3 + \cdots = 2n, \\ \text{with } i_0,\, i_1,\, i_2,\, i_3,\, \ldots \geq 0,\, i_1,\, i_3,\, i_5,\, \ldots \text{ even} \end{matrix} \right\}.$$

The unipotent classes of G are parametrized by the set X_{2n}.

Start with a partition

$$\gamma = (0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m) = (0i_0 + 1i_1 + 2i_2 + 3i_3 + \cdots)$$

in X_{2n}. By adding zeros we may arrange so that $\gamma_0 = 0$ and the number $m+1$ of parts of γ is odd, say $m + 1 = 2r + 1 = i_0 + i_1 + i_2 + \cdots$. Hence, we have $2r - (i_1 + i_2 + i_3 + \cdots) = i_0 - 1 \geq 0$, since $\gamma_0 = 0$. As in [13, 11.6], let $z_1 \leq z_2 \leq \cdots \leq z_{2r}$ be the sequence containing the number j exactly i_j times (for any $j \geq 1$) and the number 0 exactly $(i_0 - 1)$ times. Hence, we have

$$z_i = \gamma_i, \quad \text{for any } i \in \{1, 2, \ldots, 2r+1\},$$

$$z_1 = z_2 = \cdots = z_{i_0 - 1} = 0, \quad \text{if } i_0 \geq 2,$$

$$z_{i_0+i_1+\cdots+i_j} = z_{i_0+i_1+\cdots+i_j+1} = \cdots = z_{i_0+i_1+\cdots+i_j+i_{j+1}-1} = j+1,$$

for any $j \geq 0$. Let $z'_1 < z'_2 < \cdots < z'_{2r}$ be the sequence definred by $z'_i := z_i + (i-1)$. That sequence contains r even numbers $2y_1 < 2y_2 < \cdots < 2y_r$ and r odd numbers $2y'_1 + 1 < 2y'_2 + 1 < \cdots < 2y'_r + 1$. We set

$$\theta_1^\gamma := \begin{pmatrix} 0 < y'_1 + 2 < y'_2 + 3 < \cdots < y'_r + (r+1) \\ y_1 + 1 < y_2 + 2 < \ldots < y_r + r \end{pmatrix}.$$

Then θ_1^γ is a distinguished element in Ψ_{2n}.

We set

$$\begin{cases} c_0^\gamma := 0; & \\ c_i^\gamma := y_{\frac{i+1}{2}} + \dfrac{i+1}{2}, & \text{if } i \text{ odd;} \\ c_i^\gamma := y'_{\frac{i}{2}} + \dfrac{i}{2} + 1, & \text{if } i \text{ even.} \end{cases} \tag{a}$$

We get

$$\theta_1^\gamma = \begin{pmatrix} c_0^\gamma < c_2^\gamma < c_4^\gamma < \cdots, c_{2r}^\gamma \\ c_1^\gamma < c_3^\gamma < c_5^\gamma, \cdots < c_{2r-1}^\gamma \end{pmatrix}.$$

In a same way as Lusztig and Spaltenstein did in [18], we partition the sequence $(\gamma_0, \gamma_1, \ldots, \gamma_m)$ into blocks of length one or two such that all even γ_i lie in block of length one and all odd γ_i lie in blocks of length two. It can be easily checked that, for any $i \in \{0, 1, \ldots, 2r\}$,

$$c_i^\gamma = \frac{\gamma_i}{2} + i, \text{ if } \{\gamma_i\} \text{ is a block,} \tag{b}$$

and, for any $i \in \{0, 1, \ldots, 2r-1\}$,

$$\left. \begin{aligned} c_i^\gamma &= \frac{\gamma_i + 1}{2} + i, \\ c_{i+1}^\gamma &= \frac{\gamma_{i+1} + 1}{2} + i = c_i^\gamma, \end{aligned} \right\} \text{ if } \gamma_i,\, \gamma_{i+1} \text{ lie in one block.} \tag{c}$$

For any $h \in \{0, 1, \ldots, r\}$, $l \in \{1, 2, \ldots, r\}$, we set

$$a^\gamma(h) := c^\gamma_{2h} + c^\gamma_{2h+2} + \cdots + c^\gamma_{2r},$$

$$b^\gamma(l) := c^\gamma_{2l-1} + c^\gamma_{2l+1} + \cdots + c^\gamma_{2r-1},$$

and $b(r+1) := 0$. For any $i \in \{0, 1, \ldots, 2r+1\}$, we then set

$$c^\gamma(i) := \begin{cases} a^\gamma\left(\left[\frac{i}{2}\right]\right) + b^\gamma\left(\left[\frac{i}{2}\right]+1\right) & \text{if } i \text{ is even,} \\ \\ a^\gamma\left(\left[\frac{i}{2}\right]+1\right) + b^\gamma\left(\left[\frac{i}{2}\right]+1\right) & \text{if } i \text{ is odd.} \end{cases} \tag{f}$$

Hence, for any $h \in \{0, 1, \ldots, r\}$, $l \in \{1, 2, \ldots, r\}$, we have

$$c^\gamma(2h) = a^\gamma(h) + b^\gamma(h+1) = c^\gamma_{2h} + c^\gamma_{2h+1} + c^\gamma_{2h+2} + \cdots + c^\gamma_{2r-1} + c^\gamma_{2r},$$

and

$$c^\gamma(2l-1) = a^\gamma(l) + b^\gamma(l) = c^\gamma_{2l-1} + c^\gamma_{2l} + c^\gamma_{2l+1} + \cdots + c^\gamma_{2r-1} + c^\gamma_{2r}.$$

For any $i \in \{0, 1, 2, \ldots, 2r+1\}$, we set

$$\gamma(i) := \gamma_i + \gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_{2r-1} + \gamma_{2r}.$$

We set

$$d^\gamma_i := 2c^\gamma_i - 2i, \tag{h}$$

for $i \in \{0, 1, \ldots, 2r\}$. If $\{\gamma_i\}$ is a block, where $i \in \{0, 1, \ldots, 2r\}$, then $\gamma_i = d^\gamma_i$. If $\{\gamma_i, \gamma_{i+1}\}$ is a block, where $i \in \{0, 1, \ldots, 2r-1\}$, then

$$\begin{aligned} \gamma_i + \gamma_{i+1} &= (2c^\gamma_i - 2i - 1) + (2c^\gamma_{i+1} - 2i - 1) \\ &= (2c^\gamma_i - 2i) + (2c^\gamma_{i+1} - 2i - 2) \\ &= d^\gamma_i + d^\gamma_{i+1}. \end{aligned}$$

It follows that if $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks then

$$\gamma(i) = d^\gamma_i + d^\gamma_{i+1} + \cdots + d^\gamma_{2r}. \tag{i}$$

Lemma 3.7. *We have*

$$\gamma(i) = \begin{cases} 2c^\gamma(i) - 2r(2r+1) + i(i-1), & \textit{if } \{\gamma_i, \ldots, \gamma_{2r}\} \textit{ is a union of blocks,} \\ 2c^\gamma(i) - 2r(2r+1) + i(i-1) + 1, & \textit{otherwise.} \end{cases}$$

Proof. If $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks, using (i), we get

$$\gamma(i) = 2(c^\gamma_i + c^\gamma_{i+1} + \cdots + c^\gamma_{2r}) - 2\sum_{h=i}^{2r} h = 2c^\gamma(i) - 2\left(\sum_{h=i}^{2r} h - \sum_{h=i}^{2r} h\right),$$

and the result follows. If $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is not a union of blocks, then it implies that $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$ is union of blocks, that γ_i is odd, and that $\{\gamma_{i-1}, \gamma_i\}$ is a block. Applying the lemma to $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$, and using the fact that $\gamma_i = 2c^\gamma_i - 2(i-1) - 1 = 2c^\gamma_i - 2i + 1$, we get

$$\gamma(i) = \gamma_i + \gamma(i+1) = 2c^\gamma_i - 2i + 1 + 2c^\gamma(i+1) - 2r(2r+1) + (i+1)i,$$

that is, since $c^\gamma_i + c^\gamma(i+1) = c^\gamma(i)$,

$$\gamma(i) = 2c^\gamma(i) - 2r(2r+1) + i(i-1) + 1.$$

□

By definition of the order $\leq_{\Psi_{2n,1}}$, we have

$$\theta_1^\gamma \underset{\Psi_{2n,1}}{\leq} \theta_1^{\tilde\gamma} \text{ if and only if } \begin{cases} a^\gamma(h) \leq a^{\tilde\gamma}(h) & \text{for any } h \in \{0,1,\dots,r\}, \\ b^\gamma(l) \leq b^{\tilde\gamma}(l) & \text{for any } l \in \{1,2,\dots,r\}. \end{cases}$$

Lemma 3.8. *Let $\gamma \in X_{2n}$, $\tilde\gamma \in X_{2n}$. If $\theta_1^\gamma \underset{\Psi_{2n,1}}{\leq} \theta_1^{\tilde\gamma}$, then*

(a) $c^\gamma(i) \leq c^{\tilde\gamma}(i)$, *for any* $i \in \{0,1,2,\dots,2r\}$*;*
(b) $c_{i-1}^\gamma + c^\gamma(i+1) \leq c_{i-1}^{\tilde\gamma} + c^{\tilde\gamma}(i+1)$, *for any* $i \in \{1,2,\dots,2r-1\}$.

Proof. The first assertion follows from (f) and the definition of the order on X_{2n}. We will now prove (b). Let $i \in \{1,2,\dots,2r-1\}$.

(i) We assume first that i is odd. Using the definition of the order on X_{2n}, we then get

$$\begin{cases} a^\gamma\left(\frac{i-1}{2}\right) \leq a^{\tilde\gamma}\left(\frac{i-1}{2}\right), \\ b^\gamma\left(\frac{i+3}{2}\right) \leq b^{\tilde\gamma}\left(\frac{i+3}{2}\right). \end{cases} \quad (*)$$

Since

$$a^\gamma\left(\frac{i-1}{2}\right) = c_{i-1}^\gamma + a^\gamma\left(\frac{i+1}{2}\right),$$

and (using (f))

$$a^\gamma\left(\frac{i+1}{2}\right) + b^\gamma\left(\frac{i+1}{2}+1\right) = c^\gamma(i+1),$$

by adding terms by terms the inequalities $(*)$, we obtain

$$c_{i-1}^\gamma + c^\gamma(i+1) \leq c_{i-1}^{\tilde\gamma} + c^{\tilde\gamma}(i+1).$$

(ii) We now assume that i is even. Using the definition of the order on X_{2n}, we then get

$$\begin{cases} a^\gamma\left(\frac{i}{2}\right) \leq a^{\tilde\gamma}\left(\frac{i}{2}\right), \\ b^\gamma\left(\frac{i}{2}\right) \leq b^{\tilde\gamma}\left(\frac{i}{2}\right). \end{cases} \quad (*)$$

Since

$$b^\gamma\left(\frac{i}{2}\right) = c_{i-1}^\gamma + b^\gamma\left(\frac{i+2}{2}\right),$$

and (using (f))

$$a^\gamma\left(\frac{i}{2}+1\right) + b^\gamma\left(\frac{i}{2}+1\right) = c^\gamma(i+1),$$

by adding terms by terms the inequalities $(*)$, we obtain

$$c_{i-1}^\gamma + c^\gamma(i+1) \leq c_{i-1}^{\tilde\gamma} + c^{\tilde\gamma}(i+1).$$

□

By definition of the order on X_{2n}, we have

$$\gamma \leq \tilde\gamma \text{ if and only if } \gamma(i) \leq \tilde\gamma(i), \text{ for any } i \in \{0,1,\dots,2r\}.$$

Proposition 3.9. *Let* $\gamma \in X_{2n}$, $\tilde{\gamma} \in X_{2n}$. *If* $\theta_1^{\gamma} \underset{\Psi_{2n,1}}{\leq} \theta_1^{\tilde{\gamma}}$, *then* $\gamma \leq \tilde{\gamma}$, *where* $\leq$ *is the natural partial on partitions recalled in 3.2.1 (d).*

Proof. Let $\gamma \in X_{2n}$, $\tilde{\gamma} \in X_{2n}$. The following cases can occur. We assume that $\theta_1^{\gamma} \underset{\Psi_{2n,1}}{\leq} \theta_1^{\tilde{\gamma}}$.

(1) If both $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$, $\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}, \ldots, \tilde{\gamma}_{2r}\}$ are unions of blocks, then it follows from Lemma 3.7 and from Lemma 3.8 (a) that $\gamma(i) \leq \tilde{\gamma}(i)$.

(2) If neither $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ nor $\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}, \ldots, \tilde{\gamma}_{2r}\}$ are unions of blocks, then it again follows from Lemma 3.7 and from Lemma 3.8 (a) that $\gamma(i) \leq \tilde{\gamma}(i)$.

(3) We assume now that the set $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks, and that the set $\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}, \ldots, \tilde{\gamma}_{2r}\}$ is not. Then using Lemma 3.7 and Lemma 3.8 (a), we get

$$\gamma(i) \leq \tilde{\gamma}(i) - 1.$$

(4) We assume now that the set $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is not a union of blocks, and that the set $\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}, \ldots, \tilde{\gamma}_{2r}\}$ is a union of blocks. It implies that $\{\gamma_{i-1}, \gamma_i\}$ is a block. Hence $\{\gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$, $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$ are unions of blocks, $\gamma_{i-1} = \gamma_i$ is odd, and $2c_i^{\gamma} = 2c_{i-1}^{\gamma} = \gamma_i + 2i - 1$. The fact that $\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}, \ldots, \tilde{\gamma}_{2r}\}$ is a union of blocks implies that if $\tilde{\gamma}_{i-1}$ is odd, then $\{\tilde{\gamma}_{i-2}, \tilde{\gamma}_{i-1}\}$ is a block. It follows that

$$2c_{i-1}^{\tilde{\gamma}} = \begin{cases} \tilde{\gamma}_{i-1} + 2i - 2 & \text{if } \tilde{\gamma}_{i-1} \text{ is even;} \\ \tilde{\gamma}_{i-1} + 2i - 3 & \text{if } \tilde{\gamma}_{i-1} \text{ is odd.} \end{cases}$$

(a) We assume first that $\{\tilde{\gamma}_{i+1}, \tilde{\gamma}_{i+2}, \ldots, \tilde{\gamma}_{2r}\}$ is a union of blocks. Using Lemma 3.7 (a), we then get

$$\begin{cases} 2c^{\gamma}(i+1) = \gamma(i+1) + 2r(2r+1) - i(i+1), \\ 2c^{\tilde{\gamma}}(i+1) = \tilde{\gamma}(i+1) + 2r(2r+1) - i(i+1). \end{cases}$$

It follows that

$$\begin{aligned} 2c_{i-1}^{\gamma} + 2c^{\gamma}(i+1) = &\quad \gamma(i) + 2i - 1 + 2r(2r+1) - i(i+1) \\ = &\quad \gamma(i) + 2r(2r+1) - i(i-1) - 1, \end{aligned}$$

and that $2c_{i-1}^{\tilde{\gamma}} + 2c^{\tilde{\gamma}}(i+1)$ equals

$$\begin{cases} \tilde{\gamma}_{i-1} + \tilde{\gamma}(i+1) + 2r(2r+1) - i(i-1) - 2, & \text{if } \tilde{\gamma}_{i-1} \text{ even;} \\ \tilde{\gamma}_{i-1} + \tilde{\gamma}(i+1) + 2r(2r+1) - i(i-1) - 3, & \text{otherwise.} \end{cases}$$

Since $\tilde{\gamma}_{i-1} \leq \tilde{\gamma}_i$, applying Lemma 3.8 (b), we obtain

$$\gamma(i) + 2r(2r+1) - i(i-1) \leq \tilde{\gamma}(i) + 2r(2r+1) - i(i-1) - 1,$$

that is

$$\gamma(i) \leq \tilde{\gamma}(i) - 1.$$

(b) We assume now that $\{\tilde{\gamma}_{i+1}, \tilde{\gamma}_{i+2}, \ldots, \tilde{\gamma}_{2r}\}$ is not a union of blocks. Using Lemma 3.7, we then get

$$\begin{cases} 2c^{\gamma}(i+1) = \gamma(i+1) + 2r(2r+1) - i(i+1), \\ 2c^{\tilde{\gamma}}(i+1) = \tilde{\gamma}(i+1) + 2r(2r+1) - i(i+1) - 1. \end{cases}$$

It follows that

$$\begin{aligned} 2c^{\gamma}_{i-1} + 2c^{\gamma}(i+1) = &\quad \gamma(i) + 2i - 1 + 2r(2r+1) - i(i+1) \\ = &\quad \gamma(i) + 2r(2r+1) - i(i-1) - 1, \end{aligned}$$

and that $2c^{\tilde\gamma}_{i-1} + 2c^{\tilde\gamma}(i+1)$ equals

$$\begin{cases} \tilde\gamma_{i-1} + \tilde\gamma(i+1) + 2r(2r+1) - i(i-1) - 3, & \text{if } \tilde\gamma_{i-1} \text{ even;} \\ \tilde\gamma_{i-1} + \tilde\gamma(i+1) + 2r(2r+1) - i(i-1) - 4, & \text{otherwise.} \end{cases}$$

Since $\tilde\gamma_{i-1} \le \tilde\gamma_i$, applying Lemma 3.8 (b), we obtain

$$\gamma(i) + 2r(2r+1) - i(i-1) \le \tilde\gamma(i) + 2r(2r+1) - i(i-1) - 2,$$

that is

$$\gamma(i) \le \tilde\gamma(i) - 2.$$

□

3.9.2. *The odd dimensional orthogonal group.* Assume that G is simple of adjoint type B_n. Let X'_{2n+1} be the set of partitions of $2n+1$ such that any even part occurs an even number of times, that is

$$X'_{2n+1} = \left\{ \begin{matrix} \text{partition } \gamma = 0i_0 + 1i_1 + 2i_2 + 3i_3 + \cdots = 2n+1, \\ \text{with } i_0,\, i_1,\, i_2,\, i_3,\, \ldots \ge 0,\, i_2,\, i_4,\, i_6,\, \ldots \text{ even} \end{matrix} \right\}.$$

The unipotent classes of G are parametrized by the set X'_{2n+1}. We may arrange that the number $m+1$ of parts of γ is odd, say $m+1 = 2r+1 = i_0+i_1+i_2+\cdots$. As in [13, 11.7], let $z_1 \le z_2 \le \cdots \le z_{2r+1}$ be the sequence containing the number j exactly i_j times (for any $j \ge 1$) and the number 0 exactly i_0 times. Let $z'_1 < z'_2 < \cdots < z'_{2r+1}$ be the sequence defined by $z'_i := z_i + (i-1)$. That sequence contains r even numbers $2y_1 < 2y_2 < \cdots < 2y_r$ and $(r+1)$ odd numbers $2y'_1+1 < 2y'_2+1 < \cdots < 2y'_{r+1}+1$. We set

$$\theta'^{\gamma}_1 := \begin{pmatrix} y'_1 < y'_2 + 1 < \cdots < y'_{r+1} + r \\ y_1 < y_2 + 1 < \cdots < y_r + r - 1 \end{pmatrix}.$$

Then θ'^{γ}_1 is a distinguished element in Ψ'_{2n+1}. We set

$$c'^{\gamma}_i := \begin{cases} y'_{\frac{i+2}{2}} + \dfrac{i}{2} & \text{if } i \in \{0, \ldots, 2r\} \text{ is even,} \\ y_{\frac{i+1}{2}} + \dfrac{i-1}{2} & \text{if } i \in \{1, \ldots, 2r-1\} \text{ is odd.} \end{cases} \qquad \text{(a)}$$

We get

$$\theta'^{\gamma}_1 = \begin{pmatrix} c'^{\gamma}_0 < c'^{\gamma}_2 < \cdots < c'^{\gamma}_{2r} \\ c'^{\gamma}_1 < c'^{\gamma}_3 < \cdots < c'^{\gamma}_{2r-1} \end{pmatrix}.$$

In a similar way as we did in case of the symplectic group, we partition the sequence $(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{2r})$ into blocks of length one or two such that all odd γ_i lie in a block of length one and the even γ_i lie in blocks of length two. It can easily be checked that

$$c'^{\gamma}_i := \frac{\gamma_i - 1}{2} + i, \text{ if } \{\gamma_i\} \text{ is a block,} \qquad \text{(b)}$$

and

$$\left. \begin{aligned} c'^{\gamma}_i &:= \frac{\gamma_i}{2} + i, \\ c'^{\gamma}_{i+1} &= \frac{\gamma_{i+1}}{2} + i = c'^{\gamma}_i, \end{aligned} \right\} \text{ if } \gamma_i,\, \gamma_{i+1} \text{ lie in one block.} \qquad \text{(c)}$$

For any $h \in \{0, 1, \ldots, r\}$, $l \in \{1, 2, \ldots, r\}$, we set

$$a'^{\gamma}(h) := c'^{\gamma}_{2h} + c'^{\gamma}_{2h+2} + \cdots + c'^{\gamma}_{2r},$$

$$b'^{\gamma}(l) := c'^{\gamma}_{2l-1} + c'^{\gamma}_{2l+1} + \cdots + c'^{\gamma}_{2r-1},$$

and $b(r+1) := 0$. For any $i \in \{0, 1, \ldots, 2r+1\}$, we then set

$$c'^{\gamma}(i) := \begin{cases} a'^{\gamma}\left(\left[\frac{i}{2}\right]\right) + b'^{\gamma}\left(\left[\frac{i}{2}\right]+1\right) & \text{if } i \text{ is even,} \\ \\ a'^{\gamma}\left(\left[\frac{i}{2}\right]+1\right) + b'^{\gamma}\left(\left[\frac{i}{2}\right]+1\right) & \text{if } i \text{ is odd.} \end{cases} \tag{f}$$

Hence, for any $h \in \{0, 1, \ldots, r\}$, $l \in \{1, 2, \ldots, r\}$, we have

$$c'^{\gamma}(2h) = a'^{\gamma}(h) + b'^{\gamma}(h+1) = c'^{\gamma}_{2h} + c'^{\gamma}_{2h+1} + c'^{\gamma}_{2h+2} + \cdots + c'^{\gamma}_{2r-1} + c'^{\gamma}_{2r},$$

and

$$c'^{\gamma}(2l-1) = a'^{\gamma}(l) + b'^{\gamma}(l) = c'^{\gamma}_{2l-1} + c'^{\gamma}_{2l} + c'^{\gamma}_{2l+1} + \cdots + c'^{\gamma}_{2r-1} + c'^{\gamma}_{2r}.$$

For any $i \in \{0, 1, 2, \ldots, 2r+1\}$, we set

$$\gamma(i) := \gamma_i + \gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_{2r-1} + \gamma_{2r}.$$

We set

$$d'^{\gamma}_i := 2c'^{\gamma}_i - 2i + 1, \tag{h}$$

for $i \in \{0, 1, \ldots, 2r\}$. If $\{\gamma_i\}$ is a block, where $i \in \{0, 1, \ldots, 2r\}$, then $\gamma_i = d'^{\gamma}_i$. If $\{\gamma_i, \gamma_{i+1}\}$ is a block, where $i \in \{0, 1, \ldots, 2r-1\}$, then

$$\begin{aligned} \gamma_i + \gamma_{i+1} = & \quad (2c'^{\gamma}_i - 2i) + (2c'^{\gamma}_{i+1} - 2i) \\ = & \quad (d'^{\gamma}_i - 1) + (d'^{\gamma}_{i+1} - 1) + 2 \\ = & \quad d'^{\gamma}_i + d'^{\gamma}_{i+1}. \end{aligned}$$

It follows that if $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks then

$$\gamma(i) = d'^{\gamma}_i + d'^{\gamma}_{i+1} + \cdots + d'^{\gamma}_{2r}. \tag{i}$$

Lemma 3.10. *We have*

$$\gamma(i) = \begin{cases} 2c'^{\gamma}(i) - 4r^2 + i(i-2) + 1, & \textit{if } \{\gamma_i, \ldots, \gamma_{2r}\} \textit{ is a union of blocks,} \\ 2c'^{\gamma}(i) - 4r^2 + i(i-2), & \textit{otherwise.} \end{cases}$$

Proof. If $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks, using (i), we get

$$\gamma(i) = 2(c'^{\gamma}_i + c'^{\gamma}_{i+1} + \cdots + c'^{\gamma}_{2r}) - 2\sum_{h=i}^{2r} h + 2r - i + 1,$$

and the result follows. If $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is not a union of blocks, then it implies that $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$ is union of blocks, that γ_i is even, and that $\{\gamma_{i-1}, \gamma_i\}$ is a block. Applying the lemma to $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$, and using the fact that $\gamma_i = 2c'^{\gamma}_i - 2i$, we get

$$\gamma(i) = \gamma_i + \gamma(i+1) = 2c'^{\gamma}_i - 2i + 2c'^{\gamma}(i+1) - 4r^2 + (i+1)(i-1) + 1,$$

that is, since $c'^{\gamma}_i + c'^{\gamma}(i+1) = c'^{\gamma}(i)$,

$$\gamma(i) = 2c'^{\gamma}(i) - 4r^2 + i(i-2).$$

□

By definition of the order $\leq_{\Psi'_{2n,1}}$, we have

$$\theta_1^{\gamma} \underset{\Psi'_{2n,1}}{\leq} \theta_1^{\tilde\gamma} \text{ if and only if } \begin{cases} a'^{\gamma}(h) \leq a'^{\tilde\gamma}(h) & \text{for any } h \in \{0,1,\dots,r\}, \\ \\ b'^{\gamma}(l) \leq b'^{\tilde\gamma}(l) & \text{for any } l \in \{1,2,\dots,r\}. \end{cases}$$

Lemma 3.11. *Let $\gamma \in X_{2n}$, $\tilde\gamma \in X_{2n}$. If $\theta_1'^{\gamma} \underset{\Psi'_{2n,1}}{\leq} \theta_1'^{\tilde\gamma}$, then*

(a) *$c'^{\gamma}(i) \leq c'^{\tilde\gamma}(i)$, for any $i \in \{0,1,2,\dots,2r\}$;*
(b) *$c'^{\gamma}_{i-1} + c'^{\gamma}(i+1) \leq c'^{\tilde\gamma}_{i-1} + c^{\tilde\gamma}(i+1)$, for any $i \in \{1,2,\dots,2r-1\}$.*

Proof. Same as those of Lemma 6.8. □

By definition of the order on X'_{2n+1}, we have

$$\gamma \leq \tilde\gamma \text{ if and only if } \gamma(i) \leq \tilde\gamma(i), \text{ for any } i \in \{0,1,\dots,2r\}.$$

Proposition 3.12. *Let $\gamma \in X'_{2n+1}$, $\tilde\gamma \in X'_{2n+1}$. If $\theta_1'^{\gamma} \underset{\Psi'_{2n,1}}{\leq} \theta_1'^{\tilde\gamma}$, then $\gamma \leq \tilde\gamma$, where $\leq$ is the natural partial on partitions recalled in 3.2.1 (d).*

Proof. Let $\gamma \in X'_{2n+1}$, $\tilde\gamma \in X'_{2n+1}$. The following cases can occur. We assume that $\theta_1'^{\gamma} \underset{\Psi'_{2n,1}}{\leq} \theta_1'^{\tilde\gamma}$.

(1) If both $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_{2r}\}$, $\{\tilde\gamma_i, \tilde\gamma_{i+1}, \dots, \tilde\gamma_{2r}\}$ are unions of blocks, then it follows from Lemma 3.10 and from Lemma 3.11 (a) that $\gamma(i) \leq \tilde\gamma(i)$.
(2) If neither $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_{2r}\}$ nor $\{\tilde\gamma_i, \tilde\gamma_{i+1}, \dots, \tilde\gamma_{2r}\}$ are unions of blocks, then it again follows from Lemma 3.10 and from Lemma 3.11 (a) that $\gamma(i) \leq \tilde\gamma(i)$.
(3) We assume now that the set $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_{2r}\}$ is not a union of blocks, and that the set $\{\tilde\gamma_i, \tilde\gamma_{i+1}, \dots, \tilde\gamma_{2r}\}$ is one. Then using Lemma 3.10 and Lemma 3.11 (a), we get
$$\gamma(i) \leq \tilde\gamma(i) - 1.$$
(4) We assume now that the set $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_{2r}\}$ is a union of blocks, and that the set $\{\tilde\gamma_i, \tilde\gamma_{i+1}, \dots, \tilde\gamma_{2r}\}$ is not. It implies that $\{\tilde\gamma_{i-1}, \tilde\gamma_i\}$ is a block. Hence $\{\tilde\gamma_{i-1}, \tilde\gamma_i, \tilde\gamma_{i+1}, \dots, \tilde\gamma_{2r}\}$, $\{\tilde\gamma_{i+1}, \tilde\gamma_{i+2}, \dots, \tilde\gamma_{2r}\}$ are unions of blocks, $\tilde\gamma_{i-1} = \tilde\gamma_i$ is even, and $2c'^{\tilde\gamma}_i = 2c'^{\tilde\gamma}_{i-1} = \tilde\gamma_i + 2i - 2$. The fact that $\{\gamma_i, \gamma_{i+1}, \dots, \gamma_{2r}\}$ is a union of blocks implies that if γ_{i-1} is even, then $\{\gamma_{i-2}, \gamma_{i-1}\}$ is a block. It follows that
$$2c'^{\gamma}_{i-1} = \begin{cases} \gamma_{i-1} + 2i - 3 & \text{if } \gamma_{i-1} \text{ is odd;} \\ \gamma_{i-1} + 2i - 4 & \text{if } \gamma_{i-1} \text{ is even.} \end{cases}$$
   (a) We assume first that $\{\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{2r}\}$ is a union of blocks. Using Lemma 3.10 (a), we then get
$$\begin{cases} 2c'^{\gamma}(i+1) = \gamma(i+1) + 4r^2 - i^2, \\ 2c'^{\tilde\gamma}(i+1) = \gamma(i+1) + 4r^2 - i^2. \end{cases}$$
   By using Lemma 3.11 (a), we then get
$$c'^{\gamma}(i) \leq c'^{\tilde\gamma}(i),$$

that is

$$2c_i'^{\gamma} + 2c'^{\gamma}(i+1) \leq 2c_i'^{\tilde{\gamma}} + 2c'^{\tilde{\gamma}}(i+1).$$

Since $\{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{2r}\}$ is a union of blocks too, the set $\{\gamma_i\}$ is a block itself. Hence $2c_i'^{\gamma} = \gamma_i + 2i - 1$. It follows that

$$\gamma(i) \leq \tilde{\gamma}(i) - 1.$$

(b) We assume now that $\{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{2r}\}$ is not a union of blocks. Hence $\{\gamma_i, \gamma_{i+1}\}$ is a block, and

$$2c_i'^{\gamma} = 2c_{i+1}'^{\gamma} = \gamma_i + 2i.$$

On the other hand,

$$2c_i'^{\tilde{\gamma}} = \tilde{\gamma}_i + 2i - 2.$$

Using Lemma 3.11, we have

$$\begin{cases} 2c'^{\gamma}(i+1) = \gamma(i+1) + 4r^2 - i^2 + 1, \\ 2c'^{\tilde{\gamma}}(i+1) = \tilde{\gamma}(i+1) + 4r^2 - i^2. \end{cases}$$

Using again Lemma 3.11 (a), we have

$$c'^{\gamma}(i) \leq c'^{\tilde{\gamma}}(i).$$

It follows that

$$\gamma(i) \leq \tilde{\gamma}(i) - 3.$$

□

3.9.3. *The even dimensional orthogonal group.* Assume that G is simple of adjoint type D_n. Let X'_{2n} be the set of partitions of $2n$ such that any even part occurs an even number of times, that is

$$X'_{2n} = \left\{ \begin{matrix} \text{partition } \gamma = 0i_0 + 1i_1 + 2i_2 + 3i_3 + \cdots = 2n, \\ \text{with } i_0,\, i_1,\, i_2,\, i_3,\, \ldots \geq 0,\, i_2,\, i_4,\, i_6,\, \ldots \text{ even} \end{matrix} \right\}.$$

The unipotent classes of G are parametrized by the set X'_{2n}, except that to any partition in X'_{2n} such that $i_1 = i_3 = i_5 = \cdots = 0$, there correspond two (degenerate) unipotent classes in G.

We may arrange that the number m of parts of γ is even, say $m = 2r = i_0 + i_1 + i_2 + \cdots$. As in [13, 11.7], let $z_1 \leq z_2 \leq \cdots \leq z_{2r}$ be the sequence containing the number j exactly i_j times (for any $j \geq 1$) and the number 0 exactly i_0 times. Let $z'_1 < z'_2 < \cdots < z'_{2r}$ be the sequence defined by $z'_i := z_i + (i-1)$. That sequence contains r even numbers $2y_1 < 2y_2 < \cdots < 2y_r$ and r odd numbers $2y'_1 + 1 < 2y'_2 + 1 < \cdots < 2y'_r + 1$. We set

$$\theta_1'^{\gamma} := \begin{pmatrix} y'_1 < y'_2 + 1 < \cdots < y'_r + r - 1 \\ y_1 < y_2 + 1 < \cdots < y_r + r - 1 \end{pmatrix}.$$

Then $\theta_1'^{\gamma}$ is a distinguished element in Ψ'_{2n}. We set

$$c_i'^{\gamma} := \begin{cases} y'_{\frac{i+2}{2}} + \dfrac{i}{2} & \text{if } i \text{ is even}, \\ y_{\frac{i+1}{2}} + \dfrac{i-1}{2} & \text{if } i \text{ is odd}. \end{cases} \tag{a}$$

We get

$$\theta_0'^{\gamma} = \begin{pmatrix} c_2'^{\gamma} < c_4'^{\gamma} < \cdots < c_{2r}'^{\gamma} \\ c_1'^{\gamma} < c_3'^{\gamma} < \cdots < c_{2r+1}'^{\gamma} \end{pmatrix}.$$

In a similar way as we did before, we partition the sequence $(\gamma_1, \gamma_1, \gamma_2, \ldots, \gamma_{2r})$ into blocks of length one or two such that all odd γ_i lie in a block of length one and the even γ_i lie in blocks of length two. It can easily be checked that

$$c_i'^{\gamma} = \frac{\gamma_i - 1}{2} + i - 1, \text{ if } \{\gamma_i\} \text{ is a block,} \tag{b}$$

and

$$c_i'^{\gamma} = c_{i+1}'^{\gamma} = \frac{\gamma_i + 1}{2} + i - 1, \text{ if } \{\gamma_i, \gamma_{i+1}\} \text{ is a block.} \tag{c}$$

One proves the following proposition in a similar way as we did for odd orthogonal groups.

Proposition 3.13. *Let $\gamma \in X'_{2n}$, $\tilde\gamma \in X'_{2n}$. If $\theta_0'^{\gamma} \underset{\Psi'_{2n,0}}{\leq} \theta_0'^{\tilde\gamma}$, then $\gamma \leq \tilde\gamma$.*

We will write $C \leq \tilde C$, for C, $\tilde C$ unipotent classes if $\gamma \leq \tilde\gamma$, where γ, $\tilde\gamma$ are the partitions corresponding to C, $\tilde C$ respectively.

3.10. Combining Proposition 3.6 and Propositions 3.9, 3.12, 3.13, we obtain the following result:

Proposition 3.14. *Let n', n'' be two integers such that $n' + n'' = n$.*

(a) *Let $\Lambda_1' \in \Phi_{n',1}$, $\Lambda_1'' \in \Phi_{n'',1}$, $\Lambda_1 \in \Phi_{n,1}$ such that $E(\Lambda_1)$ occurs in the induced representation $I(\Lambda_1', \Lambda_1'')$. Then we have $C(\sigma_n(\Lambda_1)) \leq C(\sigma_n(\Lambda_1' + \Lambda_1''))$.*
(b) *Let $\Lambda_0' \in \Phi'_{n',0}$, $\Lambda_0'' \in \Phi'_{n'',0}$, $\Lambda_0 \in \Phi'_{n,0}$ such that $E(\Lambda_0)$ occurs in the induced representation $I'(\Lambda_0', \Lambda_0'')$. Then we have $C(\sigma_n'^0(\Lambda_0)) \leq C(\sigma_n'^0(\Lambda_0' + \Lambda_0''))$.*

The proof of Theorem 1.2 (hence also the proof of Theorem 1.1) is now complete.

4. Complements

4.0.1. *First parametrization.* Let $\mathcal{T}$ be the set of triples $\tau = (L, s, E)$ such that $\hat L_s \cap \hat L^0_{\mathrm{unip}} \neq \emptyset$ with the semisimple element s of order 2, so that the group W_s is of kind $C_r \times C_{r'}$, $D_r \times B_{r'}$ or $D_r \times D_{r'}$. Such a condition is always satisfied if the set $\hat L^0_s$ is non empty (see [1]). In this way with each character sheaf in $\mathcal{T}$ we have associated an integer $t \geq 0$ and a pair $(\Lambda_1, \Lambda_1') \in \Phi_{\tilde n',1} \times \Phi_{\tilde n'',1}$ (resp. $(\Lambda_0, \Lambda_1') \in \Phi'_{\tilde n',0} \times \Phi_{\tilde n'',1}$, $(\Lambda_0, \Lambda_0') \in \Phi'_{\tilde n',0} \times \Phi'_{\tilde n'',0}$) such that $\tilde n = \tilde n' + \tilde n''$.

This gives a bijection between $\mathcal{T}$ and

- **(B_n):** the set of triples $(t, \Lambda_1, \Lambda_1')$ where $t \geq 0$, $(\Lambda_1, \Lambda_1') \in \Phi_{\tilde n',1} \times \Phi_{\tilde n'',1}$, $\tilde n' + \tilde n'' + 2(t^2 + t) = n$;
- **(C_n):** the union of the set of triples $(t, \Lambda_1, \Lambda_1')$ where $t \geq 1$, $(\Lambda_1, \Lambda_1') \in \Phi_{\tilde n',1} \times \Phi_{\tilde n'',1}$, $\tilde n' + \tilde n'' + 2(4t^2 \pm t) = n$ and the set of pairs $(\Lambda_0, \Lambda_1') \in \Phi'_{\tilde n',0} \times \Phi_{\tilde n'',1}$ where $\tilde n' + \tilde n'' = n$,
- **(D_n):** the union of the set of triples $(t, \Lambda_1, \Lambda_1')$ where $t \geq 1$, $(\Lambda_1, \Lambda_1') \in \Phi_{\tilde n',1} \times \Phi_{\tilde n'',1}$, $\tilde n' + \tilde n'' + 8t^2 = n$ and the set of pairs $(\Lambda_0, \Lambda_0') \in \Phi'_{\tilde n',0} \times \Phi'_{\tilde n'',0}$ where $\tilde n' + \tilde n'' = n$.

4.0.2. *Second parametrization.* We get the following correspondences.

(B_n): Using the bijections $\Phi_{n'-t(t+1),1} \to \Phi_{n',2t+1}$, $\Phi_{n''-t(t+1),1} \to \Phi_{n'',2t+1}$ with $n'+n''=n$ we get a 1–to–1 correspondence between the set of triples $(t,\Lambda_1',\Lambda_1'')$ where $t \geq 0$, Λ_1', Λ_1'' are symbols of defect one with $n = \mathrm{rk}(\Lambda_1') + \mathrm{rk}(\Lambda_1'') + 2(t^2+t)$ and the set of pairs (Λ',Λ'') where Λ', Λ'' are symbols of equal defect with $n = \mathrm{rk}(\Lambda') + \mathrm{rk}(\Lambda'')$. The correspondence is defined by

$$t,\ \Lambda_1' = \begin{pmatrix} \lambda_0' < \lambda_1' < \cdots < \lambda_m' \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},\ \Lambda_1'' = \begin{pmatrix} \lambda_0'' < \lambda_1'' < \cdots < \lambda_m'' \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} 0 < 1 < \cdots < 2t-1 < \lambda_0'+2t < \lambda_1'+2t < \cdots < \lambda_m'+2t \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},$$

$$\Lambda'' = \begin{pmatrix} 0 < 1 < \cdots < 2t-1 < \lambda_0''+2t < \lambda_1''+2t < \cdots < \lambda_m''+2t \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix}.$$

(C_n):

• Using the bijections $\Phi_{n'-4t^2,1} \to \Phi_{n',4t}$, $\Phi_{n''-2t(2t+1),1} \to \Phi_{n'',4t+1}$ with $n'+n''=n$ we get a 1–to–1 correspondence between the set of triples $(t,\Lambda_1',\Lambda_1'')$ where $t \geq 1$, Λ_1' Λ_1'' are symbols of defect one with $n = \mathrm{rk}(\Lambda_1') + \mathrm{rk}(\Lambda_1'') + 8t^2 + 2t$ and the set of pairs (Λ',Λ'') where Λ', Λ'' are symbols with $\mathrm{def}(\Lambda'') = \mathrm{def}(\Lambda') + 1 \neq 0$ and with $n = \mathrm{rk}(\Lambda') + \mathrm{rk}(\Lambda'')$. The correspondence is defined by

$$t,\ \Lambda_1' = \begin{pmatrix} \lambda_0' < \lambda_1' < \cdots < \lambda_m' \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},\ \Lambda_1'' = \begin{pmatrix} \lambda_0'' < \lambda_1'' < \cdots < \lambda_m'' \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} 0 < 1 < \cdots < 4t-2 < \lambda_0'+4t-1 < \lambda_1'+4t-1 < \cdots < \lambda_m'+4t-1 \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},$$

$$\Lambda'' = \begin{pmatrix} 0 < 1 < \cdots < 4t-1 < \lambda_0''+4t < \lambda_1''+4t < \cdots < \lambda_m''+4t \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix}.$$

• Using the bijections $\Phi_{n'-4t^2,1} \to \Phi_{n',4t}$, $\Phi_{n''-2t(2t-1),1} \to \Phi_{n'',4t-1}$ with $n'+n''=n$ we get a 1–to–1 correspondence between the set of triples $(t,\Lambda_1',\Lambda_1'')$ where $t \geq 1$, Λ_1' Λ_1'' are symbols of defect one with $n = \mathrm{rk}(\Lambda_1') + \mathrm{rk}(\Lambda_1'') + 8t^2 - 2t$ and the set of pairs (Λ',Λ'') where Λ', Λ'' are symbols with $\mathrm{def}(\Lambda') = \mathrm{def}(\Lambda'') - 1 \neq 0$ and with $n = \mathrm{rk}(\Lambda') + \mathrm{rk}(\Lambda'')$. The correspondence is defined by

$$t,\ \Lambda_1' = \begin{pmatrix} \lambda_0' < \lambda_1' < \cdots < \lambda_m' \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},\ \Lambda_1'' = \begin{pmatrix} \lambda_0'' < \lambda_1'' < \cdots < \lambda_m'' \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} 0 < 1 < \cdots < 4t-2 < \lambda_0'+4t-1 < \lambda_1'+4t-1 < \cdots < \lambda_m'+4t-1 \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},$$

$$\Lambda'' = \begin{pmatrix} 0 < 1 < \cdots < 4t-3 < \lambda_0''+4t-2 < \lambda_1''+4t-2 < \cdots < \lambda_m''+4t-2 \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix}.$$

(D_n): Using the bijections $\Phi_{n'-4t^2,1} \to \Phi_{n',4t}$, $\Phi_{n''-4t^2,1} \to \Phi_{n'',4t}$ with $n'+n''=n$ we get a 1–to–1 correspondence between the set of triples $(t,\Lambda_1',\Lambda_1'')$ where $t \geq 1$, Λ_1' Λ_1'' are symbols of defect one with $n = \mathrm{rk}(\Lambda_1') + \mathrm{rk}(\Lambda_1'') + 8t^2$ and the set of pairs (Λ',Λ'') where Λ', Λ'' are symbols of equal non-zero defect with $n = \mathrm{rk}(\Lambda') + \mathrm{rk}(\Lambda'')$. The correspondence is defined by

$$t,\ \Lambda_1' = \begin{pmatrix} \lambda_0' < \lambda_1' < \cdots < \lambda_m' \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},\ \Lambda_1'' = \begin{pmatrix} \lambda_0'' < \lambda_1'' < \cdots < \lambda_m'' \\ \mu_1'' < \mu_2'' < \cdots < \mu_m'' \end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} 0 < 1 < \cdots < 4t-2 < \lambda_0'+4t-1 < \lambda_1'+4t-1 < \cdots < \lambda_m'+4t-1 \\ \mu_1' < \mu_2' < \cdots < \mu_m' \end{pmatrix},$$

$$\Lambda'' = \begin{pmatrix} 0 < 1 < \cdots < 4t-2 < \lambda''_0 + 4t - 1 < \lambda''_1 + 4t - 1 < \cdots < \lambda''_m + 4t - 1 \\ \mu''_1 < \mu''_2 < \cdots < \mu''_m \end{pmatrix}.$$

4.0.3. *Third parametrization.* Combining the 1–to–1 correspondence (4.0.2) with those defined in (4.0.1), we get a bijection between $\mathcal{T}$ and

- **(B_n):** the set $\mathcal{T}(B_n)$ of pairs $(\Lambda', \Lambda'') \in \Phi_{n'} \times \Phi_{n''}$ where $n' + n'' = n$ such that $\mathrm{def}(\Lambda') = \mathrm{def}(\Lambda'')$;
- **(C_n):** the union $\mathcal{T}(C_n)$ of the set of pairs $(\Lambda', \Lambda'') \in \Phi_{n'} \times \Phi_{n''}$ where $m+m' = n$ such that $\mathrm{def}(\Lambda') = \mathrm{def}(\Lambda'') \pm 1$, $\inf(\mathrm{def}(\Lambda'), \mathrm{def}(\Lambda'')) \neq 0$ and the set of pairs $(\Lambda'_0, \Lambda''_1) \in \Phi'_{n',0} \times \Phi_{n'',1}$ where $n' + n'' = n$.
- **(D_n):** the union $\mathcal{T}(D_n)$ of the set of pairs $(\Lambda', \Lambda'') \in \Phi_{n'} \times \Phi_{n''}$ where $n'+n'' = n$ such that $\mathrm{def}(\Lambda') = \mathrm{def}(\Lambda'') \neq 0$ and the set of pairs $(\Lambda'_0, \Lambda''_0) \in \Phi'_{n',0} \times \Phi'_{n'',0}$ where $n' + n'' = n$.

4.0.4. *Type B_n.* Let G be SO_{2n+1}. Let

$$\Lambda' = \begin{pmatrix} \lambda'_0 < \lambda'_1 < \cdots < \lambda'_{m+2t} \\ \mu'_1 < \mu'_2 < \cdots < \mu'_m \end{pmatrix} \in \Phi_{\tilde{n}'+t^2+t, 2t+1},$$

$$\Lambda'' := \begin{pmatrix} \lambda''_0 < \lambda''_1 < \cdots < \lambda''_{m+2t} \\ \mu''_1 < \mu''_2 < \cdots < \mu''_m \end{pmatrix} \in \Phi_{\tilde{n}''+t^2+t, 2t+1},$$

where $\tilde{n}' = n' - t^2 - t$, $\tilde{n}'' = n'' - t^2 - t$, $n' + n'' = n$ (hence $\tilde{n}' + \tilde{n}'' + 2t^2 + 2t = n$). We set

$$\overline{\Lambda}'_1 := \begin{pmatrix} \lambda'_0 < \lambda'_1 < \cdots < \lambda'_{m+2t} \\ 0 < 1 < \cdots < 2t-1 < \mu'_1 + 2t < \mu'_2 + 2t < \cdots < \mu'_m + 2t \end{pmatrix} \in \Phi_{\tilde{n}',1},$$

$$\overline{\Lambda}'' := \begin{pmatrix} \lambda''_0 < \lambda''_1 < \cdots < \lambda''_{m+2t} \\ 0 < 1 < \cdots < 2t-1 < \mu''_1 + 2t < \mu''_2 + 2t < \cdots < \mu''_m + 2t \end{pmatrix} \in \Phi_{\tilde{n}'',1}.$$

We then set

$$\Lambda' * \Lambda'' := (\Delta'_{2n-4t^2-4t+1, 2t+1} \circ \sigma'_{n-2t^2-2t})(\overline{\Lambda}'_1 + +\overline{\Lambda}''_1) \in \Psi_{2n+1, 2t+1}. \tag{4.1}$$

4.0.5. *Type C_n.* Let G be PSp_{2n}.

(1) We assume that $\mathrm{C}_{L^*}(s)$ is of type $\mathrm{Spin}_{8t^2} \times \mathrm{Spin}_{8t^2-4t+1} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$, with $t \geq 1$. Let

$$\Lambda' := \begin{pmatrix} \lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t} \\ \mu'_1 < \mu'_2 < \cdots < \mu'_m \end{pmatrix} \in \Phi_{\tilde{n}'+4t^2, 4t},$$

$$\Lambda'' = \begin{pmatrix} \lambda''_1 < \lambda''_2 < \cdots < \lambda''_{m+4t-1} \\ \mu''_1 < \mu''_2 < \cdots < \mu''_m \end{pmatrix} \in \Phi_{\tilde{n}''+4t^2-2t, 4t-1},$$

where $\tilde{n}' = n' - 4t^2$, $\tilde{n}'' = n'' - 4t^2 + 2t$, $n' + n'' = n$ (hence $\tilde{n}' + \tilde{n}'' + 8t^2 - 2t = n$). We set

$$\overline{\Lambda}'_1 := \begin{pmatrix} \lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t} \\ 0 < 1 < \cdots < 4t-2 < \mu'_1 + 4t - 1 < \mu'_2 + 4t - 1 < \cdots < \mu'_m + 4t - 1 \end{pmatrix},$$

$$\overline{\Lambda}''_1 := \begin{pmatrix} \lambda''_0 < \lambda''_1 < \cdots < \lambda''_{m+4t-1} \\ 0 < 1 < \cdots < 4t-3 < \mu''_1 + 4t - 2 < \mu''_2 + 4t - 2 < \cdots < \mu''_m + 4t - 2 \end{pmatrix};$$

then $\overline{\Lambda}'_1 \in \Phi_{\tilde{n}',1}$ and $\overline{\Lambda}''_1 \in \Phi_{\tilde{n}'',1}$. We then set

$$\Lambda' * \Lambda'' := (\Delta_{2n-16t^2+4t, 1-4t} \circ {}^t\sigma_{n-8t^2+2t})(\overline{\Lambda}'_1 + \overline{\Lambda}''_1) \in \Psi_{2n, 1-4t}. \tag{4.2}$$

(2) We assume that $C_{L^*}(s)$ is of type $\mathrm{Spin}_{8t^2} \times \mathrm{Spin}_{8t^2+4t+1} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$, with $t \geq 1$. Let

$$\Lambda' := \binom{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t}}{\mu'_1 < \mu'_2 < \cdots < \mu'_m} \in \Phi_{\tilde{n}'+4t^2,4t},$$

$$\Lambda'' = \binom{\lambda''_0 < \lambda''_1 < \cdots < \lambda''_{m+4t}}{\mu''_1 < \mu''_2 < \cdots < \mu''_m} \in \Phi_{\tilde{n}''+4t^2+2t,4t+1},$$

where $\tilde{n}' = n' - 4t^2$, $\tilde{n}'' = n'' - 4t^2 - 2t$, $n' + n'' = n$ (hence $\tilde{n}' + \tilde{n}'' + 8t^2 - 2t = n$). We set

$$\overline{\Lambda}'_1 := \binom{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t}}{0 < 1 < \cdots < 4t-2 < \mu'_1 + 4t - 1 < \mu'_2 + 4t - 1 < \cdots < \mu'_m + 4t - 1},$$

$$\overline{\Lambda}''_1 := \binom{\lambda''_0 < \lambda''_1 < \cdots < \lambda''_{m+4t}}{0 < 1 < \cdots < 4t-1 < \mu''_1 + 4t < \mu''_2 + 4t < \cdots < \mu''_m + 4t};$$

then $\overline{\Lambda}'_1 \in \Phi_{\tilde{n}',1}$ and $\overline{\Lambda}''_1 \in \Phi_{\tilde{n}'',1}$. We then set

$$\Lambda' * \Lambda'' := (\Delta_{2n-16t^2-4t,4t+1} \circ \sigma_{n-8t^2+2t})(\overline{\Lambda}'_1 + \overline{\Lambda}''_1) \in \Psi_{2n,4t+1}. \tag{4.3}$$

4.0.6. *Type D_n*. Let G be PSO_{2n}. We assume that $C_{L^*}(s)$ is of type $\mathrm{Spin}_{8t^2} \times \mathrm{Spin}_{8t^2} \times \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$.

(1) Assume that $t \geq 1$. Let

$$\Lambda' := \binom{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t}}{\mu'_1 < \mu'_2 < \cdots < \mu'_m} \in \Phi_{\tilde{n}'+4t^2,4t},$$

$$\Lambda'' = \binom{\lambda''_1 < \lambda''_2 < \cdots < \lambda''_{m+4t-1}}{\mu''_1 < \mu''_2 < \cdots < \mu''_m} \in \Phi_{\tilde{n}''+4t^2,4t},$$

where $\tilde{n}' = n' - 4t^2$, $\tilde{n}'' = n'' - 4t^2$, $n' + n'' = n$ (hence $\tilde{n}' + \tilde{n}'' + 8t^2 = n$).

We set

$$\overline{\Lambda}'_1 := \binom{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t}}{0 < 1 < \cdots < 4t-2 < \mu'_1 + 4t - 1 < \mu'_2 + 4t - 1 < \cdots < \mu'_m + 4t - 1},$$

$$\overline{\Lambda}''_1 := \binom{\lambda''_1 < \lambda''_1 < \cdots < \lambda''_{m+4t-1}}{0 < 1 < \cdots < 4t-2 < \mu''_1 + 4t - 1 < \mu''_2 + 4t - 1 < \cdots < \mu''_m + 4t - 1};$$

then $\overline{\Lambda}'_1 \in \Phi_{\tilde{n}',1}$ and $\overline{\Lambda}''_1 \in \Phi_{\tilde{n}'',1}$. We then set

$$\Lambda' * \Lambda'' := (\Delta'_{2n-16t^2+1,4t} \circ \sigma'_{n-8t^2})(\overline{\Lambda}'_1 + \overline{\Lambda}''_1) \in \Psi'_{2n,4t}. \tag{4.4}$$

(2) Assume now that $t = 0$. Let

$$\Lambda'_0 := \binom{\lambda'_1 < \lambda'_2 < \cdots < \lambda'_{m+4t}}{\mu'_1 < \mu'_2 < \cdots < \mu'_m} \in \Phi_{\tilde{n}',0},$$

$$\Lambda''_0 = \binom{\lambda''_1 < \lambda''_2 < \cdots < \lambda''_{m+4t-1}}{\mu''_1 < \mu''_2 < \cdots < \mu''_m} \in \Phi_{\tilde{n}'',0}$$

We set

$$\Lambda' * \Lambda'' := \sigma'^0_n(\overline{\Lambda}'_0 + \overline{\Lambda}''_0) \in \Psi'^0_{2n,0}. \tag{4.5}$$

4.0.7. *The function a.* Next, following [15, (4.11)], we define a function

$$a:\ \Phi_n \to \mathbb{N}, \quad (\text{resp. } a:\ \Phi'_{n,0} \to \mathbb{N})$$

as follows.

(Φ_n) Let $\Lambda = \binom{S}{T} \in \Phi_n$. Write the entries of S and T in a single row in increasing order: $y_0 \leq y_1 \leq \cdots \leq y_{2m}$. Let $y_0^0 \leq y_1^0 \leq \cdots \leq y_{2m}^0$ be the sequence $0 \leq 0 \leq 1 \leq 1 \leq \cdots \leq m-1 \leq m-1 \leq m$. Then by definition

$$a(\Lambda) = \sum_{0\leq i<j\leq 2m} \inf(y_i, y_j) - \sum_{0\leq i<j\leq 2m} \inf(y_i^0, y_j^0).$$

($\Phi'_{n,0}$) Let $\Lambda = \binom{S}{T} \in \Phi'_{n,0}$. Write the entries of S and T in a single row in increasing order: $y_1 \leq y_2 \leq \cdots \leq y_{2m}$. Let $y_1^0 \leq y_2^0 \leq \cdots \leq y_{2m}^0$ be the sequence $0 \leq 0 \leq 1 \leq 1 \leq \cdots \leq m-1 \leq m-1$. Then by definition

$$a(\Lambda) = \sum_{1\leq i<j\leq 2m} \inf(y_i, y_j) - \sum_{1\leq i<j\leq 2m} \inf(y_i^0, y_j^0).$$

First note that $a(\Lambda)$ is indeed well-defined on symbols. Furthermore, it is clear that a is constant on similarity classes of symbols. It is straightforward to check that the functions a defined above coincide with those defined by Lusztig in [12, (4.5)] for $\Lambda \in \Phi_{n,1}$ and in [*loc. cit.*, (4.6)] for $\Lambda \in \Phi'_{n,0}$, that is,

$$a(\Lambda) = a_{E(\Lambda)}. \tag{4.6}$$

4.0.8. *The function b.* Following [15, (4.4)], we define two functions

$$b\colon \Psi'_{2n+1} \to \mathbb{N} \quad (\text{resp. } b\colon \Psi_{2n} \to \mathbb{N},\ \ b\colon \Psi'_{2n} \to \mathbb{N}),$$

as follows.

(B_n) Let $\theta = \binom{S}{T} \in \Psi'_{2n+1}$. Write the entries of S and T in a single row in increasing order: $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{2m}$. Let $x_0^0 \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_{2m}^0$ be the sequence $0 \leq 0 \leq 2 \leq 2 \leq \cdots \leq 2(m-1) \leq 2(m-1) \leq 2m$. Then by definition

$$b(\theta) := \sum_{0\leq i<j\leq 2m} \inf(x_i, x_j) - \sum_{0\leq i<j\leq 2m} \inf(x_i^0, x_j^0).$$

(C_n) Let $\theta = \binom{S}{T} \in \Psi_{2n}$. Write the entries of S and T in a single row in increasing order: $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{2m}$. Let $x_0^0 \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_{2m}^0$ be the sequence $0 \leq 1 \leq 2 \leq 3 \leq \cdots \leq 2m-1 \leq 2m$. Then by definition

$$b(\theta) := \sum_{0\leq i<j\leq 2m} \inf(x_i, x_j) - \sum_{0\leq i<j\leq 2m} \inf(x_i^0, x_j^0).$$

(D_n) Let $\theta = \binom{S}{T} \in \Psi'_{2n}$. Write the entries of S and T in a single row in increasing order: $x_1 \leq x_2 \leq \cdots \leq x_{2m}$. Let $x_1^0 \leq x_2^0 \leq \cdots \leq x_{2m}^0$ be the sequence $0 \leq 0 \leq 2 \leq 2 \leq \cdots \leq 2(m-1) \leq 2(m-1)$. Then by definition

$$b(\theta) := \sum_{1\leq i<j\leq 2m} \inf(x_i, x_j) - \sum_{1\leq i<j\leq 2m} \inf(x_i^0, x_j^0).$$

First note that $b(\theta)$ is indeed well-defined. Furthermore, b is constant on similarity classes, and we have

$$b(\theta) = d(C(\theta)) = \frac{1}{2}(\dim G - \dim C(\theta) - \mathrm{rk} G), \tag{4.7}$$

where $(C(\theta), \mathcal{E}(\theta))$ denotes the element of $\mathcal{N}_G$ corresponding to θ.

We shall denote by $A(\Lambda', \Lambda'')$ the character sheaf in $\mathcal{T}$ corresponding to such a pair $(\Lambda', \Lambda'') \in \mathcal{T}(B_n)$ (resp. $\mathcal{T}(C_n)$, $\mathcal{T}(D_n)$). We denote by $\mathcal{T}(G)$ the set $\mathcal{T}(B_n)$ (resp $\mathcal{T}(C_n)$, $\mathcal{T}(D_n)$) if G is of type B_n (resp. C_n, D_n). Let $(V', V'') \subset \mathcal{T}(G)$ be two similarity classes such that $n' + n'' = n$. Then we define a canonical Lagrangian subspace X of the symplectic vector space $V' \times V''$ by

$$X := \{(\Lambda', \Lambda'') \in V \times V' \mid a(\Lambda') + a(\Lambda'') = b(\Lambda' * \Lambda'')\}.$$

It is clear that when (Λ', Λ'') varies in X, the element $\Lambda' * \Lambda''$ remains in a fixed similarity class, say R, see [15, 4.15(c)].

Let $\chi_{A(\Lambda', \Lambda'')}$ denote the characteristic function of the character sheaf $A(\Lambda', \Lambda'')$, and let $Y_{\Lambda' * \Lambda''}$ denote the function $Y_{\mathfrak{n}}$ (see 3.6), where $\mathfrak{n} \in \mathcal{N}_G$ corresponds to $\Lambda' * \Lambda''$.

Theorem 4.1. *If $\Lambda' \in V'$, $\Lambda'' \in V''$ and if $u \in C(\Lambda' * \Lambda'')^F$ is a split element, then*

$$\chi_{A(\Lambda', \Lambda'')}(u) = \begin{cases} (-1)^n \, q^{b(\Lambda' * \Lambda'') + \frac{1}{2} \mathrm{r_s}(\mathrm{C}_{L^*}(s))} & \text{if } (\Lambda' \Lambda'') \in X, \\ 0 & \text{if } (\Lambda', \Lambda'') \notin X. \end{cases}$$

Proof. We shall argue as Lusztig in [15, 4.13, 4.15]. Let C' be a unipotent class in G such that $a(\Lambda') + a(\Lambda'') \geq d(C')$, and let $u' \in C'$. We assume that u' is F–stable. Using (3.7), we obtain

$$\chi_{A(\Lambda', \Lambda'')}(u') = (-1)^n \, q^{b(\Lambda * \Lambda') + \frac{1}{2} \mathrm{r_s}(\mathrm{C}_{L^*}(s))} \, Y_{\Lambda' * \Lambda''}(u'). \tag{4.8}$$

If $u' \in C(\Lambda' * \Lambda'')$ we have $Y_{\Lambda' * \Lambda''}(u') = \chi_{\Lambda' * \Lambda''}(u')$.

Assume now that $C' \neq C(\Lambda' * \Lambda'')$. Since

$$d(C') \leq a(\Lambda') + a(\Lambda') \leq b(\Lambda' * \Lambda'') = d(C(\Lambda' * \Lambda'')), \tag{4.9}$$

we have $\dim C' \geq \dim C(\Lambda' * \Lambda'')$ and since $C' \neq C(\Lambda' * \Lambda'')$, the class C' is not contained in the closure of the class $C(\Lambda' * \Lambda'')$. Hence $Y_{\Lambda' * \Lambda''}(u') = 0$. Hence we get

$$\chi_{A(\Lambda', \Lambda'')}(u') = \begin{cases} (-1)^n \, q^{b(\Lambda * \Lambda') + \frac{1}{2} \mathrm{r_s}(\mathrm{C}_{L^*}(s))} \, \chi_{\Lambda' * \Lambda''}(u') & \text{if } C' = C(\Lambda' * \Lambda''), \\ 0 & \text{if } C' \neq C(\Lambda' * \Lambda''). \end{cases}$$

Now note that the condition $C' = C(\Lambda' * \Lambda'')$ implies

$$b(\Lambda' * \Lambda'') \leq a(\Lambda') + a(\Lambda'') \leq b(\Lambda' * \Lambda''), \tag{4.10}$$

hence implies that

$$b(\Lambda' * \Lambda'') = a(\Lambda') + a(\Lambda'') = d(C'). \tag{4.11}$$

In particular, if $a(\Lambda') + a(\Lambda'') \neq d(C')$, then $\chi_{A(\Lambda', \Lambda'')} = 0$. Hence the result follows. □

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