Exact Minimum Eigenvalue Distribution of a Correlated Complex Non-Central Wishart Matrix

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Abstract— We derive the exact cumulative distribution function (c.d.f.) of the minimum eigenvalue of a correlated complex non-central Wishart matrix. This result is in the form of a simple infinite series with fast convergence, and applies for the important case where the non-centrality matrix has rank one. Simplified asymptotic expressions for the c.d.f. are given for large matrix dimensions, as well as first-order expansions around the origin. The eigenvalue distributions in this paper have various important applications to multiple-input multiple-output (MIMO) communication systems, as well as other scientific areas such as econometrics and multivariate statistics.

I. INTRODUCTION

Eigenvalue distributions of Wishart matrices arise in many scientific fields. Prominent examples include multiple-input multiple-output (MIMO) wireless communication systems [1]–[5], synthetic aperture radar (SAR) signal processing [6], econometrics [7], statistical physics [8], and multivariate statistical analysis [9]. In many cases, the Wishart matrices of interest are complex, correlated, and noncentral. Such matrices arise, for example, in MIMO communication channels characterized by line-of-sight (LoS) components (i.e. Rician fading) with spatial correlation amongst the antenna elements [5].

In this paper, we consider the distribution of the minimum eigenvalue of correlated complex noncentral Wishart matrices. These distributions have direct applications to various problems in wireless communications, including the study of MIMO multi-channel beamforming systems [2], the design and analysis of adaptive MIMO multiplexing-diversity switching systems [10], [11], and the performance analysis of linear MIMO receiver structures [12]. They also have direct relevance to other fields (see, e.g. [7]).

Recently, the marginal eigenvalue distributions of random matrices have received much attention. For recent surveys, see [13]–[15]. For the extreme eigenvalues, i.e. the maximum and minimum, distributional results are now available for correlated central, uncorrelated central, and uncorrelated noncentral complex Wishart matrices [2]–[4], [16]–[21]. In the majority of cases, the standard approach has been to integrate the respective joint eigenvalue densities over suitably chosen multi-dimensional regions. For the more general class of correlated noncentral Wishart matrices considered in this paper, however, the joint eigenvalue densities are significantly more complicated, and it seems that this direct approach cannot be easily applied.

In this paper, by employing an alternative derivation technique, we derive a new exact expression for the cumulative distribution function (c.d.f.) of the minimum eigenvalue of a correlated complex noncentral Wishart matrix. This result is a simple expression involving a single infinite series with fast convergence, and can be easily and efficiently computed. For mathematical tractability, we assume symmetric Wishart matrices (i.e. where the matrix dimensionality and degrees-of-freedom are equal), and rank-one non-centrality parameters. These assumptions are practical for a number of applications. For example, symmetric Wishart matrices arise in MIMO systems with equal numbers of transmit and receive antennas, and rank one non-centrality matrices arise when there is a dominant LoS path between the transmitter and receiver.

In addition to the exact c.d.f., we also present simplified asymptotic expansions for large matrix dimensions which, interestingly, are shown to be accurate for even very small matrix sizes (e.g. $3 \times 3$). These results indicate that the statistical behavior of the minimum eigenvalue is mainly governed by the trace of the inverse of the correlation matrix. We also present simple first-order expansions of the c.d.f. around the origin, and show reductions of the main results in some special cases. Finally, our results are confirmed through simulations.

It is important to note that, to our knowledge, this paper presents the first tractable result of any form for the exact (non-asymptotic) joint or marginal eigenvalue distributions of correlated complex noncentral Wishart matrices. We note that previous related expressions have been reported in [22]; however, those expressions are extremely complicated, involving infinite series with inner summations over partitions of numbers, with each term involving invariant zonal polynomials (c.f. Section II). As such, in contrast to the results in this paper, those previous results are not practically useful or tractable from a numerical computation perspective.

The following notation is used throughout the paper. Matrices are represented as uppercase bold-face, and vectors by lowercase bold-face. The superscripts $(\cdot)^H$ and $(\cdot)^T$ indicate the Hermitian-transpose and transpose respectively. $I_p$ denotes a $p \times p$ identity matrix. We use $|\cdot|$ to represent the determinant of a square matrix, $\text{tr}(\cdot)$ to represent trace, and $\text{etr}(\cdot)$ stands for $\exp(\text{tr}(\cdot))$. Finally, $A > 0$ is used to indicate positive definiteness, and $A > B$ denotes $A - B > 0$.

II. PRELIMINARIES

In this section we provide some preliminary results and definitions in random matrix theory which will be useful for subsequent derivations.

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Definition 1: Let \( X \) be an \( n \times m \) (\( n \geq m \)) complex Gaussian random matrix distributed as \( CN_{n,m}(\Upsilon, I_m \otimes \Psi) \), where \( \Psi \in \mathbb{C}^{m \times m} > 0 \) and \( \Upsilon \in \mathbb{C}^{n \times m} \). Then \( W = X^H X \) has a correlated complex non-central Wishart distribution \( \mathcal{W}_m(n, \Psi, \Theta) \) with density function [9]

\[
f(W) = \frac{\exp(-\Theta)}{\Gamma_m(n)} \left| \Psi \right|^n \times o_\phi^T(\Theta) \prod_{\kappa,\sigma \leq 0} \Gamma_m(n - j + 1)
\]

where \( \Theta = \Psi^{-1} \Upsilon H^H \) is the non-centrality parameter, \( \Gamma_m(n) \) denotes the Bessel-type complex hypergeometric function of a matrix argument, and \( \Gamma_m(\cdot) \) represents the complex multivariate gamma function defined as

\[
\Gamma_m(n) \triangleq \prod_{j=1}^{m} \Gamma(n - j + 1)
\]

with \( \Gamma(\cdot) \) denoting the classical gamma function.

Non-central distribution problems in multivariate statistics commonly give rise to various classes of invariant polynomials [23]. The invariant polynomials described below are particularly important when dealing with eigenvalue distribution of correlated complex noncentral Wishart matrices.

Definition 2: The class of homogeneous polynomials in the elements of the \( m \times m \) Hermitian complex matrices \( X \) and \( Y \) which are invariant under simultaneous unitary transformations:

\[
X \rightarrow H^H X H, \quad Y \rightarrow H^H Y H, \quad H \in U(m)
\]

is denoted by \( C_{\phi}^{n,\sigma}(X, Y) \) [24], [25]. These polynomials satisfy the following fundamental relationship

\[
\int_{U(m)} C_{\phi}(A H^H X H, C_{\phi}(B H^H Y H) dH = \sum_{\phi \in \kappa,\sigma} \frac{C_{\phi}^{n,\sigma}(A, B) C_{\phi}(X, Y)}{\Gamma_m(I_m)}
\]

where \( C_{\phi}, C_{\sigma}, C_{\theta} \) are complex zonal polynomials indexed by the ordered partitions \( \kappa, \sigma, \phi \) of the non-negative integers \( k, s, f \) such that \( f = k + s \), into not more than \( m \) parts. Let \( G(m, C) \) represent the linear group of \( m \times m \) nonsingular complex matrices. Then, the notation \( \phi \in \kappa, \sigma \) denotes the irreducible representation of \( G(m, C) \) indexed by \( 2\phi \) that occurs in the decomposition of the Kronecker product \( 2\kappa \otimes 2\sigma \) of the irreducible representations indexed by \( 2\kappa \) and \( 2\sigma \) [24].

The next lemma presents the joint eigenvalue distribution of a correlated complex non-central Wishart matrix, in terms of invariant polynomials.

Lemma 1: The joint density of the ordered eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0 \) of the correlated complex non-central Wishart matrix \( W \sim W_m(n, \Psi, \Theta) \) is given by [22]

\[
g(A) = \frac{\pi^{m(m-1)} \exp(-\Theta)}{\Gamma_m(n) \prod_{k=1}^{m} \lambda_k^{n-k}} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 \
\]

\[
\times \sum_{k,s=0}^{m} \sum_{\kappa,\sigma \in \kappa,\sigma} C_{\phi}^{n,\sigma}(\Psi^{-1} \Theta^{-1} \Psi^{-1}) \frac{k! s! [n]_\sigma C_{\phi}(I_m)}{\Gamma_m(n)}
\]

where \( A \) is a diagonal matrix containing the eigenvalues of \( W \) along the main diagonal, and \([n]_\sigma\) denotes the complex hypergeometric coefficient defined as

\[
[n]_\sigma = \prod_{j=1}^{m} (n-j+1)_\sigma
\]

in which \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \) is a partition of the integer \( s \) into \( m \) parts such that \( \sum_{j=1}^{m} \sigma_j = s \) and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq 0 \). Also, \((\cdot)_k\) is the Pochhammer symbol given by

\[
(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad (a)_0 = 1.
\]

To determine the distribution of the minimum eigenvalue, the most direct approach is to integrate the joint eigenvalue probability density function (p.d.f.) given in (3) as follows:

\[
F_{\lambda_{\text{min}}}(x) = 1 - \Pr(\lambda_1 < \cdots < \lambda_m > x)
\]

\[
= 1 - \int_{\mathcal{D}} g(A) d\lambda_1 \cdots d\lambda_m
\]

where \( \mathcal{D} = \{ x < \lambda_m < \cdots < \lambda_1 \} \). This direct approach, however, is particularly difficult for two main reasons: (i) due to the presence of the invariant polynomials in (3), and (ii) due the unbounded upper limit of the integrals which makes term-by-term integration intractable. To circumvent these complexities, in the following we adopt an alternative derivation approach based on integrating directly over the correlated complex noncentral Wishart p.d.f. in (1).

III. NEW MINIMUM EIGENVALUE DISTRIBUTION

A. Exact Minimum Eigenvalue C.D.F.

The following theorem presents the main result of the paper; namely, the exact formula for the minimum eigenvalue distribution.

Theorem 1: Let \( X \) be an \( m \times m \) complex square matrix distributed as \( X \sim CN_{m,m}(\Upsilon, I_m \otimes \Psi) \), where \( \Psi \in \mathbb{C}^{m \times m} \) has rank one. Then the c.d.f. of the minimum eigenvalue \( \lambda_{\text{min}} \) of the correlated complex non-central Wishart matrix \( W = X^H X \sim W_m(n, \Psi, \Theta) \) is given by

\[
F_{\lambda_{\text{min}}}(x) = 1 - \exp(-\lambda \mu) \prod_{\text{tr}(\Theta)} (-x \Psi^{-1})
\]

\[
\times \sum_{k=0}^{\infty} \left( \frac{x \mu^k}{k!} \right) \sum_{t=0}^{k} \binom{k}{t} (m)_t \left( \frac{\lambda}{x \mu} \right)^t
\]

where \( \lambda = \text{tr}(\Theta) \) and \( \mu = \text{tr}(\Theta \Psi^{-1}) \).

Proof: In general terms, the method of proof is to first determine \( \Pr(\lambda_1 < \cdots < \lambda_m > x) \), and then apply the substitution \( \Upsilon = x^H x \) [22], [24], [26] to yield \( \Pr(\lambda_{\text{min}} > x) \). Then, using

\[
F_{\lambda_{\text{min}}}(x) = \Pr(\lambda_{\text{min}} < x) = 1 - \Pr(\lambda_{\text{min}} > x)
\]

we arrive at the desired c.d.f. of the minimum eigenvalue.
We start by substituting \( m = n \) into (1) to yield the joint density function of \( W \) of our interest as

\[
f(W) = \frac{\exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \text{etr} (-\Psi^{-1}W) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \sum_{k=0}^{\infty} \Theta \Psi^{-1} \text{etr} (-\Psi^{-1}W) dW \right) \right)
\]

Now, \( \text{Pr}(W > \Omega) \) can be written as

\[
\text{Pr}(W > \Omega) = \int_{W, \Omega} f(W) dW.
\]

Since \( \lambda \min > x \) is equivalent to \( W > xI_{m} \), we can rewrite (9) by choosing \( \Omega = xI_{m} \) as

\[
\text{Pr}(\lambda_{\min} > x) = \frac{\exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \int_{W > xI_{m}} \text{etr} (-\Psi^{-1}W) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \Theta \Psi^{-1} \text{etr} (-\Psi^{-1}W) dW \right) \right)
\]

Introducing the change of variables \( W = x(I_{m} + Y) \) with its differential form \( dW = x^m dY \) in (10) yields

\[
\text{Pr}(\lambda_{\min} > x) = \frac{x^m \exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \int_{Y > 0} \text{etr} (-\Psi^{-1}Y) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \Theta \Psi^{-1} (I_{m} + Y) dY \right) \right)
\]

Further manipulation in this form is highly difficult. Therefore, at this point, it is convenient to expand the hypergeometric function with its equivalent zonal polynomial series expansion to give

\[
\text{Pr}(\lambda_{\min} > x) = \frac{x^m \exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \int_{Y > 0} \text{etr} (-\Psi^{-1}Y) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \Theta \Psi^{-1} (I_{m} + Y) dY \right) \right)
\]

where \( \kappa = (\kappa_1, \ldots, \kappa_m) \) is a partition of \( k \) into not more than \( m \) parts such that \( \kappa_1 \geq \ldots \geq \kappa_m > 0 \).

Before proceeding, it is noteworthy to observe that the matrix \( \Theta \Psi^{-1} \) is nonnegative definite Hermitian with rank one. Thus, its can be represented using eigen decomposition as

\[
\Theta \Psi^{-1} = \mu \alpha \alpha^H
\]

where \( \alpha \in \mathbb{C}^{m \times 1} \) and \( \alpha^H \alpha = 1 \). Recalling that zonal polynomials depend only on the eigenvalues of their matrix arguments, and noting that the matrix \( \Theta \Psi^{-1} (I_{m} + Y) \) is also rank one, we can write (12) with the aid of (13) as follows

\[
\text{Pr}(\lambda_{\min} > x) = \frac{x^m \exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \int_{Y > 0} \text{etr} (-\Psi^{-1}Y) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \text{etr} (-\Psi^{-1}Y) \sum_{k=0}^{\infty} \Theta \Psi^{-1} (I_{m} + Y) dY \right) \right)
\]

Applying the complex analogue of [27, Corollary 7.2.4], we see that since \( \alpha^H (I_{m} + Y) \alpha \) is rank one, then for all partitions \( \kappa \) having more than one non-zero part it follows that \( C_k(x\mu \alpha^H (I_{m} + Y) \alpha) = 0 \). Hence we get

\[
C_k(x\mu \alpha^H (I_{m} + Y) \alpha) = (x\mu \alpha^H (I_{m} + Y) \alpha)^k.
\]

Therefore, we can write (14) as

\[
\text{Pr}(\lambda_{\min} > x) = \frac{x^m \exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \times \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1}Y \sum_{k=0}^{\infty} \Theta \Psi^{-1} (I_{m} + Y) dY \right) \right)
\]

Now, simple manipulation yields \( \alpha^H (I_{m} + Y) \alpha = 1 + \alpha^H Y \alpha \) and we may use the binomial expansion to obtain

\[
\alpha^H (I_{m} + Y) \alpha = (1 + \alpha^H Y \alpha)^k = \sum_{t=0}^{k} \binom{k}{t} \alpha^H Y \alpha^t
\]

where \( \binom{k}{t} = \frac{k!}{t!(k-t)!} \). Substituting (17) into (16) gives

\[
\text{Pr}(\lambda_{\min} > x) = \frac{x^m \exp(-\lambda)}{\Gamma_m(m)|\Psi|^m} \times \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \sum_{t=0}^{k} \binom{k}{t} J(x)
\]

where

\[
J(x) = \int_{Y > 0} \text{etr} (-\Psi^{-1}Y) \sum_{k=0}^{\infty} \frac{(x\alpha)^k}{k!} \left( \text{etr} \left( -\Psi^{-1} \Theta \Psi^{-1} (I_{m} + Y) dY \right) \right)
\]

Finally, substituting (24) into (18) and applying some manipulations using (7), we arrive at the result in (6).
Note that the c.d.f. expression (6) allows the correlation matrix $\Psi$ to have eigenvalues with arbitrary multiplicities.

In some cases it is convenient to re-sum the infinite series in (6) as a power series in $x$ as follows

$$
\sum_{k=0}^{\infty} \frac{(x \mu)^k}{k!(m)_k} \sum_{t=0}^{\infty} \binom{k}{t} (m)_t \left( \frac{\lambda}{x \mu} \right)^t
$$

$$
= \sum_{k=0}^{\infty} \frac{(x \mu)^k}{k!(m)_k} \sum_{t=0}^{\infty} (m)_{t-k}(t-k)! \lambda^{t-k}
$$

$$
= \sum_{k=0}^{\infty} \frac{(x \mu)^k}{k!(m)_k} F_1 (m; m + k; \lambda)
$$

$$
= \exp(\lambda) \sum_{k=0}^{\infty} \frac{(x \mu)^k}{k!(m)_k} \Gamma(m; k; m + k; -\lambda)
$$

where $\Gamma(\cdot)$ is the confluent hypergeometric function. Note that the final equality was obtained by using Kummer’s transformation [29, eq. 9.212.1]. As such, we have the following alternative formula for the minimum eigenvalue c.d.f.

$$
F_{\lambda_{\min}}(x) = 1 - \exp(\lambda) \sum_{k=0}^{\infty} \frac{(x \mu)^k}{k!(m)_k} \Gamma(m; k; m + k; -\lambda).
$$

(26)

**B. Expansions and Special Cases**

In this section, we provide some simplified asymptotic expansions of the minimum eigenvalue c.d.f., and examine some special cases.

The following corollary presents a very simple representation for the minimum eigenvalue c.d.f. as the matrix dimension $m$ grows large.

**Corollary 1:** For large $m$, the c.d.f. (26) becomes

$$
F_{\lambda_{\min}}(x) \approx 1 - \exp(-x \Psi^{-1})
$$

(27)

**Proof:** The result follows by noting that, for sufficiently large $m$, the infinite summation in (26) is dominated by its first term.

Our numerical results will confirm that this asymptotic expression is accurate even for small values of $m$ (eg. $m = 3$). It should be noted that if $\text{tr}(\Psi^{-1}) < \infty$ as $m \rightarrow \infty$, then (27) represents a finite upper bound for the minimum eigenvalue c.d.f.

The following corollary presents a first-order expansion in $x$ for the minimum eigenvalue c.d.f. This result is particularly useful for deriving the diversity order and array gain of MIMO multichannel beamforming wireless systems (see, eg. [2]).

**Corollary 2:** For small $x$, the c.d.f. (26) becomes

$$
F_{\lambda_{\min}}(x) \approx \left( \text{tr}(\Psi^{-1}) - \frac{\mu}{m} F_1 (1; m + 1; -\lambda) \right) x.
$$

(28)

**Proof:** The result follows by substituting $k = 1$ into (26), expanding the exponential factor, and then summing first order terms.

**Corollary 3:** For $\Psi = I_m$ (i.e., the uncorrelated case), the c.d.f. (6) becomes

$$
F_{\lambda_{\min}}(x) = 1 - \exp(-mx - \lambda M)
$$

$$
\times \sum_{k=0}^{\infty} \frac{(\lambda M x)^k}{k!(m)_k} \sum_{t=0}^{\infty} \binom{k}{t} (m)_t \left( \frac{1}{x} \right)^t
$$

(29)

where $\lambda M = \text{tr}(\Psi^H \Psi)$.

Proof: From (6), $\Psi = I_m$ implies that $\lambda = \mu = \lambda M$. The result follows by direct substitution. 

An alternative, more complicated, expression for the same scenario has been obtained in [2, eq. 57-61].

**Corollary 4:** For $\Psi = 0$ (i.e., the correlated central Wishart case), the c.d.f. (6) becomes

$$
F_{\lambda_{\min}}(x) = 1 - \exp(-x \Psi^{-1}).
$$

(30)

**Proof:** The result follows by direct substitution. 

This result, for correlated central Wishart matrices, agrees exactly with a very recent expression in [18, eq. 2.15].

**IV. NUMERICAL RESULTS**

This section presents simulation results to confirm the accuracy of our analytical minimum eigenvalue c.d.f. expressions, and to investigate the joint effect of correlation and non-zero mean. For our simulations, we model the mean matrix $\Psi$ and correlation matrix $\Sigma$ as follows

$$
\Psi = \sum_{i,j=1}^{K} a_{i,j} R_{i,j}
$$

(31)

where $K \in [0, \infty)$ is a constant, we choose $a = (1 0 .... 0)^T_{m \times 1}$, and the $(i,j)$'th entry of $R$ is obtained via the popular exponential correlation model [30] as

$$
R_{i,j} = \rho^{|i-j|}, \quad 0 \leq \rho < 1, \quad i, j = 1, 2, ..., m.
$$

(32)

Note that this structure is related to the correlated Rician faded MIMO channel model (see, eg. [5]).

Fig. 1 shows the minimum eigenvalue c.d.f. for different dimensions $m$. The “Analytical” curves were generated based on (6), with the infinite series truncated to only 20 terms. The “Analytical (Asymptotic)” curves were generated based on (27). We have used $K = 1$ and $\rho = 0.1$. We clearly see a precise agreement between the exact analytical curves and their simulated counterparts. Moreover, the asymptotic curves, formally valid for large $m$, are seen to very accurately approximate the exact c.d.f.s for even low matrix dimensions (eg. $m = 3$). It is also interesting to note that the c.d.f.s become steeper as $m$ increases. This behavior can be readily seen from (27), upon noting that the factor $\text{tr}(\Psi^{-1})$ is increasing with $m$ for the exponential correlation model.

Analogous results for the uncorrelated central Wishart case can also be found in [15], [31].

Fig. 2 investigates the effect of correlation and $K$ factor on the minimum eigenvalue c.d.f. Results are shown for $m = 3$. The “Analytical” curves were generated based on (6), again using only 20 terms in the infinite series. We see that the c.d.f.s become steeper when either the correlation coefficient $\rho$ or the $K$ factor increase.

**V. CONCLUSION**

We have derived a new exact expression for the minimum eigenvalue of a correlated complex non-central Wishart matrix. Our main result, which applies for symmetric Wishart scenarios with rank-one non-centrality parameters, is expressed in the

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Note that the presence of zero-valued entries in $a$ is not required. We choose to adopt this basic model here for simplicity.
form of a simple infinite series which converges quickly and is easy and efficient to compute. This distribution is important for a variety of different applications; including MIMO communications, SAR signal processing, statistical physics, and econometrics. We also presented a new simplified asymptotic expression for the c.d.f. for large matrix dimensions which, interestingly, was shown to be accurate for even very low dimensions. These results indicated that the statistical behavior of the minimum eigenvalue is mainly governed by the trace of the inverse of the correlation matrix. We also presented simple first-order expansions of the c.d.f. around the origin. Our results were confirmed through simulations.

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