Lattices associated with distance-regular graphs

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Abstract

Let $\mathcal{L}$ be a finite set associated with cliques of a distance-regular graph of order $(s, t)$, with $d$-cliques of Johnson graphs and antipodal distance-regular graphs of diameter $d$, respectively. If we partially order $\mathcal{L}$ by the ordinary inclusion, three families of finite atomic lattices are obtained. This article discusses their geometricity, and computes their characteristic polynomials.

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1. Introduction

In this section we recall some terminology and definitions concerning finite posets and lattices. For more information, the readers may consult [1].

Let $P$ be a poset. For $a, b \in P$, we say $a$ covers $b$, denoted by $b < \cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If $P$ has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1) and say that $P$ is a poset with 0 (resp. 1). Let $P$ be a finite poset with 0. By a rank function on $P$, we mean a function $r$ from $P$ to the set of all the nonnegative integers such that

(i) $r(0) = 0$.
(ii) $r(a) = r(b) + 1$ whenever $b < \cdot a$.

Let $P$ be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

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is called the characteristic polynomial of $P$, where $\mu$ is the Möbius function and $r$ is the rank function of $P$.

A poset $P$ is said to be a lattice if both $a \lor b := \sup\{a, b\}$ and $a \land b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let $P$ be a finite lattice with 0. By an atom in $P$, we mean an element in $P$ covering 0. We say $P$ is atomic if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice $P$ is said to be a geometric lattice if $P$ admits a rank function $r$ satisfying

$$r(a \land b) + r(a \lor b) \leq r(a) + r(b), \quad \forall a, b \in P.$$ 

In a series of papers [3–8], Huo and Wan et al. constructed lattices from orbits of subspaces under finite classical groups, computed their characteristic polynomials and discussed their geometricity. In this paper, we construct finite atomic lattices from the cliques of a distance-regular graph of order $(s, t)$, from $d$-cliques of Johnson graphs and antipodal distance-regular graphs of diameter $d$, respectively. Moreover we discuss their geometricity, and compute their characteristic polynomials.

2. Distance-regular graphs

In this section, we first recall some definitions concerning distance-regular graphs; for more information, the readers may consult [2].

Let $\Gamma = (X, R)$ be a connected graph. For vertices $u$ and $v$ in $X$, let $\partial(u, v)$ denote the distance between $u$ and $v$. An $l$-subset $A$ of $X$ is said to be a $d$-clique with size $l$ if any two distinct vertices of $A$ are at distance $d$. The empty set $\emptyset$ is regarded as the clique (resp. $d$-clique) with size 0. A 1-clique is said to be a clique.

A connected graph $\Gamma = (X, R)$ is said to be distance-regular whenever for all integers $h, i, j$ and for all $u, v \in X$ with $\partial(u, v) = h$, the number $|\{w \in X \mid \partial(u, w) = i, \partial(w, v) = j\}|$ is independent of the choice of $u$ and $v$. For any vertex $u$, let $\Gamma_i(u) = \{v \in X \mid \partial(u, v) = i\}$. In this case, the size of $\Gamma_i(u)$ depends only on $i$, denoted by $k_i$.

A distance-regular graph $\Gamma = (X, R)$ is said to be of order $(s, t)$ if, for each vertex $u \in X$, the induced subgraph on $\Gamma(u)$ is a disjoint union of $t + 1$ cliques with size $s$. Then each maximal clique is of size $s + 1$, and each vertex is contained in $t + 1$ maximal cliques.

**Lemma 2.1.** Let $\Gamma = (X, R)$ be a distance-regular graph of order $(s, t)$ with $n$ vertices. Then the following hold.

(i) The number of cliques with size $l \geq 2$ is $n^{(t+1)s+1} \binom{s+1}{l}$. 

(ii) The number of cliques with size $l \geq 2$ containing a given vertex is $(t + 1) \binom{s}{l-1}$.

**Proof.** (i) By computing the number of pairs $(u, C)$ in two ways, where $C$ is a maximal clique containing a vertex $u$, the number of the maximal cliques of $\Gamma$ is $\frac{n^{(t+1)s+1}}{s+1}$. Since any two distinct maximal cliques have at most one common vertex, (i) follows.

(ii) By computing the number of pairs $(u, C)$ in two ways, where $C$ is a clique with size $l$ containing a vertex $u$, (ii) follows. $\blacksquare$

Let $V$ be a set of size $n \geq 2d$, and let $X = \{A \subset V \mid |A| = d\}$. Let $J(n, d)$ denote the Johnson graph defined on $X$ such that $A$ and $B$ are adjacent if $|A \cap B| = d - 1$. Since $\partial(A, B) = d$ if and only if $A \cap B = \emptyset$, the number of $d$-cliques of $J(dm, d)$ with size $l$ is

$$u(m, d; l) = \binom{dm}{dl} (dl)!/(dl)!l!.$$
Lemma 2.2. Let $\Gamma$ be the Johnson graph $J(dm, d)$. For a given $d$-clique $A$ with size $l_1 \geq 1$, let $u(m, d; l_1, l_2)$ denote the number of $d$-cliques with size $l_2$ containing $A$. Then

$$u(m, d; l_1, l_2) = \frac{u(m, d; l_2)}{u(m, d; l_1)}.$$

Proof. The proof is similar to that of Lemma 2.1 and will be omitted. □

A distance-regular graph $\Gamma$ of diameter $d \geq 2$ is said to be antipodal if $\partial(x, y) = \partial(x, z) = d$ implies that $y = z$ or $\partial(y, z) = d$. The following results are obvious.

Lemma 2.3. Let $\Gamma = (X, R)$ be an antipodal distance-regular graph of diameter $d$ with $n$ vertices.

(i) The numbers of $d$-cliques with size $l$ is $u(n, l) = \binom{n\cdot k}{l}$. 

(ii) The numbers of $d$-cliques with size $l$ containing a given $d$-clique with size $i$ is $\binom{k-1}{l-i}$.

3. Lattices associated with distance-regular graphs of order $(s, t)$

In this section, we always assume that $\Gamma = (X, R)$ is a distance-regular graph of order $(s, t)$ with $n$ vertices. Let $C$ denote the set of all the cliques of $\Gamma$, and let $\mathcal{L}(s, t) = C \cup \{X\}$. If we partially order $\mathcal{L}(s, t)$ by the ordinary inclusion, then $\mathcal{L}(s, t)$ is a finite atomic lattice, denoted by $\mathcal{L}_O(s, t)$.

Theorem 3.1. $\mathcal{L}_O(s, t)$ is a geometric lattice if and only if $t = 0$.

Proof. For any $A \in \mathcal{L}_O(s, t)$, define

$$r(A) = \begin{cases} s + 2, & \text{if } A = X, \\ |A|, & \text{otherwise}. \end{cases}$$

Then $r$ is the rank function of $\mathcal{L}_O(s, t)$.

If $t = 0$, then $\Gamma$ is a clique, and so $\mathcal{L}_O(s, t)$ is a geometric lattice. Now suppose that $t \geq 1$. Then the diameter of $\Gamma$ is at least two. For any two vertices $A$ and $B$ at distance two, we have

$$r(A \wedge B) + r(A \vee B) = s + 2 > 2 = r(A) + r(B).$$

Hence $\mathcal{L}_O(s, t)$ is not a geometric lattice whenever $t \geq 1$. □

Lemma 3.2. The Möbius function of $\mathcal{L}_O(s, t)$ is

$$\mu(A, B) = \begin{cases} \frac{(-1)^{|B| - |A|}}{n - nst - s - 1}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\ 1, & \text{if } \emptyset = A < B = X, \\ -1, & \text{if } A < B = X \text{ and } |A| = 1, \\ 0, & \text{if } A < B = X \text{ and } |A| = s + 1, \text{ otherwise}. \end{cases}$$
Proof. The Möbius function of $\mathcal{L}_O(s, t)$ is
\[
\mu(A, B) = \begin{cases} 
(-1)^{|B|-|A|}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\
- \sum_{A \leq C < X} (-1)^{|C|-|A|}, & \text{if } A < B = X, \\
0, & \text{otherwise}.
\end{cases}
\]
By Lemma 2.1 we obtain
\[
\mu(\emptyset, X) = -1 + n - \frac{n(t + 1)}{s + 1} \sum_{i=2}^{s+1} (-1)^i \binom{s + 1}{i} = \frac{n - nst - s - 1}{s + 1}.
\]
If $|A| = 1$, by Lemma 2.1 again we obtain
\[
\mu(A, X) = -1 - (t + 1) \sum_{i=1}^{s} (-1)^i \binom{s}{i} = t.
\]
If $2 \leq |A| < s + 1$, then
\[
\sum_{A \leq C < X} (-1)^{|C|-|A|} = \sum_{i=0}^{s+1-|A|} (-1)^i \binom{s + 1 - |A|}{i} = 0.
\]
If $|A| = s + 1$, then $\sum_{A \leq C < X} (-1)^{|C|-|A|} = 1$.
Hence the desired result follows. □

Theorem 3.3. The characteristic polynomial of $\mathcal{L}_O(s, t)$ is
\[
\chi(\mathcal{L}_O(s, t), x) = x^{s+2} + \frac{n - nst - s - 1}{s + 1} - n x^{s+1} + \frac{n(t + 1)}{s + 1} \sum_{i=2}^{s+1} (-1)^i \binom{s + 1}{i} x^{s+2-i}.
\]

Proof. It is immediate by Lemma 3.2. □

4. Lattices associated with Johnson graphs

In this section we always assume that $\Gamma = (X, R)$ is the Johnson graph $J(dm, d)$ with $m \geq 2$. Let $C$ be the set of all the $d$-cliques of $\Gamma$, and let $\mathcal{L}(d, m) = C \cup \{X\}$. If we partially order $\mathcal{L}(d, m)$ by the ordinary inclusion, then $\mathcal{L}(d, m)$ is a finite atomic lattice, denoted by $\mathcal{L}_O(d, m)$.

Theorem 4.1. $\mathcal{L}_O(d, m)$ is a geometric lattice if and only if $d = 1$.

Proof. For any $A \in \mathcal{L}_O(s, t)$, define
\[
r(A) = \begin{cases} 
m + 1, & \text{if } A = X, \\
|A|, & \text{otherwise}.
\end{cases}
\]
Note that $r$ is the rank function of $\mathcal{L}_O(d, m)$.
If $d = 1$, $J(dm, d)$ is a clique, and so $\mathcal{L}_O(d, m)$ is a geometric lattice. Now suppose $d \geq 2$.
Pick two adjacent vertices $A$ and $B$ of $\Gamma$. Then
\[
r(A \land B) + r(A \lor B) = m + 1 > 2 = r(A) + r(B).
\]
Hence $\mathcal{L}_O(d, m)$ is not a geometric lattice whenever $d \geq 2$. □
Lemma 4.2. The Möbius function of $L_O(d, m)$ is

$$
\mu(A, B) = \begin{cases} 
(-1)^{|B|-|A|}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\
\sum_{i=0}^{m} (-1)^{i+1} u(m, d; i), & \text{if } \emptyset = A < B = X, \\
\sum_{i=0}^{m-|A|} (-1)^{i+1} u(m, d; |A|, |A| + i), & \text{if } \emptyset \neq A < B = X, \\
0, & \text{otherwise.}
\end{cases}
$$

Proof. The Möbius function of $L_O(d, m)$ is

$$
\mu(A, B) = \begin{cases} 
(-1)^{|B|-|A|}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\
- \sum_{A \leq C < X} (-1)^{|C|-|A|}, & \text{if } A < B = X, \\
0, & \text{otherwise.}
\end{cases}
$$

By Lemma 2.2 we obtain

$$
\mu(\emptyset, X) = - \sum_{\emptyset \leq C < X} (-1)^{|C|} = \sum_{i=0}^{m} (-1)^{i+1} u(m, d; i),
$$

and

$$
\sum_{\emptyset \neq A \leq C < X} (-1)^{|C|-|A|} = \sum_{i=0}^{m-|A|} (-1)^i u(m, d; |A|, |A| + i).
$$

Hence the lemma holds. \(\square\)

Theorem 4.3. The characteristic polynomial of $L_O(d, m)$ is

$$
\chi(L_O(d, m), x) = \sum_{i=0}^{m} (-1)^i u(m, d; i)(x^{m+1-i} - 1).
$$

Proof. It is immediate by Lemma 4.2. \(\square\)

5. Lattices associated with antipodal distance-regular graphs

In this section, we always assume that $\Gamma = (X, R)$ is an antipodal distance-regular graph of diameter $d$ with $n$ vertices. Let $C$ be the set of all the $d$-cliques of $\Gamma$, and let $L(d) = C \cup \{X\}$. If we partially order $L(d)$ by the ordinary inclusion, then $L(d)$ is a finite atomic lattice, denoted by $L_O(d)$.

Theorem 5.1. $L_O(d)$ is not a geometric lattice.

Proof. For any $A \in L_O(d)$, define

$$
r(A) = \begin{cases} 
k_d + 2, & \text{if } A = X, \\
|A|, & \text{otherwise.}
\end{cases}
$$
Then \( r \) is the rank function of \( L_0(d) \). For any two adjacent vertices \( A \) and \( B \) of \( \Gamma \), we have
\[
r(A \land B) + r(A \lor B) = k_d + 2 > 2 = r(A) + r(B).
\]
Hence \( L_0(d) \) is not geometric. □

**Lemma 5.2.** The Möbius function of \( L_0(d) \) is
\[
\mu(A, B) = \begin{cases} 
(-1)^{|B| - |A|}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\
\frac{n - k_d - 1}{k_d + 1}, & \text{if } \emptyset = A < B = X, \\
-1, & \text{if } A < B = X \text{ and } |A| = k_d + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** The Möbius function of \( L_0(d) \) is
\[
\mu(A, B) = \begin{cases} 
(-1)^{|B| - |A|}, & \text{if } A \leq B \neq X \text{ or } A = B = X, \\
\sum_{A \leq C < X} (-1)^{|C| - |A|}, & \text{if } A < B = X, \\
0, & \text{otherwise}.
\end{cases}
\]

By Lemma 2.3 we obtain
\[
\mu(\emptyset, X) = -1 - \frac{n}{k_d + 1} \sum_{i=1}^{k_d+1} (-1)^i \binom{k_d + 1}{i} = \frac{n - k_d - 1}{k_d + 1}.
\]
If \( 1 \leq |A| < k_d + 1 \), by Lemma 2.3 again we obtain
\[
\sum_{A \leq C < X} (-1)^{|C| - |A|} = \sum_{i=0}^{k_d+1-|A|} (-1)^i \binom{k_d + 1 - |A|}{i} = 0.
\]
If \( |A| = k_d + 1 \), then \( \sum_{A \leq C < X} (-1)^{|C| - |A|} = 1 \).
Hence the desired result follows. □

**Theorem 5.3.** The characteristic polynomial of \( L_0(d) \) is
\[
\chi(L_0(d), x) = x^{k_d+2} + \frac{n - k_d - 1}{k_d + 1} + \frac{n}{k_d + 1} \sum_{i=1}^{k_d+1} (-1)^i \binom{k_d + 1}{i} x^{k_d+2-i}.
\]

**Proof.** It is immediate by Lemma 5.2. □

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