Vacuum properties in the presence of quantum fluctuations of the quark condensate.*

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Abstract

The quantum fluctuations of the quark condensate are calculated using a regulated Nambu Jona-Lasinio model. The corresponding quantum fluctuations of the chiral fields are compared to those which are predicted by an "equivalent" sigma model. They are found to be large and comparable in size but they do not restore chiral symmetry. The restoration of chiral symmetry is prevented by an "exchange term" of the pion field which does not appear in the equivalent sigma model. A vacuum instability is found to be dangerously close when the model is regulated with a sharp 4-momentum cut-off.

1 Introduction.

This lecture discusses the modifications of vacuum properties which could arise due to quantum fluctuations of the chiral field, more specifically, due to the quantum fluctuations of the quark condensate. The latter is found to be

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surprisingly large, the root mean square deviation of the quark condensate attaining and exceeding 50% of the condensate itself. We shall discuss two distinct modifications of the vacuum: restoration of chiral symmetry due to quantum fluctuations of the chiral field, as heralded by Kleinert and Van den Boosche [1], and a vacuum instability not related to chiral symmetry restoration [2].

2 Chiral symmetry restoration due to quantum fluctuations of the chiral field.

2.1 The linear sigma model argument.

The physical vacuum with a spontaneously broken chiral symmetry is often described by the linear sigma model, which, in the chiral limit ($m_\pi = 0$), has a euclidean action of the form:

$$I = \int d_4x \left(\frac{1}{2} \left(\partial_\mu \sigma\right)^2 + \frac{1}{2} \left(\partial_\mu \pi_i\right)^2 + \frac{\kappa^2}{8} \left(\sigma^2 + \pi_i^2 - f_\pi^2\right)^2 \right) \tag{1}$$

Classically, we have (for translationally invariant fields):

$$\sigma^2 + \pi_i^2 = f_\pi^2 \tag{2}$$

and the vacuum stationary point is:

$$\sigma = f_\pi \qquad \pi_i = 0 \tag{3}$$

We assume that κ^2 is large enough (and the σ-meson is heavy enough) not to have to worry about the quantum fluctuations of the σ field. So we quantize the pion field while the σ field remains classical. We may then say that: $\sigma^2 = f_\pi^2 - \langle \pi_i^2 \rangle$. Classically, $\langle \pi_i^2 \rangle = 0$ but the quantum fluctuations of the pion field make $\langle \pi_i^2 \rangle > 0$ and therefore $\sigma^2 < f_\pi^2$.

Let us estimate the fluctuation $\langle \pi_i^2 \rangle$ of the pion field. A system of free pions of mass m_π is described by the partition function:

$$Z = \int D(\pi)\, e^{-\frac{1}{2}\int d_4x \pi_i \left(-\partial_\mu^2 + m_\pi^2\right)\pi_i} = e^{-\frac{1}{2} tr \ln\left(-\partial_\mu^2 + m_\pi^2\right)} \tag{4}$$

It follows that:

$$\frac{1}{2}\int d_4x \left\langle \pi_i^2 (x) \right\rangle = -\frac{\partial \ln Z}{\partial m_\pi^2} = \frac{\partial}{\partial m_\pi^2} \frac{1}{2} tr \ln\left(-\partial_\mu^2 + m_\pi^2\right) = \frac{1}{2}\Omega \left(N_f^2 - 1\right) \sum_{k<\Lambda} \frac{1}{k^2 + m_\pi^2} \tag{5}$$

where the sum is regularized using a 4-momentum cut-off and where $\Omega = \int d_4 x$.is the euclidean space-time volume. In the chiral limit $m_\pi = 0$, we have:

$$\left\langle \pi_i^2 (x) \right\rangle = \frac{1}{2\Omega} \left(N_f^2 - 1 \right) \sum_{k<\Lambda} \frac{1}{k^2} = \left(N_f^2 - 1 \right) \frac{\Lambda^2}{16\pi^2} \tag{6}$$

so that:

$$\sigma^2 = f_\pi^2 - \left(N_f^2 - 1 \right) \frac{\Lambda^2}{16\pi^2} \tag{7}$$

If we had evaluated this quantity with a 3-momentum cut-off, we would have obtained $\langle \pi_i^2 \rangle = \left(N_f^2 - 1 \right) \frac{\Lambda^2}{8\pi^2}$. Let us pursue with a 4-momentum cut-off. We have:

$$\frac{\langle \pi_i^2 \rangle}{f_\pi^2} = \left(N_f^2 - 1 \right) \frac{\Lambda^2}{16\pi^2 f_\pi^2} \tag{8}$$

We deduce that chiral symmetry restoration will occur when $\sigma = 0$, that is, when $\frac{\langle \pi_i^2 \rangle}{f_\pi^2} > 1$:

$$\frac{\langle \pi_i^2 \rangle}{f_\pi^2} = \left(N_f^2 - 1 \right) \frac{\Lambda^2}{16\pi^2 f_\pi^2} > 1 \tag{9}$$

With $f_\pi = 93\ MeV$ and with $N_f^2 - 1 = 3$ pions, the condition reads:

$$\Lambda^2 > \frac{1}{2.20 \times 10^{-6}} \qquad \Lambda > 674\ MeV \tag{10}$$

In most calculations which use the Nambu Jona-Lasinio model, this condition is fulfilled. We conclude that the quantum fluctuations of the pion do indeed restore chiral symmetry. If we had used a 3-momentum cut-off, chiral symmetry would be restored when $\Lambda > 477\ MeV$.

2.2 The non-linear sigma model argument.

We now argue that this is precisely what is claimed by Kleinert and Van den Boosche [1], although it is said in a considerably different language. They argue as follows. If κ^2 (and therefore the σ mass) is large enough, the action can be thought of as the action of the non-linear sigma model, which in turn can be viewed as an action with N_f^2 fields, namely (σ, π_i), subject to the constraint:

$$\sigma^2 + \pi_i^2 = f_\pi^2 \tag{11}$$

The way to treat the non-linear sigma model is in the textbooks [3]. We work with the action:

$$I_\lambda(\sigma,\pi)=\int d_4x\left(\frac{1}{2}(\partial_\mu\sigma)^2+\frac{1}{2}(\partial_\mu\pi_i)^2+\lambda\left(\sigma^2+\pi_i^2-f_\pi^2\right)\right) \tag{12}$$

in which we add a constraining parameter λ. The action is made stationary with respect to variations of λ. We integrate out the π field, to get the effective action:

$$I_\lambda(\sigma)=\int d_4x\left(\frac{1}{2}(\partial_\mu\sigma)^2+\lambda\left(\sigma^2-f_\pi^2\right)\right)+\frac{1}{2}tr\ln\left(-\partial_\mu^2+\lambda\right) \tag{13}$$

The action is stationary with respect to variations of λ and σ if:

$$\lambda\sigma=0 \qquad \sigma^2=f_\pi^2-\frac{1}{2}\left(N_f^2-1\right)\sum_k\frac{1}{k^2+\lambda} \tag{14}$$

So either $\lambda=0$ and $\sigma\neq0$, in which case we have:

$$\sigma^2=f_\pi^2-\frac{1}{2}\left(N_f^2-1\right)\sum_k\frac{1}{k^2} \tag{15}$$

or $\lambda\neq0$ and $\sigma=0$.

The condition (15) is exactly the same as the condition (7). Thus, the "stifness factor", discussed in Ref.[1], is nothing but a measure of $\frac{\left\langle\pi_i^2\right\rangle}{f_\pi^2}$.

3 Quantum fluctuations of the quark condensate calculated in the Nambu Jona-Lasinio model.

We now show that the quantum fluctuations of the chiral field are indeed large in the Nambu Jona-Lasinio model, but that chiral symmetry is far from being restored. The regularized Nambu Jona-Lasinio model is defined in section 4. We begin by giving some results.

In the Nambu Jona-Lasinio model, the chiral field is composed of a scalar field S and N_f^2-1 pseudoscalar fields P_i. They are related to the quark bilinears:

$$S=V\left(\bar\psi\psi\right) \qquad P_i=V\left(\bar\psi i\gamma_5\tau_i\psi\right) \tag{16}$$

where $V = -\frac{g^2}{N_c}$ is the 4-quark interaction strength. The quark propagator in the vacuum is:

$$\frac{1}{k_\mu \gamma_\mu + M_0 r_k^2} \tag{17}$$

and the model is regularized using either a sharp 4-momentum cut-off or a soft gaussian cut-off function:

$$\begin{aligned} r_k &= 1 \; if \; k^2 < \Lambda^2 \quad r_k = 0 \; if \; k > \Lambda \quad (sharp \; cut-off) & (18) \\ r_k &= e^{-\frac{k^2}{2\Lambda^2}} \quad (gaussian \; regulator) & (19) \end{aligned}$$

. Let $\varphi_0 = M_0$ be the strength of the scalar field in the physical vacuum. We shall show results obtained with typical parameters. If we choose $M_0 = 300 \; MeV$ and $\Lambda = 750 \; MeV$, then $\frac{M_0}{\Lambda} = 0.4$. We then obtain $f_\pi = 94.6 \; MeV$ with a sharp cut-off and $f_\pi = 92.4 \; MeV$ with a gaussian cut-off (in the chiral limit). The interaction strengths are:

$$V = -9.53 \; \Lambda^{-2} \; (sharp \;\; cut-off) \qquad V = -18.4 \; \Lambda^{-2} \; (gaussian \;\; cut-off) \tag{20}$$

and the squared pseudo-scalar field has the expection value

$$\left\langle P_i^2 \right\rangle = V^2 \left\langle \left(\bar{\psi} i \gamma_5 \tau_i \psi \right)^2 \right\rangle \tag{21}$$

At low q we identify the pion field as:

$$\pi_i = \sqrt{Z_\pi} P_i \qquad f_\pi = \sqrt{Z_\pi} M_0 \tag{22}$$

so that, in the Nambu Jona-Lasinio model:

$$\frac{\langle \pi_i^2 \rangle}{f_\pi^2} = \frac{V^2 \left\langle \left(\bar{\psi} i \gamma_5 \tau_i \psi \right)^2 \right\rangle}{M_0^2} \tag{23}$$

where $\left\langle \left(\bar{\psi} i \gamma_5 \tau_i \psi \right)^2 \right\rangle$ is the pion contribution to the squared condensate.

3.1 Results obtained for the quark condensate and for the quantum fluctuations of the chiral field.

Let us examine the values of the quark condensates and of the quantum fluctuations of the chiral field calculated in the chiral limit.

- The quark condensate calculated with a sharp cut off is:

$$\left\langle \bar{\psi}\psi \right\rangle^{\frac{1}{3}} = -0.352 \times \Lambda = 263\ MeV \quad (sharp\ cut - off) \tag{24}$$

  is about 25 % smaller when it is calculated with a soft gaussian regulator:

$$\left\langle \bar{\psi}\psi \right\rangle^{\frac{1}{2}} = -0.280 \times \Lambda = 210\ MeV \quad (gaussian\ regulator) \tag{25}$$

- The magnitude of the quantum fluctuations of the pion field can be measured by the mean square deviation Δ^2 of the condenstate from its classical value:

$$\Delta^2 = \left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle - \left\langle \bar{\psi}\psi \right\rangle^2 \tag{26}$$

  The relative root mean square fluctuation of the condensate Δ is:

$$\frac{\Delta}{\left|\left\langle \bar{\psi}\psi \right\rangle\right|} = 0.41 \quad (sharp\ cut - off) \qquad \frac{\Delta}{\left\langle \bar{\psi}\psi \right\rangle} = 0.77 \quad (gaussian\ regulator) \tag{27}$$

  These are surprisingly large numbers, certainly larger than $1/N_c$. The linear sigma model estimate did give us a fair warning that this might occur.

- This feature also applies to the ratio $\frac{\left\langle \pi_i^2 \right\rangle}{f_\pi^2} = \frac{V^2\left\langle \left(\bar{\psi}i\gamma_5\tau_i\psi\right)^2 \right\rangle}{M_0^2}$ which was so crucial for the linear sigma model estimate of the restoration of chiral symmetry. We find:

$$\frac{\left\langle \pi_i^2 \right\rangle}{f_\pi^2} = 0.38 \quad (sharp\ cut - off) \qquad \frac{\left\langle \pi_i^2 \right\rangle}{f_\pi^2} = 0.85 \quad (gaussian\ regulator) \tag{28}$$

- In spite of these large quantum fluctuations of the chiral field, the quark condensates change by barely a few percent. This is shown in tables 1 and 2 where various contributions to the quark condensate are given in units of Λ^3. The change in the quark condensate is much smaller than $1/N_c$.

$\left\langle \bar{\psi}\psi \right\rangle$	σ-contribution	π-contribution	total
classical	-0.04187	0	-0.04187
exchange term	0.00158	-0.00475	-0.00317
ring graphs	0.00014	0.00134	0.00148
total contribution	-0.04015	-0.00341	-0.04356

Table 1: Various contributions to the quark condensate calculated with a sharp 4-momentum cut-off and with $M_0/\Lambda = 0.4$. The quark condensate is expressed in units of Λ^3.

$\left\langle \bar{\psi}\psi \right\rangle$	σ-contribution	π-contribution	total
classical	-0.02178	0	-0.02178
exchange term	0.00162	-0.00486	-0.00324
ring graphs	0.00077	0.00228	0.00305
total contribution	-0.0193	-0.00258	-0.02197

Table 2: Various contributions to the quark condensate calculated with a gaussian cut-off function and with $M_0/\Lambda = 0.4$. The quark condensate is expressed in units of Λ^3.

3.2 The effect and meaning of the exchange terms.

The tables 1 and 2 show that, among the $1/N_c$ corrections, the exchange terms dominate. The exchange and ring graphs are illustrated on figures 1 and 2. The way in which they arise is explained in section 4.1. The exchange graphs contribute 2-3 times more than the remaining ring graphs. Furthermore, the pion contributes about three times more to the condensate than the sigma, so that the sigma field contributes about as much to the exchange term as any one of the pions. However, the exchange term in the pion channel enhances the quark condensate instead of reducing it. As a result of this there is a very strong cancellation between the exchange terms and the ring graphs. This is why the sigma and pion loops contribute so little to the quark condensate. They increase the condensate by 4% when a sharp cut-off is used, and by 1% when a gaussian regulator is used. This is about ten times less than $1/N_c$.

The ring graphs reduce the condensate (in absolute value) in both the sigma and pion channels. This can be expected. Indeed, the ring graphs promote quarks from the Dirac sea negative energy orbits (which contribute negative values to the condensate) to the positive energy orbits (which contribute positive values to the condensate). The net result is a positive contribution to the condensate which reduces the negative classical value.

What then is the meaning of the exchange terms? The exchange terms have the special feature of belonging to first order perturbation theory (see figures 1 and 2). Their contribution to the energy is not due to a modification of the Dirac sea. It is simply the exchange term arising in the expectation value of the quark-quark interaction in the Dirac sea.

However, the contribution of the exchange term to the quark condensate does involve $q\bar{q}$ excitations. These excitations are due to a modification of the constituent quark mass which is expressed in terms of quark-antiquark excitations of the Dirac sea. The exchange term is modifying (increasing in fact) the constituent quark mass and therefore the value of f_π.

These results suggest that, in order to reduce the Nambu Jona-Lasinio model to an equivalent sigma model, it might be better to include the exchange term in the constituent quark mass, which is another way of saying that, in spite of the $1/N_c$ counting rule, it may be better to do Hartree-Fock theory than Hartree theory. The exchange (Fock) term should be included in the gap equation. The direct (Hartree) term is, of course, included in the classical bosonized action.

In the equivalent sigma model, f_π is proportional to the constituent quark mass. Failure to notice that that the constituent quark mass is altered by the exchange term is what lead Kleinert and Van den Boosche to conclude erronously in Ref.[1] that chiral symmetry would be restored in the Nambu Jona-Lasinio model. They were right however in expecting large quantum fluctuations of the quark condensate.

4 The regularized Nambu Jona-Lasinio model.

The condensates quoted in section 3.1 were calculated with a regularized Nambu Jona-Lasinio model which is defined by the euclidean action:

$$I_m\left(q,\bar{q}\right) = \int d_4x \, \left[\bar{q}\left(-i\partial_\mu\gamma_\mu\right)q + m\bar{\psi}\psi - \left(\frac{g^2}{2N_c}+j\right)\left(\bar{\psi}\Gamma_a\psi\right)^2\right] \qquad (29)$$

The euclidean Dirac matrices are $\gamma_\mu = \gamma^\mu = (i\beta, \vec{\gamma})$. The matrices $\Gamma_a = (1, i\gamma_5\vec{\tau})$ are defined in terms of the $N_f^2 - 1$ generators $\vec{\tau}$ of flavor rotations. Results are given for $N_f = 2$ flavors. The coupling constant $\frac{g^2}{N_c}$ is taken to be inversely proportional to N_c in order to reproduce the N_c counting rules. The current quark mass m is introduced as a source term used to calculate the regularized quark condensate $\left\langle \bar{\psi}\psi \right\rangle$. We have also introduced a source term $\frac{1}{2}j\left(\bar{\psi}\Gamma_a\psi\right)^2$ which is used to calculate the squared quark condensate $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle$.

The quark field is $q(x)$ and the $\psi(x)$ fields are *delocalized* quark fields, which are defined in terms of a *regulator* r as follows:

$$\psi(x) = \int d_4x \ \left\langle x \left| r \right| y \right\rangle \ q(y) \tag{30}$$

The regulator r is diagonal in k-space: $\left\langle k \left| r \right| k' \right\rangle = \delta_{kk'} r(k)$ and its explicit form in given in Eq.(18). The use of a sharp cut-off function is tantamount to the calculation of Feynman graphs in which the quark propagators are cut off at a 4-momentum Λ - a most usual practise. The regulator r, introduced by the delocalized fields, makes all the Feynman graphs converge. A regularization of this type results when quarks propagate in a vacuum described by in the instanton liquid model of the QCD (see Ref.[4] and further references therein). A Nambu Jona-Lasinio model regulated in this manner with a gaussian regulator was first used in Ref.[5], and further elaborated and applied in both the meson and the soliton sectors [5],[6],[7],[8], [9],[2]. Its properties are also discussed in [10].

With one exception. In this work, as in Ref.[2], the regulator multiplies the current quark mass. The introduction of the regulator in the current quark mass term $m\bar{\psi}\psi = m\bar{q}r^2q$ requires some explanation. The current quark mass m is used as a source term to calculate the quark condensate $\left\langle \bar{\psi}\psi \right\rangle$ which, admittedly, would be finite (by reason of symmetry) even in the absence of a regulator - and, indeed, values of quark condensates are usually calculated with an unregularized source term in the Nambu Jona-Lasinio action. However, when we calculate the *fluctuation* $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle - \left\langle \bar{\psi}\psi \right\rangle^2$ of the quark condensate, the expectation value $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle$ diverges. It would be inconsistent and difficult to interpret the fluctuation $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle - \left\langle \bar{\psi}\psi \right\rangle^2$

if $\left\langle \bar{\psi}\psi \right\rangle$ were evaluated using a bare source term and $\left\langle \left(\bar{\psi}\Gamma_a\psi \right)^2 \right\rangle$ using a regulator. When a regularized source term $m\bar{\psi}\psi = m\bar{q}r^2q$ is used, the current quark mass m can no longer be identified with the current quark mass term appearing in the QCD lagrangian. Of course, when a sharp cut-off is used, it makes no difference if the current quark mass term is multiplied by the regulator or not. We have seen in section 3.1 that the leading order contribution to the quark condensate $\left\langle \bar{\psi}\psi \right\rangle^{1/3}$ diminishes by only 20% when the sharp cut-off is replaced by a gaussian regulator. (This statement may be misleading because when the sharp cut-off is replaced by a gaussian regulator, the interaction strength V is also modified so as to fit f_π. If we use a gaussian regulator, the quark condensate calculated with a regulated source term mr^2 is $\left\langle \bar{\psi}\psi \right\rangle = -0.0218\ \Lambda^3$ whereas the quark condensate calculated with a bare source term m is $\left\langle \bar{\psi}\psi \right\rangle = -0.0505\ \Lambda^3$.)

The way in which the current quark mass of the QCD lagrangian appears in the low energy effective theory is model dependent and it has been studied in some detail in Ref.[11] within the instanton liquid model of the QCD vacuum [12],[13],[4].

An equivalent bosonized form of the Nambu Jona-Lasinio action (29) is:

$$I_{j,m}(\varphi) = -Tr\ln\left(-i\partial_\mu\gamma_\mu + r\varphi_a\Gamma_a r\right) - \frac{1}{2}(\varphi - m)(V-j)^{-1}(\varphi - m) \quad (31)$$

The first term is the quark loop expressed in terms of the chiral field φ, which is a chiral 4-vector $\varphi_a = (S, P_i)$ so that $\varphi_a\Gamma_a = S + i\gamma_5\tau_i P_i$. In the second term, the chiral 4-vector $m_a \equiv (m, 0, 0, 0)$ is the current quark mass and V is the local interaction:

$$\langle xa|V|yb\rangle = -\frac{g^2}{N_c}\delta_{ab}\delta(x-y) \quad (32)$$

The partition function of the system, in the presence of the sources j and m is given by the expression:

$$e^{-W(j,m)} = \int D(\varphi)\, e^{-I_{j,m}(\varphi) - \frac{1}{2}tr\ln(V-j)} \quad (33)$$

The quark condensate $\left\langle \bar{\psi}\psi \right\rangle$ and the squared quark condensates $\left\langle \left(\bar{\psi}\Gamma_a\psi \right)^2 \right\rangle$ can be calculated from the partition function $W(j,m)$ using the expressions:

$$\left\langle \bar{\psi}\psi \right\rangle = \frac{1}{\Omega}\frac{\partial W(j,m)}{\partial m} \qquad \frac{1}{2}\left\langle \left(\bar{\psi}\Gamma_a\psi \right)^2 \right\rangle = -\frac{1}{\Omega}\frac{\partial W(j,m)}{\partial j} \quad (34)$$

where Ω is the space-time volume $\int d_4x = \Omega$.

The stationary point $\varphi_a = (M, 0, 0, 0)$ of the action is given by the gap equation:

$$(V - j)^{-1} = -4N_cN_f\frac{M}{M-m}g_M \tag{35}$$

This equation relates the constituent quark mass M to the interaction strength $V - j$.

4.1 The exchange and ring contributions.

The second order expansion of the action $I_{jm}(\varphi)$ around the stationary point reads:

$$I_{jm}(\varphi) = I_{jm}(M) + \frac{1}{2}\varphi\left(\Pi + (V-j)^{-1}\right)\varphi \tag{36}$$

where $I_{jm}(M)$ is the action calculated at the stationary point $\varphi = (M, 0, 0, 0)$ and where Π is the polarization function (often referred to as the Lindhardt function):

$$\langle xa\,|\Pi|\,yb\rangle = -\frac{\delta}{\delta\varphi_a(x)\,\delta\varphi_b(y)}Tr\ln\left(-i\partial_\mu\gamma_\mu + r\varphi_a\Gamma_a r\right) \tag{37}$$

Substituting this expansion into the partition function (33), we can calculate the partition function $W(j, m)$ using gaussian integration with the result:

$$W(j,m) = I_{jm}(M) + \frac{1}{2}tr\ln\left(1 - \Pi(V-j)\right) \tag{38}$$

The first term of the action (38) is what we refer to as the "classical" action. The values labelled "classical" in the tables displayed in section 3.1 are obtained by calculating the condensates (34) while retaining only the term I_{jm} in the partition function (38). The logarithm in (38) is what we refer to as the loop contribution. The expansion of the logarithm expresses the loop contribution in terms of the Feynman graphs shown on figure 1.

The first term of the loop expansion is what we call the "exchange term", also referred to as the Fock term:[1]:

$$W_{exch} = -\frac{1}{2}tr\Pi(V-j) \tag{39}$$

The remaining terms are what we call the ring graphs.

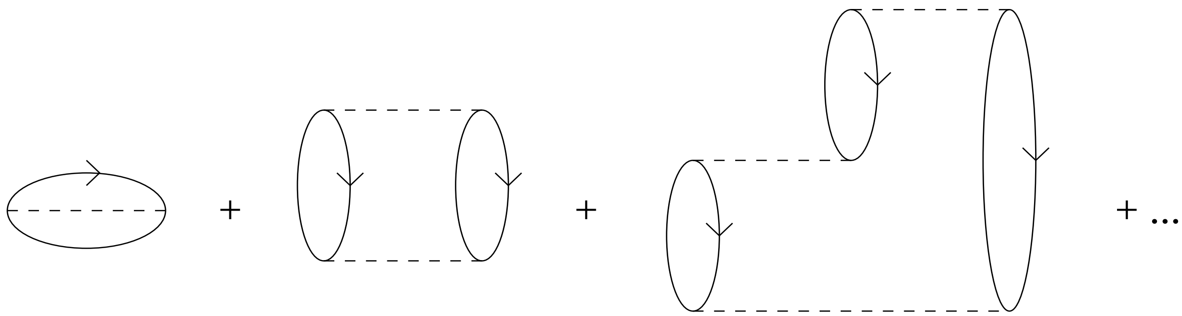

Figure 1: The Feynman graphs which represent the meson loop contribution to the partition function. The first graph is the exchange graph and the remaining graphs are the ring graphs.

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It is simple to show that the inverse meson propagators are given by:

$$K^{-1} = \Pi + (V - j)^{-1} \tag{40}$$

They are diagonal in momentum and flavor space: $\langle qa \left| K^{-1} \right| k'q \rangle = \delta_{ab}\delta_{kk'} K_a(q)$ and a straighforward calculation yields the following explicit expressions for the S (sigma) and P (pion) inverse propagators:

$$K_S^{-1}(q) = 4N_cN_f \left(\frac{1}{2} q^2 f_M^{22}(q) + M^2 \left(f_M^{26}(q) + f_M^{44}(q) \right) - g_M(q) + \frac{M}{M-m} g_M \right) \tag{41}$$

$$K_P^{-1}(q) = 4N_cN_f \left(\frac{1}{2} q^2 f_M^{22}(q) + M^2 \left(f_M^{26}(q) - f_M^{44}(q) \right) - g_M(q) + \frac{M}{M-m} g_M \right) \tag{42}$$

where the loop integrals are:

$$f_M^{np}(q) = \frac{1}{\Omega} \sum_k \frac{r_{k-\frac{q}{2}}^n r_{k+\frac{q}{2}}^p}{\left(\left(k - \frac{q}{2} \right)^2 + r_{k-\frac{q}{2}}^4 M^2 \right) \left(\left(k + \frac{q}{2} \right)^2 + r_{k+\frac{q}{2}}^4 M^2 \right)} \tag{43}$$

and:

$$g_M(q) = \frac{1}{\Omega} \sum_k \frac{r_{k-\frac{q}{2}}^2}{\left(k - \frac{q}{2} \right)^2 + r_{k-\frac{q}{2}}^4 M^2} r_{k+\frac{q}{2}}^2 \qquad g_M \equiv g_M(q=0) \tag{44}$$

These are the expressions which are obtained from the second order expansion of the action (31) retaining the regulators from the outset and throughout.

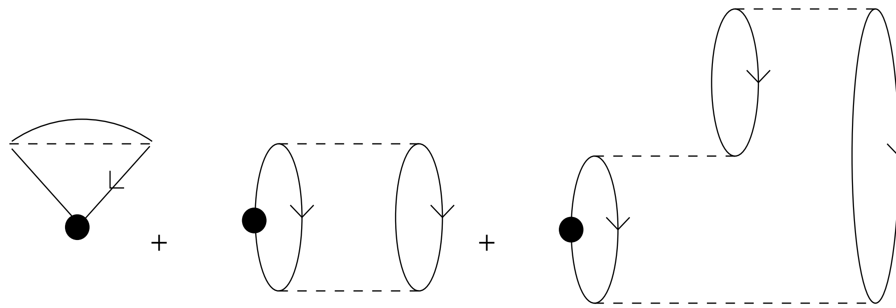

Figure 2: The contribution to the quark condensate of the Feynman graphs shown on figure 1. The black blob represents the operator $\bar{\psi}\psi$. The first graph (which is the dominating contribution) is the contribution of the exchange term. It represents $q\bar{q}$ excitations which describe a change in mass of the Dirac sea quarks. This exchange graph would not appear in a Hartree-Fock approximation, which would include the exchange graph in the gap equation.

Innumerable papers have been published (including some of my own) in which the meson propagators are derived from the unregulated Nambu Jona-Lasinio action:

$$I_{j,m}(\varphi) = -Tr\ln\left(-i\partial_\mu\gamma_\mu + \varphi_a\Gamma_a\right) - \frac{1}{2}(\varphi - m)(V-j)^{-1}(\varphi - m) \quad (45)$$

The expressions obtained for the propagators are then:

$$K_S^{-1}(q) = 4N_cN_f\left(\frac{1}{2}\left(q^2+4M^2\right)f_M(q) + \frac{m}{M-m}g_M\right) \quad (46)$$

$$K_P^{-1}(q) = 4N_cN_f\left(\frac{1}{2}q^2 f_M(q) + \frac{m}{M-m}g_M\right) \quad (47)$$

where the loop integrals are:

$$f_M(q) = \frac{1}{\Omega}\sum_{k<\Lambda}\frac{1}{\left(\left(k-\frac{q}{2}\right)^2+M^2\right)\left(\left(k+\frac{q}{2}\right)^2+M^2\right)} \quad (48)$$

and:

$$g_M = \frac{1}{\Omega}\sum_{k<\Lambda}\frac{1}{\left(k-\frac{q}{2}\right)^2+M^2} \quad (49)$$

[1]The direct (Hartree) term is included in the "classical" action $I_{j,m}$.

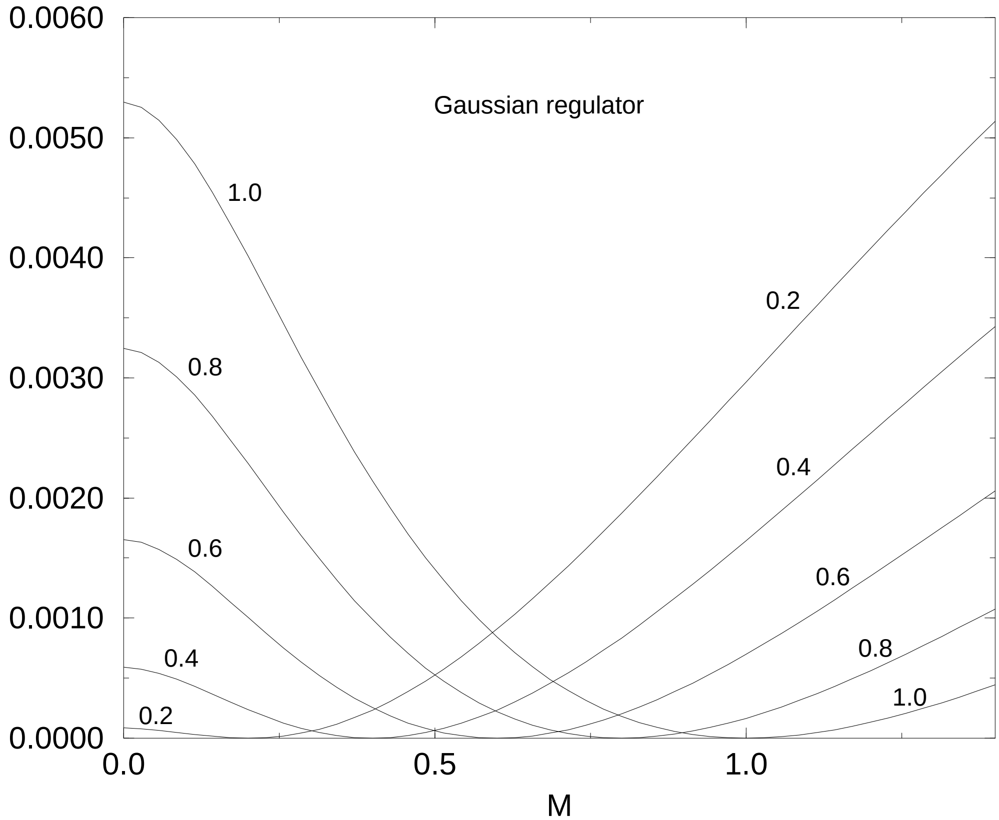

Figure 3: The effective potential plotted against M when a soft gaussian cut-off function is used. The potential is expressed units of Λ^4.

The table 3 shows the low q behaviour of the S and P inverse propagators in various approximations. They are calculated in the chiral limit.

5 An instability of the vacuum.

The partition function (38) can also be used to calculate the effective potential:

$$\Gamma = W(j,m) - j\frac{\partial W(j,m)}{\partial j} = W(j,m) + \frac{1}{2} j \left\langle \left(\bar{\psi}\Gamma_a \psi\right)^2 \right\rangle \tag{50}$$

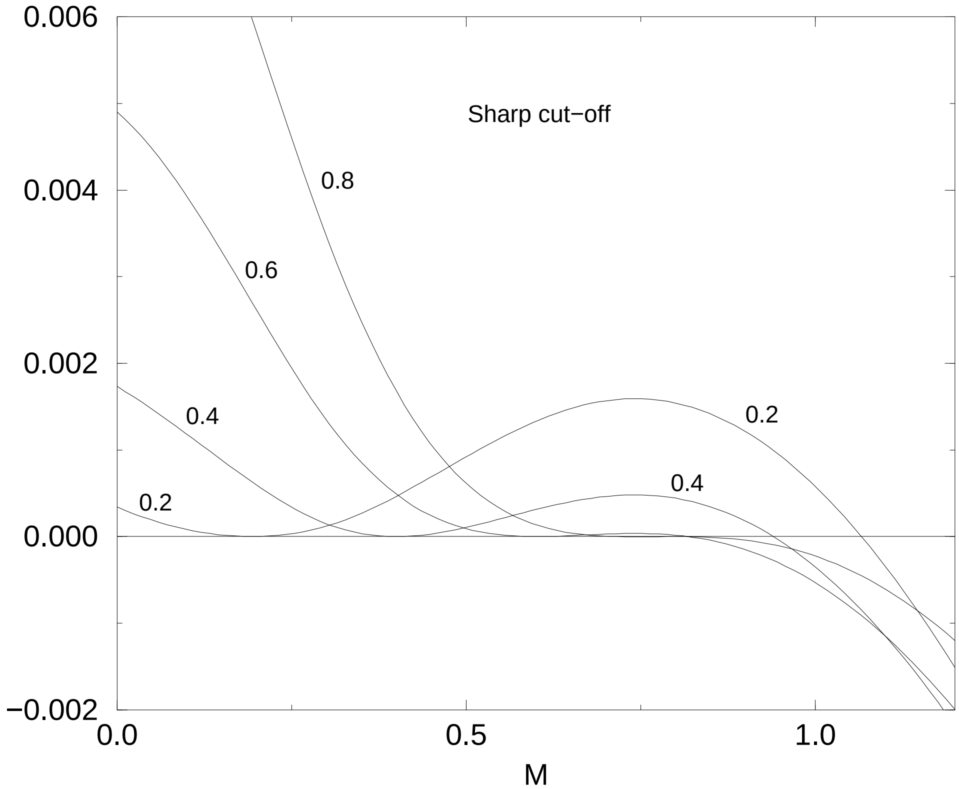

Figure 4: The effective potential plotted against M when a sharp cut-off is used.The effective potential is expressed in units of Λ^4.

As we vary j, the squared condensate $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle$ changes. Thus, when we plot the effective potential against j, we discover how the energy of the system varies when the system is forced to modify the squared condensate $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle$. The effective potential has a stationary point at $j = 0$, that is, in the absence of a constraint. If the stationary point of the effective potential is a minimum, the system is (at least locally) stable against fluctuations of $\left\langle \left(\bar{\psi}\Gamma_a\psi\right)^2 \right\rangle$. If it is an inflection point, it is unstable and we shall indeed find that this can easily occur when a sharp cut-off is used.

When j is varied, the constituent quark mass M also changes, according to the gap equation (35). One finds that M is a monotonically increasing function of j so that the effective potential can be plotted against M equally

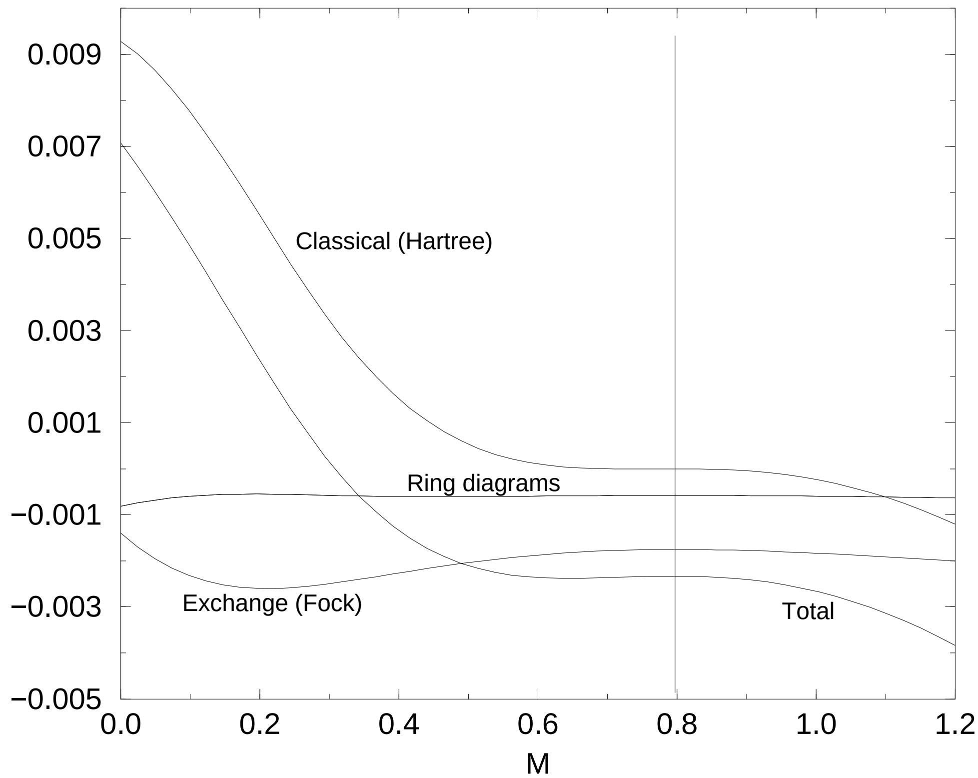

Figure 5: Various contributions to the effective potential calculated with a sharp cut-off and $M_0/\Lambda = 0.8$. The contributions are expressed in units of Λ^4.

well. The vacuum constituent quark mass is the mass M_0 obtained with $j = 0$. The contribution of each Feynman graph to the effective potential is stationary at the point $M = M_0$ and this is why plots of the the effective potential against M are nicer to look at than plots against j. The vacuum constituent quark mass M_0 is a measure of the interaction strength V, to which it is related by the gap equation. For a given shape of the regulator, the occurrence of an instability depends on only one parameter, namely M_0/Λ.

Figure 3 shows the effective potential calculated with a gaussian cut-off for various values of M_0/Λ. The ground state appears to be stable within the range of reasonable values of M_0/Λ.

Figure 4 shows the effective potential plotted against M when a sharp cut-off is used. When $M_0/\Lambda > 0.74$ the ground state develops an instability

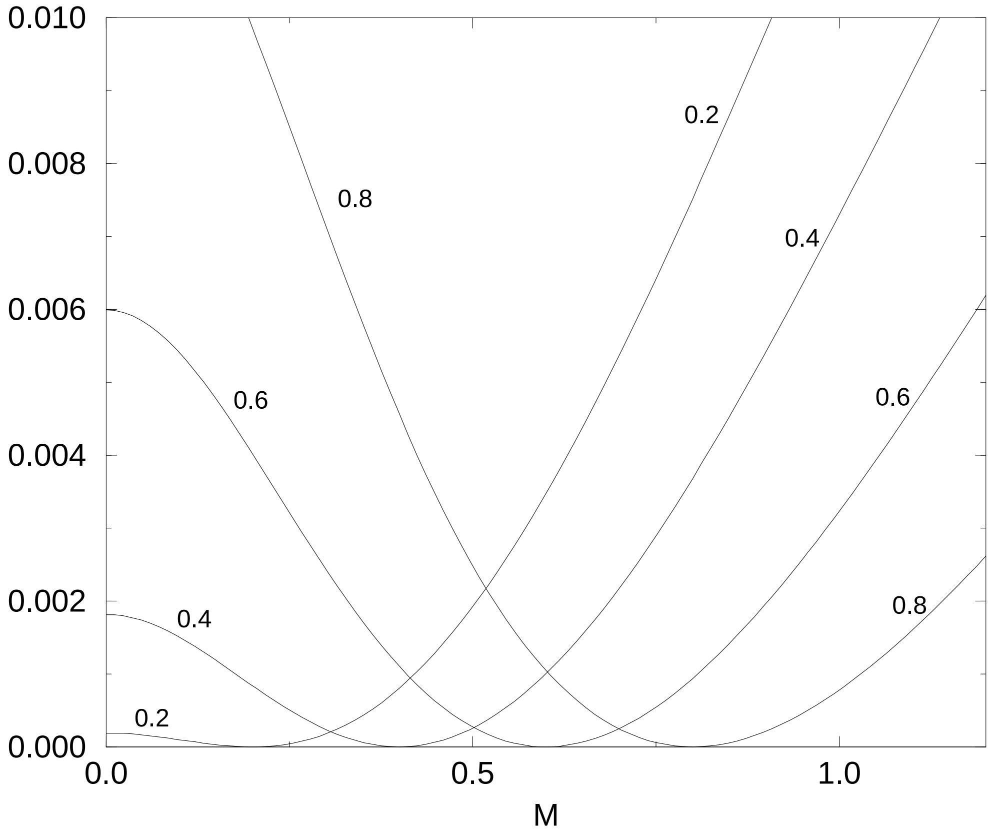

Figure 6: The effective potential calculated with a sharp 3-momentum cut-off plotted against M. It is expressed in units of Λ^4.

with respect to increasing values of M. This instability is not related to the restoration of chiral symmetry and, indeed, the pion remains a Goldstone boson for all values of M. As shown on Fig.5, the instability is due to the classical action and the meson loop contributions do not modify it.

Figure 6 shows the effective potential calculated with a sharp 3-momentum cut-off. No instability appears. This provides a clue as to the cause of the instability which arises when a sharp 4-momentum cut-off is used. Indeed, when a 3-momentum cut-off is used, the Nambu Jona-Lasinio model defines a time-independent hamiltonian and the 3-momentum cut-off simply restricts the Hilbert space available to the quarks. This allows a quantum mechanical interpretation of the results. If H is the Nambu Jona-Lasinio hamiltonian,

inverse propagators	$K_P^{-1}(q=0)$	$Z_\pi = \left.\frac{dK_P^{-1}}{dq^2}\right\vert_{q=0}$	$K_S^{-1}(q=0)$	$\left.\frac{dK_S^{-1}}{dq^2}\right\vert_{q=0}$
regulated action	0	0.0995	0.0546	0.0592
regulated $f(q)$	0	0.0850	0.0544	0.0448
$f(q) = f(0)$	0	0.0850	0.0544	0.0850

Table 3: Three approximations to the inverse S and P propagators, calculated with a sharp 4-momentum cut-off and with $M_0/\Lambda = 0.4$. The first row gives the values obtained from an regularized action (31). The second row gives the values obtained from a unregularized action and by subsequently regularizing the loop integrals. The last row gives the results obtained by neglecting the q dependence of the loop integral $f(q)$. The inverse quark propagators are given in units of Λ^2 and $\frac{dK^{-1}}{dq^2}$ is dimensionless.

then the ground state wavefunction $|j\rangle$ is calculated with the hamiltonian

$$\bar{H}_j = H - j \int d_3x \, \left(\bar{\psi}\Gamma_a\psi\right)^2 \tag{51}$$

containing the constraint proportional to j. The effective potential Γ is then equal to the energy $E(j) = \langle j|H|j\rangle$ of the system and it displays a stationary point when $j = 0$ or, equivalently, when $M = M_0$. The Nambu Jona-Lasinio model, regularized with a 3-momentum cut-off, has been used in Refs.[14] and [15] for example.

The use of a 3-momentum cut-off has another important feature. The meson propagators have only poles on the imaginary axis where they should. When a 4-momentum cut-off is used, unphysical poles appear in the complex energy plane, as they do when proper-time regularization is used for the quark loop [16].

The fact that the instability occurs when the model is regularized with a 4-momentum cut-off and not when a 3-momentum cut-off is used, strongly suggests that the instability is due to the unphysical poles introduced by the regulator. This conclusion is corroborated by the observation that the instability also occurs when a gaussian cut-off is used, but at the much higher values $M_0/\Lambda \geq 2.93$ where the cut-off is too small to be physically meaningful. With a gaussian regulator and in the relevant range of parameters $0.4 < M_0/\Lambda < 0.8$, one needs to probe the system with values as high as $M/\Lambda > 4$ before it becomes apparent that the energy is not bounded from below. The instability is an unpleasant feature of effective theories which use relatively low cut-offs. However, the low value of the cut-off is dictated by the vacuum properties and we need to learn to work with it. Further details are found in Ref.[2].

We conclude from this analysis that it is much safer to use a soft regulator, such as a gaussian, than a sharp cut-off.

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