On some cycles in Wenger Graphs

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Abstract

Let p be a prime, q be a power of p, and let $\mathbb{F}_q$ be the field of q elements. For any positive integer n, the Wenger graph $W_n(q)$ is defined as follows: it is a bipartite graph with the vertex partitions being two copies of the $(n+1)$-dimensional vector space $\mathbb{F}_q^{n+1}$, and two vertices $p = (p(1), \ldots, p(n+1))$, and $l = [l(1), \ldots, l(n+1)]$ being adjacent if $p(i) + l(i) = p(1)l(1)^{i-1}$, for all $i = 2, 3, \ldots, n+1$.

In 2008, Shao, He and Shan showed that for $n \geq 2$, $W_n(q)$ contains a cycle of length $2k$ where $4 \leq k \leq 2p$ and $k \neq 5$. In this paper we extend their results by showing that

(i) for $n \geq 2$ and $p \geq 3$, $W_n(q)$ contains cycles of length $2k$, where $4 \leq k \leq 4p+1$ and $k \neq 5$;

(ii) for $q \geq 5$, $0 < c < 1$, and every integer k, $3 \leq k \leq q^c$, if $1 \leq n < (1 - c - \frac{7}{3}\log_q 2)k - 1$, then $W_n(q)$ contains a $2k$-cycle. In particular, $W_n(q)$ contains cycles of length $2k$, where $n + 2 \leq k \leq q^c$, provided q is sufficiently large.

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1 Introduction

All graph theory notions that we use do but not define can be found in Bollobás [1]. Let $q = p^e$, where p is a prime and $e \geq 1$ is an integer, and let $\mathbb{F}_q$ be the field of q elements. For every $i \geq 2$, we define $f_i : \mathbb{F}_q^2 \to \mathbb{F}_q$ to be an arbitrary function. For any integer n with $n \geq 1$, the graph $\Gamma_n(q) = \Gamma(q; f_2, f_3, \ldots, f_{n+1})$ is defined as follows. The vertex set of $\Gamma_n(q)$ is the disjoint union of two copies of the $(n+1)$-dimensional vector space $\mathbb{F}_q^{n+1}$ over the finite field $\mathbb{F}_q$, one denoted by P_{n+1} and the other by L_{n+1}. Elements of P_{n+1} will be called *points* and those of L_{n+1} *lines*. For $1 \leq i \leq n+1$, we denote by $p(i)$ and $l(i)$ the ith coordinate of the point p and the line l, respectively. A point p and a line l are adjacent in $\Gamma_n(q)$, if

$$p(i) + l(i) = f_i(p(1), l(1)) \quad i = 2, \ldots, n+1. \tag{1}$$

It is important to observe that, for each point p and each $\alpha \in \mathbb{F}_q$, there is exactly one line l adjacent to p with $l(1) = \alpha$. Likewise, for each line l, there is exactly one point p adjacent to l with $p(1) = \alpha$. In particular the graphs $\Gamma_n(q)$ have $2q^{n+1}$ vertices and are q-regular. Hence, they have q^{n+2} edges.

If $f_i(p(1), l(1)) = p(1)l(1)^{i-1}$, i.e., the condition of adjacency is

$$p(i) + l(i) = p(1)l(1)^{i-1},\ i = 2, \ldots, n+1, \tag{2}$$

we call $\Gamma_n(q)$ the *Wenger* graph, and denote it by $W_n(q)$.

The introduction of these graphs by Wenger [22] was motivated by the problem in extremal graph theory of determining the greatest number of edges in a graph of order v containing no cycle of length $2k$. This number is denoted by $\mathrm{ex}(v, C_{2k})$. It follows from a theorem by Bondy and Simonovits [2], that $\mathrm{ex}(v, C_{2k}) = O(v^{1+1/k})$, $v \to \infty$. Lower bounds of magnitude $v^{1+1/k}$ for $\mathrm{ex}(v, C_{2k})$ were known (and still are) for $k = 2, 3, 5$ only, and $W_k(p)$, $k = 2, 3, 5$, provided new and simpler examples of extremal graphs of this magnitude. For many results on $\mathrm{ex}(n, C_{2k})$, see Verstraëte [23], Pikhurko [18], Füredi and Simonovits [7], Bukh and Jiang [4], Terlep and Williford [21], and references therein.

In [10], Lazebnik and Ustimenko, using a construction based on a certain Lie algebra, arrived at a family of bipartite graphs over $\mathbb{F}_q$ which were generalizations of the graphs in [22] and were isomorphic to them for prime q. Other representations of Wenger graphs, including the one we use in this paper, appeared in Lazebnik and Viglione [12] and Viglione [24]. For more details on various definitions of Wenger graphs and related isomorphisms, see a recent paper by Cioabă, Lazebnik and Li [6].

It was shown in [10] that the graph $W_1(q)$ is isomorphic to an induced subgraph of the Levi graph (the point-line incidence graph) of the projective plane $PG(2, q)$, the graph $W_2(q)$ is isomorphic to an induced subgraph of the Levi graph of the generalized quadrangle $Q(4, q)$, and $W_3(q)$ is a homomorphic image of an induced subgraph of the Levi graph of the generalized hexagon $H(q)$. For the definition of generalized polygons, see, e.g., A. Brouwer, A. Cohen and A. Neumaier [3].

It was also shown in [10] that the automorphism group of $W_n(q)$ acts transitively on each of P_{n+1} and L_{n+1}, and on the set of edges of $W_n(q)$. In other words, the graphs $W_n(q)$ are point-, line-, and edge-transitive. A more detailed study, see [12], also showed that $W_1(q)$ is vertex-transitive for all q; $W_2(q)$ is vertex-transitive for even q; and for any $n \geq 3$ and $q \geq 3$, and for $n = 2$ and odd q, the graphs $W_n(q)$ are not vertex-transitive. For a recent generalization of these results, see Cara, Rottey and Voorde [5]. Another result of [12] is that $W_n(q)$ is connected when $1 \leq n \leq q-1$, and disconnected when $n \geq q$, in which case it has q^{n-q+1} components, each isomorphic to $W_{q-1}(q)$. In [25], Viglione proved that when $1 \leq n \leq q-1$, the diameter of $W_n(q)$ is $2n+2$. We wish to note that the statement about the number of components of $W_n(q)$ becomes apparent from the definition. Indeed, as $l(1)^i = l(1)^{i+q-1}$, all points and lines in a component have the property that their coordinates indexed by i, j, $i \equiv j \mod (q-1)$ are equal. Hence, points p, having $p(1) = \ldots = p(q) = 0$, and at least one distinct coordinate $p(i)$, $q+1 \leq i \leq n+1$, belong to different components. This shows that the number of components is at least q^{n-q+1}. As $W_{q-1}(q)$ is connected and $W_n(q)$ is edge-transitive, all components are isomorphic to $W_{q-1}(q)$. Hence there are exactly q^{n-q+1} of them.

Li and Lih [14] used Wenger graphs to determine the asymptotic behavior of the Ramsey number $r_n(C_{2k}) = \Theta(n^{k/(k-1)})$ when $k \in \{2, 3, 5\}$ and $n \to \infty$; the Ramsey number $r_n(G)$ equals the minimum integer N such that in any edge-coloring of the complete graph K_N with n colors, there is a monochromatic G.

Our definition of Wenger graphs points to a relation between them and the moment curve $t \mapsto (0, 1, t, t^2, t^3, ..., t^n)$, $t \in \mathbb{F}_q$, and, hence, with the Vandermonde's determinant, which was explicitly used in [22]. It is implicit in some geometric constructions by Mellinger and Mubayi [16] of extremal graphs without short even cycles. See also the aforementioned paper [5].

In [15], Li, Lu and Wang showed that the graphs $W_n(q)$, $n = 1, 2$, are Ramanujan, by computing the eigenvalues of graphs $D(k, q)$, $k = 2, 3$, from [11]. The result follows from the fact that $W_1(q) \simeq D(2, q)$, and $W_2(q) \simeq D(3, q)$. In [6], the complete spectrum of $W_n(q)$ was determined, and the results imply that Wenger graphs $W_n(q)$ are expanders for fixed n and large q. For more details on expanders and Ramanujan graphs, see Hoory, Linial and Wigderson [9], Murty [17], and references therein.

Shao, He and Shan [19] proved that for any $n \geq 2$, and any integer k with $k \neq 5$, $4 \leq k \leq 2p$, $W_n(q)$ contains cycles of length $2k$, and for any vertex of the graph, there is a cycle of length $2k$ containing it. We wish to remark that the edge-transitivity of $W_n(q)$ implies the existence of a $2k$-cycle through any edge, a stronger statement. With a little more work, one can show that every path of length three in $W_n(q)$ is a subgraph of a $2k$-cycle of $W_n(q)$. The graphs $W_n(q)$ are sparse: if $v = v(W_n(q))$ and $e = e(W_n(q))$, then $e = v^{1+1/n} = o(v^2)$, $v \to \infty$. Therefore various methods of proving the existence of long cycles in $W_n(q)$, in particular, hamiltonian cycles (see Gould [8]), do not apply. In this paper we extend the result from [19] in the following ways.

Theorem 1 *Let $n \geq 2$ and let p be a prime, q be a power of p with $p \geq 3$. For any integer k, where $k = 4$ or $6 \leq k \leq 4p + 1$, $W_n(q)$ contains even cycles of length $2k$.*

Theorem 2 *Let q be a prime power with $q \geq 5$, c be a real number such that $0 < c < 1$, and k be an integer such that $3 \leq k \leq q^c$. If $1 \leq n < (1 - c - \frac{7}{3}\log_q 2)k - 1$, then $W_n(q)$ contains a $2k$-cycle. In particular, $W_n(q)$ contains cycles of length $2k$, where $n + 2 \leq k \leq q^c$, provided q is sufficiently large.*

2 Proof of Theorem 1

For $q = p = 3$, since $2 \leq n \leq q - 1$, we have $n = 2$. In this case the existence of cycles in $W_2(3)$ of length $2k$, $4 \leq k \leq 4 \cdot 3 + 1 = 13$ is verified by computer. Otherwise, as q is a power of p, $p \geq 3$, then $q \neq 4$. Therefore, in this proof, we can assume that $q \geq 5$ whenever it is needed.

Though the main goal of this paper is understanding the cycle lengths of $W_n(q)$, Lemma 1 below holds for any graph $\Gamma_n(q)$, and so it can be used to study cycles in this class of graphs. Let $k, n \geq 1$, and $P = p_1 l_1 p_2 l_2 \ldots p_k l_k$ be a $p_1 l_k$-walk of length $2k - 1$ in $\Gamma_n(q)$, where all $p_i \in P_{n+1}$ and all $l_i \in L_{n+1}$. Suppose

$$g_i(P) = f_i(p_1(1), l_1(1)) + \ldots + f_i(p_{k-1}(1), l_{k-1}(1)) + f_i(p_k(1), l_k(1)),$$

and

$$h_i(P) = f_i(p_2(1), l_1(1)) + \ldots + f_i(p_k(1), l_{k-1}(1)) + f_i(p_1(1), l_k(1)),$$

for $i = 2, 3, \ldots, n + 1$.

Lemma 1 *The walk P is a closed walk of length $2k$ in $\Gamma_n(q)$ if and only if $g_i(P) = h_i(P)$ for all $i = 2, \ldots, n + 1$.*

Proof We are told that $P = p_1 l_1 p_2 l_2 \ldots p_k l_k$ is a walk, so p_j is adjacent to l_j for $j = 1, \ldots, k$ and l_j is adjacent to p_{j+1} for $j = 1, \ldots, k - 1$. Hence, for $2 \leq i \leq n + 1$, we have

$$\begin{aligned} f_i(p_j(1), l_j(1)) &= p_j(i) + l_j(i) \quad \text{for } j = 1, \ldots, k, \text{ and} \\ f_i(p_{j+1}(1), l_j(1)) &= p_{j+1}(i) + l_j(i) \quad \text{for } j = 1, \ldots, k - 1\,. \end{aligned}$$

Therefore, for $2 \leq i \leq n + 1$, we have

$$\begin{aligned} g_i(P) &= (p_1(i) + l_1(i)) + \ldots + (p_k(i) + l_k(i)) \\ &= p_1(i) + \ldots + p_k(i) + l_1(i) + \ldots + l_k(i) \\ &= (p_2(i) + l_1(i)) + \ldots + (p_k(i) + l_{k-1}(i)) + (p_1(i) + l_k(i)) \\ &= h_i(P) - f_i(p_1(1), l_k(1)) + (p_1(i) + l_k(i))\,. \end{aligned}$$

Thus the condition $g_i(P) = h_i(P)$ for $2 \le i \le n+1$ is equivalent to $f_i(p_1(1), l_k(1)) = p_1(i) + l_k(i)$ for $2 \le i \le n+1$, which is equivalent to p_1 being adjacent to l_k, and hence equivalent to P being a closed walk. □

From now on we concentrate on Wenger graphs, and provide an explicit description of their $2k$-cycles. Our main idea is to use Lemma 1 to construct a closed $2k$-walk in $W_n(q)$ by giving an explicit description of the first coordinates of points and lines of the walk. Then we argue that the obtained closed walk is actually a $2k$-cycle.

Our arguments will slightly depend on the residue class of k modulo 4. In what follows we will give all the details for the cases $k \equiv 0, 2 \pmod 4$ only. In the two other cases, we indicate modifications in the construction of our closed walk, but omit the verifications that the obtained $2k$-walk is indeed a $2k$-cycle. These verifications follow the same pattern as in the case $k \equiv 0, 2 \pmod 4$.

Lemma 2 *Let $n \ge 2$ and let p be a prime, q be a power of p with $p \ge 3$. For any integer k with $1 \le k \le p-1$, $W_n(q)$ contains cycles of length $8k+4$.*

Proof The observation after equation (1) means that, given a point p_1 and elements $\alpha_1, \dots, \alpha_j, \beta_1, \dots, \beta_j \in \mathbb{F}$, there is a unique walk $p_1 l_1 \dots p_j l_j$ with $p_i(1) = \alpha_i$ and $l_i(1) = \beta_i$. Now choose two distinct elements α_1, α_2 in $\mathbb{F}_q$, and three distinct elements β_1,β_2,β_3 in $\mathbb{F}_q$. Let p_1 be a point with $p_1(1) = \alpha_1$. There is then a unique walk $A = p_1 l_1 p_2 l_2 \dots p_{4k+2} l_{4k+2}$ with

$$p_i(1) = \begin{cases} \alpha_1, & \text{for } i \text{ odd}, \\ \alpha_2, & \text{for } i \text{ even}, \end{cases}$$

and with

$$l_i(1) = \begin{cases} \beta_1, & i = 1, 2k+2, \\ \beta_2, & i = 2, 4, \dots, 2k, 2k+3, 2k+5, \dots, 4k+1, \\ \beta_3, & i = 3, 5, \dots, 2k+1, 2k+4, 2k+6, \dots, 4k+2. \end{cases}$$

The first coordinates of p_i and l_i are exhibited in the following table.

i	1	2	3	4	5	...	$2k+2$	$2k+3$	$2k+4$	$2k+5$	$2k+6$	...
$p_i(1)$	α_1	α_2	α_1	α_2	α_1	...	α_2	α_1	α_2	α_1	α_2	...
$l_i(1)$	β_1	β_2	β_3	β_2	β_3	...	β_1	β_2	β_3	β_2	β_3	...

Table 1: The first coordinates of p_i and l_i in A

For $i = 2, 3, \dots, n+1$ we obtain

$$\begin{aligned} g_i(A) &= f_i(\alpha_1, \beta_1) + f_i(\alpha_2, \beta_1) + k(f_i(\alpha_1, \beta_2) + f_i(\alpha_2, \beta_2) + f_i(\alpha_1, \beta_3) + f_i(\alpha_2, \beta_3)) \\ &= h_i(A). \end{aligned}$$

By Lemma 1, A is a closed $(8k+4)$-walk in $W_n(q)$. We want to show that no two vertices of A are the same. It is sufficient to show this for any two points. Indeed, if any two lines of A are the same, their neighboring points with the first coordinate α_1 must also be the same.

So let us assume the contrary, i.e., that two points of A are the same. This means that there exist j_1 and j_2, $1 \le j_1 < j_2 \le 4k+1$, such that $p_{j_1} = p_{j_2}$. We break our proof into several cases, depending on the values of j_1 and j_2. As $p_{j_1}(1) = p_{j_2}(1)$, we can consider two cases, depending on whether this common value is α_1 or α_2. It is easy to see that it is sufficient to consider the case $p_{j_1}(1) = p_{j_2}(1) = \alpha_1$ only: the case $p_{j_1}(1) = p_{j_2}(1) = \alpha_2$ can be reduced to it by listing the vertices of A starting with p_{2k+2} and keeping the corresponding indices of β's.

Therefore we assume that $p_{j_1}(1) = p_{j_2}(1) = \alpha_1$. In this case $A_1 = p_{j_1} l_{j_1} \ldots p_{j_2-1} l_{j_2-1}$ is a closed subwalk of A, and, by Lemma 1, $g_i(A_1) = h_i(A_1)$ for all $i = 2, \ldots, n+1$. We now use the Wenger property $f_i(x,y) = xy^{i-1}$.

Case 1: $j_1 = 1$, $j_2 \in \{3, 5, \ldots, 2k+1\}$. Then, for $i = 2, \ldots, n+1$,

$$\begin{aligned} 0 = g_i(A_1) - h_i(A_1) &= f_i(\alpha_1,\beta_1) - f_i(\alpha_2,\beta_1) + \frac{j_2-1}{2}(f_i(\alpha_2,\beta_2) - f_i(\alpha_1,\beta_2)) \\ &\quad + \frac{j_2-3}{2}(f_i(\alpha_1,\beta_3) - f_i(\alpha_2,\beta_3)) \\ &= (\beta_1^{i-1} - \beta_3^{i-1} + \frac{j_2-1}{2}(\beta_3^{i-1} - \beta_2^{i-1}))(\alpha_1 - \alpha_2). \end{aligned}$$

Since $\alpha_1 \ne \alpha_2$, we have

$$\beta_1^{i-1} - \beta_3^{i-1} + \frac{j_2-1}{2}(\beta_3^{i-1} - \beta_2^{i-1}) = 0.$$

As $n \ge 2$ and $p \ge 3$, substituting $i = 2, 3$ in this equation, we obtain

$$\beta_1 - \beta_3 + \frac{j_2-1}{2}(\beta_3 - \beta_2) = 0,$$

$$\beta_1^2 - \beta_3^2 + \frac{j_2-1}{2}(\beta_3^2 - \beta_2^2) = 0.$$

These two equations imply $(\beta_1 - \beta_3)(\beta_1 - \beta_2) = 0$, a contradiction to our assumption that all three β_i are distinct.

Case 2: $j_1 = 1$ and $j_2 \in \{2k+3, 2k+5, \ldots, 4k+1\}$. Then, for $i = 2, \ldots, n+1$,

$$\begin{aligned} 0 = g_i(A_1) - h_i(A_1) &= \frac{4k+3-j_2}{2}(f_i(\alpha_1,\beta_3) + f_i(\alpha_2,\beta_2) - f_i(\alpha_1,\beta_2) - f_i(\alpha_2,\beta_3)) \\ &= \frac{4k+3-j_2}{2}(\beta_3^{i-1} - \beta_2^{i-1})(\alpha_1 - \alpha_2). \end{aligned}$$

For $i = 2$, this gives
$$\frac{4k+3-j_2}{2}(\beta_3-\beta_2)(\alpha_1-\alpha_2) = 0.$$
Note that if $k \leq p-1$, then $2 \leq 4k+3-j_2 \leq 2k \leq 2p-2$. As $4k+3-j_2$ is even, it is not equal to p. Hence, p does not divide $(4k+3-j_2)/2$. Since $\alpha_1 \neq \alpha_2$ and $\beta_2 \neq \beta_3$, we obtain a contradiction.

Case 3: $j_1, j_2 \in \{3, 5, \ldots, 2k+1\}$. Then, for $i = 2, \ldots, n+1$,

$$\begin{aligned} 0 = g_i(A_1) - h_i(A_1) &= \frac{j_2-j_1}{2}(f_i(\alpha_1,\beta_3)+f_i(\alpha_2,\beta_2)-f_i(\alpha_2,\beta_3)-f_i(\alpha_1,\beta_2)) \\ &= \frac{j_2-j_1}{2}(\beta_3^{i-1}-\beta_2^{i-1})(\alpha_1-\alpha_2). \end{aligned}$$

For $i = 2$, this gives
$$\frac{j_2-j_1}{2}(\beta_3-\beta_2)(\alpha_1-\alpha_2) = 0.$$

As $0 < j_2 - j_1 \leq 2k-2 \leq 2p-4$ and $j_2 - j_1$ is even, it is not p, and so p does not divide $(j_2-j_1)/2$. Since $\alpha_1 \neq \alpha_2$, and $\beta_2 \neq \beta_3$, we obtain a contradiction.

Case 4: $j_1 \in \{3, 5, \ldots, 2k+1\}$ and $j_2 \in \{2k+3, 2k+5, \ldots, 4k+1\}$. Then, for $i = 2, \ldots, n+1$,

$$\begin{aligned} 0 = g_i(A_1) - h_i(A_1) &= f_i(\alpha_2,\beta_1)-f_i(\alpha_1,\beta_1)+f_i(\alpha_1,\beta_3)-f_i(\alpha_2,\beta_3) \\ &\quad+\frac{4k+4-j_1-j_2}{2}(f_i(\alpha_1,\beta_3)+f_i(\alpha_2,\beta_2)-f_i(\alpha_1,\beta_2)-f_i(\alpha_2,\beta_3)) \\ &= (\beta_3^{i-1}-\beta_1^{i-1}+\frac{4k+4-j_1-j_2}{2}(\beta_3^{i-1}-\beta_2^{i-1}))(\alpha_1-\alpha_2), \end{aligned}$$

For $\alpha_1 \neq \alpha_2$, this gives

$$\beta_3^{i-1}-\beta_1^{i-1}+\frac{4k+4-j_1-j_2}{2}(\beta_3^{i-1}-\beta_2^{i-1}) = 0,$$

For $i = 2, 3$, we obtain

$$\beta_3-\beta_1+\frac{4k+4-j_1-j_2}{2}(\beta_3-\beta_2) = 0,$$

$$\beta_3^2-\beta_1^2+\frac{4k+4-j_1-j_2}{2}(\beta_3^2-\beta_2^2) = 0.$$

These two equations imply $(\beta_1-\beta_3)(\beta_1-\beta_2) = 0$, a contradiction to our assumption that all three β_i are distinct.

Case 5: $j_1, j_2 \in \{2k+3, 2k+5, \ldots, 4k+1\}$. Then, for $i = 2, \ldots, n+1$,

$$\begin{aligned} 0 = g_i(A_1) - h_i(A_1) &= \frac{j_2 - j_1}{2}(f_i(\alpha_2, \beta_3) + f_i(\alpha_1, \beta_2) - f_i(\alpha_1, \beta_3) - f_i(\alpha_2, \beta_2)) \\ &= \frac{j_2 - j_1}{2}(\beta_2^{i-1} - \beta_3^{i-1})(\alpha_1 - \alpha_2), \end{aligned}$$

As $k \leq p-1$, $2 \leq j_2 - j_1 \leq 2k-2 \leq 2p-4$. Since $j_2 - j_1$ is even, p does not divide it. So p does not divide $(j_2 - j_1)/2$, and substituting $i = 2$ in the equation above we obtain $(\alpha_1 - \alpha_2)(\beta_2 - \beta_3) = 0$, a contradiction. This completes the proof of Lemma 2. □

Lemma 3 *Let $n \geq 2$ and let p be a prime, q be a power of p with $p \geq 3$. For any integer k with $1 \leq k \leq p-1$, $W_n(q)$ contains cycles of length $8k+6$.*

Proof Let $\alpha_1, \alpha_2, \alpha_3$ be three distinct elements and $\beta_1, \beta_2, \beta_3$ be three distinct elements in $\mathbb{F}_q$, respectively. Let $B = p_1 l_1 p_2 l_2 \ldots p_{4k+3} l_{4k+3}$ be a $p_1 l_{4k+3}$-walk in $W_n(q)$, where all $p_i \in P_{n+1}$ and all $l_i \in L_{n+1}$, and where

$$p_i(1) = \begin{cases} \alpha_1, & i = 1, 2k+3, \\ \alpha_2, & i = 2, 4, \ldots, 4k+2, \\ \alpha_3, & i = 3, 5, \ldots, 2k+1, 2k+5, \ldots, 4k+3, \end{cases}$$

and

$$l_i(1) = \begin{cases} \beta_1, & i = 1, 2k+2, \\ \beta_2, & i = 2, 4, \ldots, 2k, 2k+3, 2k+5, \ldots, 4k+3, \\ \beta_3, & i = 3, 5, \ldots, 2k+1, 2k+4, 2k+6, \ldots, 4k+2. \end{cases}$$

We can also exhibit the first coordinates of p_i and l_i in the following table.

i	1	2	3	4	5	$\ldots$	$2k+2$	$2k+3$	$2k+4$	$2k+5$	$2k+6$	$\ldots$	$4k+3$
$p_i(1)$	α_1	α_2	α_3	α_2	α_3	$\ldots$	α_2	α_1	α_2	α_3	α_2	$\ldots$	α_3
$l_i(1)$	β_1	β_2	β_3	β_2	β_3	$\ldots$	β_1	β_2	β_3	β_2	β_3	$\ldots$	β_2

Table 2: The first coordinates of p_i and l_i in B

Using the same method as in Lemma 2, we obtain that the walk B is also a closed walk without repeated vertices, so we have cycles of length $8k+6$. □

Lemma 4 *Let $n \geq 2$ and let p be a prime, q be a power of p with $q \geq 5$ and $p \geq 3$. For any integer k with $2 \leq k \leq p$, $W_n(q)$ contains cycles of length $8k$.*

Proof Let α_1, α_2 be distinct elements in $\mathbb{F}_q$ and $\beta_1, \beta_2, \beta_3, \beta_4$ be four distinct elements in $\mathbb{F}_q$, respectively. Let $C = p_1 l_1 p_2 l_2 \ldots p_{4k} l_{4k}$ be a $p_1 l_{4k}$-walk in $W_n(q)$, where all $p_i \in P_{n+1}$ and all $l_i \in L_{n+1}$. and where

$$p_i(1) = \begin{cases} \alpha_1, & \text{for } i \text{ odd}, \\ \alpha_2, & \text{for } i \text{ even}, \end{cases}$$

and

$$l_i(1) = \begin{cases} \beta_1, & i = 1, 2k, \\ \beta_2, & i = 3, 5, \ldots, 2k-1, 2k+2, 2k+4, \ldots, 4k-2, \\ \beta_3, & i = 2, 4, \ldots, 2k-2, 2k+3, 2k+5, \ldots, 4k-1, \\ \beta_4, & i = 2k+1, 4k. \end{cases}$$

We can also exhibit the first coordinates of p_i and l_i in the following table.

i	1	2	3	$\ldots$	$2k$	$2k+1$	$2k+2$	$2k+3$	$\ldots$	$4k$
$p_i(1)$	α_1	α_2	α_1	$\ldots$	α_2	α_1	α_2	α_1	$\ldots$	α_2
$l_i(1)$	β_1	β_3	β_2	$\ldots$	β_1	β_4	β_2	β_3	$\ldots$	β_4

Table 3: The first coordinates of p_i and l_i in C

To simplify the argument, we check the existence of particular $8k$-cycles, by setting $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 0$, $\beta_2 = 1$, $\beta_4 \neq \beta_3 + 1$, and all β_i are distinct.

Applying Lemma 1, and using that $f_i(0, y) = f_i(x, 0) = 0$ for Wenger graphs, we obtain that C is a closed $8k$-walk in $W_n(q)$. To show that no two vertices of C are the same, it is sufficient to show this for any two points (same explanation as in our proof of Lemma 2). So let us assume the contrary, i.e., that two points of C are the same. This means that there exist j_1 and j_2, $1 \leq j_1 < j_2 \leq 4k$, such that $p_{j_1} = p_{j_2}$. Then $C_1 = p_{j_1} l_{j_1} \ldots p_{j_2-1} l_{j_2-1}$ is a closed subwalk of C, and, by Lemma 1, $g_i(C_1) = h_i(C_1)$ for all $i = 2, \ldots, n+1$.

Case 1. $p_{j_1}(1) = p_{j_2}(1) = \alpha_1$.

Case 1.1: $j_1 = 1$, $j_2 \in \{3, 5, \ldots, 2k-1\}$. Then, for $i = 2, \ldots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - 3}{2}(\beta_3^{i-1} - 1) + \beta_3^{i-1}.$$

As $n \geq 2$ and $p \geq 3$, substituting $i = 2, 3$ in this equation, we obtain $\beta_3 = 0$, a contradiction (as $\beta_3 \neq \beta_1$).

Case 1.2: $j_1 = 1$, $j_2 = 2k+1$. Then, for $i = 2, \ldots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = (k-1)(\beta_3^{i-1} - 1).$$

For $i = 2$, as $0 < k - 1 < p$, it implies $\beta_3 = 1$, a contradiction (as $\beta_3 \neq \beta_2$).

Case 1.3: $j_1 = 1$ and $j_2 \in \{2k+3, 2k+5, \dots, 4k-1\}$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{4k+1-j_2}{2}(\beta_3^{i-1} - 1) + 1 - \beta_4^{i-1}.$$

Substituting $i = 2, 3$ in this equation, we obtain that $\beta_4 = 1$ or $\beta_4 = \beta_3$, a contradiction.

Case 1.4: $j_1, j_2 \in \{3, 5, \dots, 2k-1\}$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - j_1}{2}(\beta_3^{i-1} - 1).$$

For $i = 2$, the difference $j_2 - j_1$ is even and is at most $2k - 4$. Hence, it is not p, and $1 \leq (j_2 - j_1)/2 \leq k - 2 < p$. This implies $\beta_3 = 1$, a contradiction.

Case 1.5: $j_1 \in \{3, 5, \dots, 2k-1\}$ and $j_2 = 2k+1$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{2k-1-j_1}{2}(\beta_3^{i-1} - 1) - 1.$$

Substituting $i = 2, 3$ in this equation, we obtain $\beta_3 = 0$, a contradiction.

Case 1.6: $j_1 \in \{3, 5, \dots, 2k-1\}$ and $j_2 \in \{2k+3, 2k+5, \dots, 4k-1\}$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{4k+2-j_1-j_2}{2}(\beta_3^{i-1} - 1) - \beta_4^{i-1}.$$

Substituting $i = 2, 3$ in this equation, we obtain $\beta_4 = 0$ or $\beta_4 = \beta_3 + 1$, a contradiction.

Case 1.7: $j_1 = 2k+1$ and $j_2 \in \{2k+3, 2k+5, \dots, 4k-1\}$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - 2k - 3}{2}(1 - \beta_3^{i-1}) + 1 - \beta_4^{i-1}.$$

Substituting $i = 2, 3$ in this equation, we obtain $\beta_4 = 1$ or $\beta_3 = \beta_4$, a contradiction.

Case 1.8: $j_1, j_2 \in \{2k+3, 2k+5, \dots, 4k-1\}$. Then, for $i = 2, \dots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - j_1}{2}(1 - \beta_3^{i-1}).$$

For $i = 2$, the difference $j_2 - j_1$ is even and is at most $2k - 4$. Hence, it is not p, and $1 \leq (j_2 - j_1)/2 \leq k - 2 < p$. This implies $\beta_3 = 1$, a contradiction.

Case 2. $p_{j_1}(1) = p_{j_2}(1) = \alpha_2$.

Case 2.1: $j_1, j_2 \in \{2, 4, \ldots, 2k\}$. Then, for $i = 2, \ldots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - j_1}{2}(\beta_3^{i-1} - 1).$$

For $i = 2$, the difference $j_2 - j_1$ is even and is at most $2k - 2$. Hence, it is not p, and $1 \leq (j_2 - j_1)/2 \leq k - 1 < p$. This implies $\beta_3 = 1$, a contradiction.

Case 2.2: $j_1 \in \{2, 4, \ldots, 2k\}$ and $j_2 \in \{2k+2, 2k+4, \ldots, 4k\}$. Then, for $i = 2, \ldots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{4k + 2 - j_1 - j_2}{2}(\beta_3^{i-1} - 1) - \beta_4^{i-1}.$$

Substituting $i = 2, 3$ in this equation, we obtain $\beta_4 \neq 0$ or $\beta_4 = \beta_3 + 1$, a contradiction.

Case 2.3: $j_1, j_2 \in \{2k+2, 2k+4, \ldots, 4k\}$. Then, for $i = 2, \ldots, n+1$,

$$0 = g_i(C_1) - h_i(C_1) = \frac{j_2 - j_1}{2}(1 - \beta_3^{i-1}).$$

For $i = 2$, the difference $j_2 - j_1$ is even and is at most $2k - 2$. Hence, it is not p, and $1 \leq (j_2 - j_1)/2 \leq k - 2 < p$. This implies $\beta_3 = 1$, a contradiction.

This completes the proof of Lemma 4. □

Lemma 5 *Let $n \geq 2$ and let p be a prime, q be a power of p with $q \geq 5$ and $p \geq 3$. For any integer k with $2 \leq k \leq p+1$, $W_n(q)$ contains cycles of length $8k+2$.*

Proof Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be four distinct elements in $\mathbb{F}_q$ and $\beta_1, \beta_2, \beta_3$ be three distinct elements in $\mathbb{F}_q$, respectively. Let $D = p_1 l_1 p_2 l_2 \ldots p_{4k+1} l_{4k+1}$ be a $p_1 l_{4k+1}$-walk in $W_n(q)$, where all $p_i \in P_{n+1}$ and all $l_i \in L_{n+1}$, and where

$$p_i(1) = \begin{cases} \alpha_1, & i = 1, 2k+1, \\ \alpha_2, & i = 3, 5, \ldots, 2k-1, 2k+3, 2k+5, \ldots, 4k-1, \\ \alpha_3, & i = 4, 6, \ldots, 2k, 2k+2, \ldots, 4k-2, 4k+1, \\ \alpha_4, & i = 2, 4k, \end{cases}$$

and

$$l_i(1) = \begin{cases} \beta_1, & i = 1, 4, 6, \ldots, 2k, 2k+3, 2k+5, \ldots, 4k-3, 4k, \\ \beta_2, & i = 2, 2k+1, 4k-1, 4k+1, \\ \beta_3, & i = 3, 5, \ldots, 2k-1, 2k+2, 2k+4, \ldots, 4k-2. \end{cases}$$

We can also exhibit the first coordinates of p_i and l_i in the following table.

i	1	2	3	$\dots$	$2k$	$2k+1$	$2k+2$	$\dots$	$4k-1$	$4k$	$4k+1$
$p_i(1)$	α_1	α_4	α_2	$\dots$	α_3	α_1	α_3	$\dots$	α_2	α_4	α_3
$l_i(1)$	β_1	β_2	β_3	$\dots$	β_1	β_2	β_3	$\dots$	β_2	β_1	β_2

Table 4: The first coordinates of p_i and l_i in D

To simplify it, we take $\alpha_2 = 0$, $\alpha_3 = 1$, $\beta_1 = 0$, $\beta_3 = 1$ and $\alpha_1 \neq \alpha_4 + 1$ in $\mathbb{F}_q$. Using the same method as in Lemma 4, we obtain that D is a closed walk without repeated vertices, so we have cycles of length $8k+2$. $\square$

Combining the results of Lemma 2, 3, 4 and 5, we obtain Theorem 1.

3 Proof of Theorem 2

Using a result of Verstraëte [23, Theorem 8] for an upper bound on the size $e(G)$ of a $2k$-cycle–free bipartite graph G of order v, we obtain

$$e(G) \leq 2kv^{1+1/k}.$$

If $W_n(q)$ were $2k$-cycle–free, then we would have

$$q^{n+2} \leq 2k(2q^{n+1})^{1+1/k},$$

which is equivalent to

$$q^{1-\frac{n+1}{k}} \leq 2^{2+\frac{1}{k}}k.$$

Since $n < (1 - c - \frac{7}{3}\log_q 2)k - 1$ and $3 \leq k \leq q^c$, we have

$$q^{1-\frac{n+1}{k}} > 2^{\frac{7}{3}}q^c \geq 2^{2+\frac{1}{k}}k,$$

a contradiction. $\square$

4 Concluding Remarks

For some n, q and p, we have established the existence of cycles in $W_n(q)$ of some lengths different from those covered by Theorems 1 and 2. The methods used in the related arguments are different from the ones discussed in this paper, and this is a reason for excluding those results.

For example, for $p \geq 3$, $W_1(q)$ contains cycles of length $2k$, where $3 \leq k \leq q^2 - q + 1$. We also know that for $n \geq 1$ and $p \geq 3$, $W_n(q)$ contains cycles of length $2k$, where

$k = p^2 - p$. For $1 \le n \le p-2$, $W_n(q)$ contains at least p^{n-1} cycles of length $2p^2$, and for $n = p-1 \ge 2$, $W_n(q)$ contains at least p^{n-2} cycles of length $2p^3$. Joining some cycles in $W_n(q)$ by special paths, we could also obtain many other cycle lengths not mentioned in this paper.

A result of Sudakov and Verstraëte [20, Theorem 2.2], implies that the set of consecutive even cycle lengths in $W_n(q)$ is of size $\Omega(q^3)$, provided the order of the graph is sufficiently large. Hence, the graph must contain cycles of length $\Omega(q^3)$. We have mentioned that $W_1(q)$ is an induced subgraph of the Levi graph of $PG(2,q)$. In [13], Lazebnik, Mellinger and Vega showed that Levi graph of any finite projective plane of order q (which is sparse), contains cycles of length $2k$ for all k, $3 \le k \le q^2+q+1$. We believe that a similar statement holds for Wenger graphs $W_n(q)$ for all $q \ge 3$, though they are much sparser for $n \ge 2$.

Based on all of the above, and on results obtained by computer, we make the following conjecture.

Conjecture *For every $n \ge 1$, and every prime power q, $q \ge 3$, $W_n(q)$ contains cycles of length $2k$, where $4 \le k \le q^{n+1}$ and $k \ne 5$.*

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