BLOWING UP AT ZERO POINTS OF POTENTIAL FOR AN INITIAL BOUNDARY VALUE PROBLEM

JONG-SHENQ GUO AND MASAHIKO SHIMOJO

ABSTRACT. We study nonnegative radially symmetric solutions for a semilinear heat equation in a ball with spatially dependent coefficient which vanishes at the origin. Our aim is to construct a solution that blows up at the origin where there is no reaction. For this, we first prove that the blow-up is complete, if the origin is not a blow-up point and if there is no blow-up point on the boundary. Then we prove that a threshold solution exists such that it blows up in finite time incompletely and there is no blow-up point on the boundary. On the other hand, we prove that any zero of nonnegative potential is not a blow-up point for a more general problem under the assumption that the solution is monotone in time.

1. INTRODUCTION

In this paper, we study the blow-up phenomena for the following initial boundary value problem:

$$\begin{cases} u_t = \Delta u + |x|^\sigma u^p, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u = u_0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $p > 1$, $\sigma > 0$, Ω is a bounded smooth domain in $\mathbb{R}^N$ with N a positive integer and u_0 is a nonnegative bounded smooth function in $\bar{\Omega}$ with $u_0 = 0$ on $\partial\Omega$.

It is known that for each initial datum u_0 as above, (1.1) has a nonnegative classical solution u for $t \in [0, T)$ for some maximal existence time $T \in (0, \infty]$. If $T < \infty$, then we have

$$\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$$

and we say that the solution u *blows up* in finite time with the *blow-up time* T. For a given solution u that blows up at $t = T < \infty$, a point $a \in \bar{\Omega}$ is called a blow-up point if there exists a sequence $\{(x_n, t_n)\}$ in $Q_T := \Omega \times (0, T)$ such that $x_n \to a$, $t_n \uparrow T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$. The set of all blow-up points is called the *blow-up set*.

The phenomena of blow-up have attracted a lot of attention for past years. Most literature are concerned with equations without spatially dependent coefficient. The main concerns are about criteria of blow-up, locations of blow-up points, blow-up rate and continuation after blow-up. For example, for the spatially homogeneous equation, we refer the reader to [2, 3, 5, 7, 8, 9, 11, 12, 13, 17, 20, 21, 24, 26, 32] and so on. The authors of [15, 29, 33] considered the Cauchy problem for spatially inhomogeneous equation in (1.1). They obtained the existence and nonexistence of global nonnegative solutions.

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Corresponding author: J.-S. Guo, Tel.: +886 2 2930 9036; fax: +886 2 2933 2342.

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Note that there is no reaction for the problem (1.1) at $x = 0$. It is interesting to see that whether $x = 0$ is a blow-up point when u blows up. Since there is no reaction for the equation (1.1) at $x = 0$, it seems that $x = 0$ cannot be a blow-up point. In fact, it is known that, for nonnegative radially symmetric solutions of (1.1) with $\Omega = B_R = \{x \in \mathbb{R}^N; r := |x| < R\}$, $x = 0$ is not a blow-up point under certain conditions (see [14]). A more general theorem in this direction shall be given in section 4. In particular, we shall prove that $x = 0$ is not a blow-up point of (1.1) under the assumption $u_t > 0$.

On the other hand, for the Cauchy problem for the equation in (1.1), it is shown in [6] that there are self-similar solutions with the origin as a blow-up point. This surprising result contradicts our intuition, although the domain under consideration is the whole space. An interesting question arises immediately, namely, what happen if the domain is bounded. Our first aim of this paper is to construct a radially symmetric blow-up solution of (1.1) with $x = 0$ as a blow-up point. More precisely, when $N = 3$, without the assumption $u_t > 0$, we shall construct a radially symmetric solution of (1.1) that blows up at the origin, if $p > p_s := (N + 2 + 2\sigma)/(N - 2) = 5 + 2\sigma$ (when $N = 3$). Note that the range of p is super-critical in the Sobolev sense.

Let us now give a brief description of the main idea of this construction which is originally used for the homogeneous equation: $u_t = \Delta u + u^p$. Our solution that blows up at the origin is characterized as the limit of an increasing sequence of global classical solutions $0 < u_1 < u_2 < u_3 < \cdots$ such that each u_k belongs to the domain of attraction of the stable stationary solution $u = 0$ and such that $u^* = \lim_{k\to\infty} u_k$ lies on the boundary of this domain of attraction. The monotonicity of this sequence and Kaplan type argument about the problem (1.1) yield the uniform boundedness of u^* on certain integrals and this limit function u^* is indeed a time-global weak solution. Furthermore, this solution is proved to be unbounded in L^∞-sense on the time interval $[0, \infty)$. See, e.g., [27]. Hence either u^* blows up in finite time (cf. [9, 25, 21]), or u^* exists globally in time and tends to infinity as $t \to \infty$. Under certain restriction we prove that u^* blows up incompletely in finite time by using the method of [9]. On the other hand, we prove that the solution cannot be extended beyond the blow-up time as a weak solution, if $x = 0$ is not a blow-up point. Therefore, we conclude that u^* blows up at the origin. See [23, 21] for the spatially homogeneous equation.

This paper is organized as follows. In section 2, we shall prove that the blow-up is complete, if $x = 0$ is not a blow-up point and if there is no blow-up point on ∂B_R for a nonnegative radially symmetric solution for $p > 1 + 2\sigma/N$. The construction of a solution u^* that blows up incompletely at the origin for $N = 3$ and $p > 5 + 2\sigma$ is carried out in section 3. Finally, in section 4 we shall prove that any zero of nonnegative potential is not a blow-up point for a more general problem (4.1) than (1.1) under the assumption $u_t > 0$.

2. CRITERION OF COMPLETENESS

In this section, we study the continuation beyond the blow-up time. There are at least three different ways to consider continuation of the solution after blow-up time. The first way of continuation is called the proper extension (cf. [2]), the second one is a minimal L^1-continuation introduced in [21] and the third way is the L^1-weak solution introduced in [27] for example.

In the following, we set $f(x, u) := |x|^\sigma u^p$. Note that $f_u(x, u) \geq 0$ and $f(x, 0) = 0$. First, we define the minimal L^1-continuation as follows.

Definition 2.1. *The function $\tilde{u}$ is called the minimal L^1-solution of* (1.1) *with initial datum u_0 in the maximal existence time interval $[0, T^*)$, if there exists a sequence $\{\tilde{u}_{0,n}\}_{n\in\mathbb{N}} \subset C(\bar{\Omega})$ with*

$$0 \le \tilde{u}_{0,1} \le \tilde{u}_{0,2} \le \tilde{u}_{0,3} \le \cdots \to u_0 \quad \text{in } C(\bar{\Omega})$$

and $u_{0,n} \not\equiv u_0$ for all n such that classical solution $\tilde{u}_n$ of (1.1) *with initial datum $u_{0,n}$ exists for $0 \le t < T^*$ $(\forall n)$ and satisfies*

$$\lim_{n\to\infty} \|\tilde{u}_n(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^1(\Omega)} = 0, \quad \forall t \in [0, T^*), \tag{2.1}$$

$$\lim_{n\to\infty} \|f(x,\tilde{u}_n) - f(x,\tilde{u})\|_{L^1(\Omega\times(0,t))} = 0, \quad \forall t \in [0, T^*). \tag{2.2}$$

Let u be a classical solution (1.1) with initial value u_0 which blows up at time T and let $\tilde{u}$ be the minimal L^1-solution with initial datum u_0 in $[0, T^*)$ for some $T^* \ge T$. The well-posedness of the problem (1.1) implies $\tilde{u}_n(\cdot,t) \to u(\cdot,t)$ for all $0 \le t < T$. We call $\tilde{u}$ as the *minimal L^1-continuation* of u. We say that the blow-up is *complete* if $T = T^*$; and is *incomplete* if $T < T^*$. If $T^* = \infty$, we call $\tilde{u}$ as an *L^1-global minimal continuation.*

Remark 2.1. The above definition of the completeness is the same as the standard one using proper extension as in [2, 17] for spatially homogeneous equation. See [21] for this fact.

Remark 2.2. For the equation $u_t = \Delta u + u^p$, It is known ([2, 9]) that if $1 < p < p_s := (N+2)/(N-2)$, then the blow-up is complete for any initial datum. However, in the supercritical case $p > p_s$, there is a solution whose blow-up is incomplete (cf. [9, 21]).

The following complete blow-up result for radially symmetric solutions was proved in [21, 31] for the equation $u_t = \Delta u + u^p$.

Theorem 1. *Let $p > 1 + 2\sigma/N$ and u be a nonnegative radially symmetric solution of* (1.1) *with $\Omega = B_R$. Assume $x = 0$ is not blow-up point and there is no blow-up point on ∂B_R. Then the blow-up of the solution u is complete.*

To prove Theorem 1, we first introduce the following energy functional. Suppose $u(t) \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and define

$$J[u](t) := \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\,dx - \frac{1}{p+1}\int_\Omega |x|^\sigma |u(x,t)|^{p+1}\,dx. \tag{2.3}$$

If u is a classical solution, a simple computation yields

$$\frac{d}{dt}J[u](t) = -\int_\Omega |u_t(x,t)|^2\,dx. \tag{2.4}$$

Therefore, the functional $J[u](t)$ is monotone nonincreasing in t.

By using this energy functional, we can derive some a priori estimates for the minimal L^1-continuation. These estimates were first discovered by [2] for equations without spatially dependent coefficient. In particular, the following proposition yields the completeness of blow-up, if one can show that $J[u](t) \to -\infty$ as $t \to T$.

Proposition 1. *Let $p > 1+2\sigma/N$. Suppose that the solution u of the problem* (1.1) *blows up at time $T \in (0,\infty)$. Let $\tilde{u}$ (defined in $[0,T^*)$ with $T^* \ge T$) be the minimal L^1-continuation of the solution u. Then we have $\tilde{u}_t \in L^2_{loc}((0,T^*); L^2(\Omega))$.*

Proof. We shall first prove the desired estimates for classical solutions $\tilde{u}_n$ defined on $[0,T^*)$. For notational simplicity, we shall suppress the tilde and index of $\tilde{u}_n$.

Given $\tau \in (0,T/2)$ and $\varepsilon \in (0,T^*-\tau)$. By a simple calculation, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx = -2J[u](t) + \frac{p-1}{p+1}\int_\Omega |x|^\sigma |u|^{p+1}\,dx. \tag{2.5}$$

By the Hölder inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx \geq -2J[u](t) + \frac{p-1}{p+1}\left(\int_\Omega |x|^{-\frac{2\sigma}{p-1}}\,dx\right)^{-\frac{p-1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{p+1}{2}}. \tag{2.6}$$

We set

$$g(t) := \int_\Omega u^2\,dx \quad \text{and} \quad a = a(N,p,\sigma) := \frac{p-1}{p+1}\left(\int_\Omega |x|^{-\frac{2\sigma}{p-1}}\,dx\right)^{-\frac{p-1}{2}} > 0.$$

Then (2.6) is equivalent to

$$\frac{1}{2}g'(t) \geq -2J[u](t) + ag(t)^{\frac{p+1}{2}}.$$

Note that $p > 1 + 2\sigma/N$ ensures the constant $a = a(N,p,\sigma)$ is finite.

First, we suppose that $J[u](t_0) \leq 0$ for some $t_0 \in [0,T^*)$. Note that this inequality also holds for all $t \in (t_0,T^*)$, because of the monotonicity of $J[u](t)$ in time. It follows that

$$g'(t) \geq 2ag(t)^{(p+1)/2} \quad \text{for all} \quad t \in [t_0,T^*).$$

Thus by an integration we deduce that

$$\|u(t)\|_{L^2(\Omega)} \leq f_{T^*}(t) := \{a(p-1)(T^*-t)\}^{-1/(p-1)} \tag{2.7}$$

for all $t \in [t_0,T^*)$.

Next, we assume that $J[u](t) \geq 0$ for all $t \in (0,T^*)$. For $0 < t_1 < t < T^*$, by using (2.4), we have

$$\begin{aligned}
\int_\Omega [u(x,t)-u(x,t_1)]^2dx &= \int_\Omega \Big[\int_{t_1}^t u_s(x,s)ds\Big]^2 dx \\
&\leq (t-t_1)\int_\Omega\int_{t_1}^t [u_s(x,s)]^2 ds dx \\
&= (t-t_1)\int_{t_1}^t \Big\{\int_\Omega [u_s(x,s)]^2dx\Big\}ds \\
&= (t-t_1)\int_{t_1}^t \Big\{-\frac{d}{ds}J[u](s)\Big\}ds \\
&\leq (t-t_1)J[u](t_1).
\end{aligned}$$

Hence we obtain that

$$\|u(t)\|_{L^2(\Omega)} \leq \|u(t_1)\|_{L^2(\Omega)} + (t-t_1)^{1/2}\{J[u](t_1)\}^{1/2}. \tag{2.8}$$

Thus, combining (2.8) with (2.7), we have proved

$$\|u(t)\|_{L^2(\Omega)} \leq A(t) := \max\{f_{T^*}(t),\psi(t)\} \qquad \text{for all} \quad t \in (0,T^*), \tag{2.9}$$

where

$$\psi(t) := \inf_{0\leq t_1\leq t}\{\|u(t_1)\|_{L^2(\Omega)} + (t-t_1)^{1/2}\{J[u](t_1)\}^{1/2}\}.$$

Given $0 < t_1 < t_0 < T^*$. By integrating (2.5) from t_1 to t_0, we deduce

$$\frac{1}{2}\|u(t_0)\|^2_{L^2(\Omega)} + 2\int_{t_1}^{t_0} J[u](s)\,ds \geq \frac{p-1}{p+1}\int_{t_1}^{t_0}\int_\Omega |x|^\sigma u^{p+1}\,dxds.$$

This inequality, (2.9) and the monotonicity of $J[u](t)$ in t yield

$$\frac{p-1}{p+1}\int_{t_1}^{t_0}\int_\Omega |x|^\sigma u^{p+1}\,dxds \leq \left\{\frac{1}{2}A^2(t_0) + 2(t_0-t_1)J[u](t_1)\right\}. \tag{2.10}$$

Multiplying (2.4) by $(t_0 - t)$, integrating by parts and using (2.3), we obtain

$$\begin{aligned}\int_{t_1}^{t_0}(t_0-t)\|u_t\|^2_{L^2(\Omega)}\,dt &= (t_0-t_1)J[u](t_1) - \int_{t_1}^{t_0} J[u](t)\,dt \\ &\leq (t_0-t_1)J[u](t_1) + \frac{1}{p+1}\int_{t_1}^{t_0}\int_\Omega |x|^\sigma u^{p+1}\,dxdt.\end{aligned}$$

It follows from (2.10) that

$$\int_{t_1}^{t_0}(t_0-t)\|u_t\|^2_{L^2(\Omega)}\,dt \leq \frac{p+1}{p-1}(t_0-t_1)J[u](t_1) + \frac{1}{2(p-1)}A^2(t_0). \tag{2.11}$$

Now we apply (2.11) with $t_1 = \tau$ and $t_0 = T^* - \varepsilon$ to obtain

$$\int_\tau^{T^*-\varepsilon}\|u_t\|^2_{L^2(\Omega)}\,dt \leq C_2$$

for some positive constant $C_2 = C_2(\|u(\tau)\|_{L^2(\Omega)}, J[u](\tau), \varepsilon)$.

Finally, by a limiting argument, we obtain the same results for the minimal L^1-continuation $\tilde{u}$ of u. ☐

Remark 2.3. For the equation $u_t = \Delta u + u^p$, it is known that every blow-up solution satisfies $J[u](t) \to -\infty$ as $t \to T$, if $1 < p < p_s$. See [2] for the detail. This divergence property of energy functional, however, is not true in general, when $p > p_s$ as shown in [20, 21].

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By assumption, the origin is not a blow-up point of u and there is $a \in B_R$ such that a is a blow-up point of u. Let $r_1 := |a|$. Then $r_1 \in (0, R)$. We claim that

$$\lim_{t\to T}\{(T-t)^\beta(r_1+\xi\sqrt{T-t})^{\beta\sigma}u(r_1+\xi\sqrt{T-t},t)\} = \kappa \tag{2.12}$$

uniformly on $|\xi| < C$ for any $C > 0$, where $\beta := 1/(p-1)$ and $\kappa := \beta^\beta$.

To prove this, we first consider a transformation $v(r,t) := r^{\beta\sigma}u(r,t)$. Then v satisfies the following one-dimensional problem:

$$\begin{cases} v_t = v_{rr} + v^p + \left(\frac{k-1}{r}v_r - \frac{b}{r^2}v\right), & r \in (0,R),\ t > 0, \\ v(R,t) = v(0,t) = 0,\ t \geq 0, \\ v(r,0) = v_0(r) := r^{\beta\sigma}u_0(r), & r \in (0,R), \end{cases}$$

where $k := N - 2\beta\sigma$ and $b := \beta\sigma(N-2-\beta\sigma)$. Since $x = 0$ is not a blow-up point of u, it is clear that v blows up at the same time T as u does and r_1 is a blow-up point of v. Moreover, $r = 0$ is not a blow-up point of v. Hence there exist $r_0 \in (0, R/3)$ and $\varepsilon \in (0, r_0)$ such that $B_{r_0+2\varepsilon}$ does not contain any blow-up points of v.

For any $\eta \in [r_0, R)$, we define $w := w_\eta$ by

$$w(\xi, s) = (T-t)^\beta v(r,t), \quad \xi = \frac{r-\eta}{\sqrt{T-t}}, \quad s = -\ln(T-t).$$

Then w satisfies

$$w_s = w_{\xi\xi} - \frac{\xi}{2} w_\xi - \beta w + w^p + \Big(\frac{k-1}{\xi + \eta e^{s/2}} w_\xi - \frac{bw}{(\xi+\eta e^{s/2})^2}\Big)$$

for $\xi \in (-\eta e^{s/2}, (R-\eta)e^{s/2})$ and $s > s_0 := -\ln T$.

Next, we introduce the following energy functional:

$$E[w](s) = \int_{\xi_1(s)}^{\xi_2(s)} \Big(\frac{1}{2} w_\xi^2 + \frac{\beta w^2}{2} - \frac{w^{p+1}}{p+1}\Big)(\xi, s)\rho(\xi)\, d\xi,$$

where

$$\rho(\xi) := e^{-\xi^2/4},\ \xi_1(s) := (r_* - \eta)e^{s/2},\ \xi_2(s) := (R-\eta)e^{s/2},\ r_* := r_0 - \varepsilon/2.$$

By a similar argument to that of [22], we conclude that the ω-limit set of w_η with $\eta = r_1$ is included in the set of nonnegative bounded solutions of the problem

$$U_{\xi\xi} - \frac{\xi}{2} U_\xi - \beta U + U^p = 0, \quad \xi \ge 0, \quad U_\xi(0) = 0.$$

It is also known from Theorem 1 of [11] that the only nonnegative bounded solution of this one dimensional elliptic problem is either κ or 0. Thus the ω-limit set of w_{r_1} is contained in the set $\{\kappa, 0\}$. Furthermore, the limit 0 is excluded in the ω-limit set by using a nondegeneracy result of [13]. This proves (2.12). For more details, we refer to [22] for $b = 0$ or [14, Proposition 4.1] for general b.

Now, for any $\varepsilon > 0$ small, by (2.12), there exists $t_0 \in (0, T)$ such that

$$(1-\varepsilon)\kappa(T-t)^{-\beta} \le |x|^{\beta\sigma} u(x,t) \le (1+\varepsilon)\kappa(T-t)^{-\beta} \quad \text{on} \quad S_c \times [t_0, T),$$

where

$$S_c := \{x \in \mathbb{R}^N; r_1 - c(T-t)^{1/2} \le |x| \le r_1 + c(T-t)^{1/2}\}$$

with the constant $c > 0$ such that $c(T-t_0)^{1/2} \le r_1/2$. This gives us

$$\int_{B_R} |x|^\sigma u^p\, dx \ge \int_{S_c} |x|^\sigma u^p\, dx \ge \Big(\frac{r_1}{2}\Big)^\sigma (1-\varepsilon)^p \kappa^p (T-t)^{-\frac{p}{p-1}} m(S_c) \ge C'(T-t)^{-\frac{p+1}{2(p-1)}}, \tag{2.13}$$

for all $t \in [t_0, T)$, where $m(S_c)$ is the measure of the annulus S_c and C' is a constant depending on κ, c, ε and r_1. By a simple calculation, we get

$$\int_{B_R} u_t\, dx = \int_{B_R} |x|^\sigma u^p\, dx + \int_{B_R} \Delta u\, dx = \int_{B_R} |x|^\sigma u^p\, dx - \int_{\partial B_R} |u_r|\, dS. \tag{2.14}$$

Since u does not blow up near the boundary, from (2.13) and (2.14), we may assume without loss of generality that

$$\int_{B_R} u_t\, dx \ge \frac{1}{2} \int_{B_R} |x|^\sigma u^p\, dx, \quad t \in [t_0, T). \tag{2.15}$$

On the other hand, it follows from the Hölder inequality that

$$\int_{B_R} u_t^2\, dx \ge C_R \Big(\int_{B_R} u_t\, dx\Big)^2$$

for some positive constant C_R. Combining this with (2.13) and (2.15), we obtain that

$$\int_{B_R} u_t^2\,dx \geq \frac{1}{4}C_R\Big(\int_{B_R}|x|^\sigma u^p\,dx\Big)^2 \geq \frac{1}{4}C_R C'^2(1-\varepsilon)^{2p}(T-t)^{-\frac{p+1}{p-1}}, \quad t\in[t_0,T).$$

Integrating the above inequality from t_0 to T, we get

$$\int_{t_0}^T\int_{B_R} u_t^2\,dxdt = \infty.$$

Therefore, we conclude from Proposition 1 that the blow-up of u is complete. Hence the theorem is proved. □

3. Construction of desired blow-up solutions

In this section, we denote the solution of (1.1) with initial value u_0 by $u(x,t;u_0)$ or simply by $u(t;u_0)$. We only consider the case when $\Omega = B_R$. We define the following two spaces:

$$\begin{aligned} X &:= \{v\in L^\infty(\Omega)\cap C(\bar\Omega) \,:\, v\geq 0,\ v=0 \text{ on } \partial\Omega \text{ and } v \text{ is radially symmetric}\},\\ A &:= \{u_0\in X \,:\, u(t;u_0) \text{ is global and } \lim_{t\to\infty}\|u(t;u_0)\|_{L^\infty(\Omega)} = 0\}. \end{aligned}$$

Our first aim of this section is to construct the threshold solution as in the following proposition.

Proposition 2. *Let $N=3$, $p>5+2\sigma$ and $\Omega=B_R$. Let u_μ be the radially symmetric solution of* (1.1) *with initial value μg, where $\mu>0$ and $g\in X\setminus\{0\}$ such that $rg(r)$ is decreasing in a neighborhood of $r=R$. Then there exists μ^* such that the solution u^* of* (1.1) *with the initial value $u_0=\mu^* g$ exists globally as the minimal L^1-solution but it is unbounded in L^∞-norm.*

The proof of this proposition is similar to that of [27] except for Lemma 3.3, but we give the details of the argument for the reader's convenience.

The first observation is the following lemma by the so-called Kaplan's argument [16]. Let ϕ_1 be the first eigenfunction with the first eigenvalue $\lambda_1>0$ for the Laplace operator in Ω with zero Dirichlet boundary condition such that $\|\phi_1\|_{L^1(\Omega)}=1$.

Lemma 3.1. *Let $p>1+\sigma/N$. If the solution u of* (1.1) *with initial datum u_0 exists globally in time, then*

$$\int_\Omega u\phi_1\,dx \leq C(\Omega,\sigma,p,N), \quad t\in(0,\infty),$$

where $C=C(\Omega,\sigma,p,N)$ is a constant independent of u_0.

Proof. Multiplying (1.1) by ϕ_1 and integrating, and using $p > 1 + \sigma/N$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\phi_1\,dx &= \int_\Omega \Delta u\phi_1\,dx + \int_\Omega |x|^\sigma u^p\phi_1\,dx \\ &= -\lambda_1\int_\Omega u\phi_1 dx + \int_\Omega |x|^\sigma u^p\phi_1\,dx \\ &\geq -\lambda_1\int_\Omega u\phi_1\,dx + \left(\int_\Omega u\phi_1\,dx\right)^p \left(\int_\Omega |x|^{-\sigma/(p-1)}\phi_1\,dx\right)^{-(p-1)} \\ &\geq -\lambda_1\int_\Omega u\phi_1\,dx + C'(\Omega,\sigma,p,N)\left(\int_\Omega u\phi_1\,dx\right)^p\end{aligned}$$

By the standard Kaplan type argument, we conclude that the right-hand side of the above inequality cannot be positive for all $t \in (0,\infty)$. The lemma is proved. ☐

Lemma 3.2. *The set A is nonempty and relatively open in X.*

Proof. We shall divide the proof into two steps.

Step 1. We claim that there exists $\varepsilon_0 > 0$ such that if $\|u(\tau;u_0)\|_{L^\infty(\Omega)} \leq \varepsilon_0$ for some $\tau > 0$, then $\|u(t;u_0)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. This also implies that A is nonempty.

We first prove that 0 is an isolated stationary solution of (1.1). For this, we let v be any solution of

$$-\Delta v = |x|^\sigma v^p,\quad x\in\Omega,\quad v>0,\quad x\in\Omega,\quad v=0,\quad x\in\partial\Omega \tag{3.1}$$

such that $\|v\|_{L^\infty(\Omega)} < \varepsilon_1 := (\lambda_1 R^{-\sigma})^{1/(p-1)}$. Multiplying (3.1) by ϕ_1 and integrating it over Ω, we obtain

$$0 = \int_\Omega \phi_1(\Delta v + |x|^\sigma v^p)\,dx = \int_\Omega \phi_1 v(|x|^\sigma v^{p-1} - \lambda_1)\,dx < 0,$$

a contradiction. Thus 0 is an isolated stationary solution of (1.1). Note that, by the Pohozaev argument [28], the problem (3.1) has no positive solution if $p > 5 + 2\sigma$ and $N = 3$.

Next, for $\Omega = B_R$, we let $D = B_{2R}$ and let (λ_1^D, ϕ_1^D) be the first eigen pair of $-\Delta$ in D with zero Dirichlet boundary condition. We also assume that ϕ_1^D is nonnegative and $\|\phi_1^D\|_{L^\infty(D)} = 1$. Choose $\varepsilon_2 \in (0,\varepsilon_1)$ sufficiently small such that $\varepsilon_2\phi_1^D \leq (\lambda_1^D R^{-\sigma})^{\frac{1}{p-1}}$ on D. It is easy to see that $v_2 = \varepsilon_2\phi_1^D$ is a supersolution of (1.1). We define

$$\varepsilon_0 = \min_{x\in\bar\Omega} v_2(x).$$

Note that $\varepsilon_0 > 0$. Suppose now $\|u(\tau;u_0)\|_{L^\infty(\Omega)} \leq \varepsilon_0$ for some $\tau > 0$. By the comparison principle, we obtain

$$\|u(t;u_0)\|_{L^\infty(\Omega)} \leq \|v_2\|_{L^\infty(\Omega)} \leq \varepsilon_2 < \varepsilon_1$$

for all $t > \tau$. On the other hand, by the standard theory of dynamical system with Lyapunov functional, the ω-limit set of u_0 is included in the set of stationary solutions. Therefore, $\|u(t;u_0)\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$.

Step 2. Show that A is relatively open in X.

Let $u_0 \in A$. Then there exists $\tau_0 > 0$ such that

$$\|u(\tau;u_0)\|_{L^\infty(\Omega)} \leq \varepsilon_0/2 \quad \forall\,\tau > \tau_0, \tag{3.2}$$

where $\varepsilon_0 > 0$ is the constant given in Step 1. We claim that there exists a positive constant δ depending on $\|u(\cdot,\cdot;u_0)\|_{L^\infty(\Omega\times(0,\tau_0))}$ such that $\lim_{t\to\infty}\|\hat{u}(\cdot,t)\|_{L^\infty(\Omega)} = 0$, if $\|u_0-\hat{u}_0\|_{L^\infty(\Omega)} \le \delta$, where $\hat{u}(x,t)$ is the solution of (1.1) with initial value $\hat{u}_0$.

In order to prove this, we only need to show that

$$\|u-\hat{u}\|_{L^\infty(\Omega\times(0,\tau))} \le \varepsilon_0/2 \tag{3.3}$$

for some $\tau > \tau_0$, if $\|u_0-\hat{u}_0\|_{L^\infty(\Omega)} \le \delta$. Because, if we know this, from (3.2), we get

$$\|\hat{u}(\cdot,\tau)\|_{L^\infty(\Omega)} \le \varepsilon_0.$$

Combining this with Step 1, we conclude $\lim_{t\to\infty}\|\hat{u}(\cdot,t)\|_{L^\infty(\Omega)} = 0$.

To obtain (3.3), we consider the equation for $z := u-\hat{u}$:

$$\begin{cases} z_t = \Delta z + b(x,t,z), & x\in\Omega,\ t>0, \\ z(x,0) = u_0(x)-\hat{u}_0(x), & x\in\Omega, \\ z(x,t)=0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{3.4}$$

where $b(x,t,z) := |x|^\sigma\{u(x,t)^p - [u(x,t)-z]^p\}$. We define

$$M = \|u(\cdot,\cdot;u_0)\|_{L^\infty(\Omega\times[0,2\tau_0])}.$$

Then

$$b(x,t,z) \le pR^\sigma|z|\max\left\{M^{p-1},(M-z)^{p-1}\right\} := \gamma_M(z)$$

for all $x\in\Omega$, $t\in[0,2\tau_0]$.

Let h be the solution of

$$\frac{dh}{dt} = \gamma_M(h), \quad h(0)=\delta. \tag{3.5}$$

Then h is a super-solution of (3.4) when $\|u_0-\hat{u}_0\|_{L^\infty(\Omega)} \le \delta$, since

$$\begin{aligned} &h_t = \Delta h + \gamma_M(h) \ge \Delta h + b(x,t,h), \quad x\in\Omega, t>0, \\ &h(0) \ge u_0-\hat{u}_0 \quad \text{in} \quad \Omega, \quad h>0. \end{aligned}$$

Note that $\gamma_M(h)$ is Lipschitz continuous in h, the initial value problem (3.5) for $h(t)$ is well-posed. Taking $\delta>0$ sufficiently small, we have

$$\max_{t\in[0,2\tau_0]}|h(t)| < \varepsilon_0/2.$$

By the comparison principle, we obtain

$$\max_{t\in[0,\tau]}\|u(t;u_0)-u(t;\hat{u}_0)\|_{L^\infty(\Omega)} \le \max_{t\in[0,\tau]}|h(t)| < \varepsilon_0/2$$

Thus (3.3) is established and the lemma is proved. □

We next give two estimates for the case $N=3$.

Lemma 3.3. *Let $N=3$, $p\ge\sigma+1$ and $\Omega=B_R$. Let u be a nonnegative radially symmetric global solution of (1.1) with $u_0\ge 0$ and $u_0\not\equiv 0$. Then for each $\tau>0$ there exists a constant $C=C(\Omega,\sigma,p,N,\tau,u_0)>0$ such that*

$$\int_{B_R} u\,dx \le C, \quad t\ge\tau, \tag{3.6}$$

$$\int_0^t\int_{B_R}|x|^\sigma u^p\,dxds \le C(1+\lambda_1 t), \quad t\ge\tau. \tag{3.7}$$

Proof. We consider the transformation $z(r,t) := |x|u(x,t) = ru(r,t)$, $r = |x|$. Then z satisfies

$$\begin{cases} z_t = z_{rr} + r^{\sigma+1-p} z^p, & r \in (0,R),\ t > 0, \\ z(0,t) = z(R,t) = 0, & t > 0, \\ z(r,0) = ru_0(r), & r \in (0,R). \end{cases} \tag{3.8}$$

Note that $z(r,t) > 0$ for all $r \in (0,R)$, $t > 0$. It follows from the Hopf lemma that $z_r(R,t) < 0$ for all $t > 0$. Given a fixed $\tau \geq 0$ (as long as $z_r(R,\tau) < 0$). Then by a standard reflection argument, using the fact that $r^{\sigma+1-p}$ is monotone decreasing in r (due to the assumption $p \geq \sigma + 1$), we can find a small positive constant ε, depending on τ, such that z is monotone decreasing in r on $[R-2\varepsilon, R] \times [\tau, \infty)$. Indeed, the constant ε can be chosen in such a way as long as $\varepsilon \in (0, R/2)$ and $z_r(r,\tau) < 0$ for $r \in [R-2\varepsilon, R]$.

Now, given a fixed $t \geq \tau$. Then we have

$$\begin{aligned} \int_{B_R} u(x,t)dx &= 4\pi \int_0^R rz(r,t)dr \\ &= 4\pi \int_0^{R-2\varepsilon} rz(r,t)dr + 4\pi \int_{R-2\varepsilon}^{R-\varepsilon} rz(r,t)dr + 4\pi \int_{R-\varepsilon}^{R} rz(r,t)dr. \end{aligned}$$

Setting $s = 2(R-\varepsilon)-r$ and using $[2(R-\varepsilon)-s]/s \leq C_\varepsilon := R/(R-2\varepsilon)$ for all $s \in [R-2\varepsilon, R-\varepsilon]$, from the monotonicity of z in r we have

$$\int_{R-\varepsilon}^{R} rz(r,t)dr \leq C_\varepsilon \int_{R-2\varepsilon}^{R-\varepsilon} rz(r,t)dr.$$

Therefore, for any $t \geq \tau$ we deduce that

$$\begin{aligned} \int_{B_R} u(x,t)dx &\leq (1+C_\varepsilon)4\pi \int_0^{R-\varepsilon} rz(r,t)dr \\ &= (1+C_\varepsilon) \int_{B_{R-\varepsilon}} u(x,t)dx \\ &\leq \frac{1+C_\varepsilon}{a_0} \int_{B_{R-\varepsilon}} u(x,t)\phi_1(x,t)dx, \end{aligned}$$

where $a_0 := \min_{B_{R-\varepsilon}} \phi_1 > 0$. Consequently, (3.6) follows from Lemma 3.1.

Next, we prove the second estimate (3.7). Recall

$$\frac{d}{dt}\int_{B_R} u\phi_1\, dx = -\lambda_1 \int_{B_R} u\phi_1\, dx + \int_{B_R} |x|^\sigma u^p \phi_1\, dx.$$

By an integration from 0 to t, we have

$$\int_{B_R} u\phi_1\, dx - \int_{B_R} u_0\phi_1\, dx = -\lambda_1 \int_0^t \int_{B_R} u\phi_1\, dxds + \int_0^t \int_{B_R} |x|^\sigma u^p \phi_1\, dxds$$

Combining this with Lemma 3.1, we obtain

$$\int_{B_R} u\phi_1\, dx - \int_{B_R} u_0\phi_1\, dx \geq -C_1\lambda_1 t + \int_0^t \int_{B_R} |x|^\sigma u^p \phi_1\, dxds.$$

Since u_0 and ϕ_1 are positive, this immediately yields

$$\int_0^t \int_{B_R} |x|^\sigma u^p \phi_1 \, dxds \le C_1 \lambda_1 t + \int_{B_R} u\phi_1 \, dx \le C_1(1+\lambda_1 t) \quad \forall\, t > 0.$$

Finally, we note that $|x|^\sigma u^p(x,t) = r^{\sigma-p} z^p(r,t)$ is monotone decreasing on $Q_\varepsilon := [R-2\varepsilon, R] \times [\tau, \infty)$, since $p \ge \sigma + 1$ and $z(r,t)$ is monotone decreasing on Q_ε. The rest of the argument is completely the same as that of the proof of (3.6). We omit it. Thus the lemma follows. □

Proof of Proposition 2. Fix $g \in X \setminus \{0\}$ and define $\mu^* = \sup\{\mu > 0; \mu g \in A\}$. We claim that $\mu^* < \infty$. Indeed, this follows from Lemma 3.1 and the observation $\int_\Omega (\mu g)\phi_1 \, dx = \mu \int_\Omega g\phi_1 dx \to \infty$ as $\mu \to \infty$.

Let u_μ be the solution of (1.1) with the initial datum μg for $0 < \mu < \mu^*$. From the definitions of A and μ^*, u_μ exists globally in time. The comparison principle implies that u_μ is monotone increasing in μ. Hence we are able to define

$$u^*(x,t) = \lim_{\mu \nearrow \mu^*} u_\mu(x,t), \quad x \in \Omega, t \in [0,\infty)$$

by allowing the value $+\infty$. We shall show that u^* is a minimal L^1-global solution. That is u^* satisfies the conditions (2.1) and (2.2) with $T^* = \infty$.

Since $z_r(r,0) = (rg)'(r) < 0$ in a fixed neighborhood of $r = R$, so we can choose $\tau = 0$ in the proof of of Lemma 3.3. Hence the estimate (3.6) in Lemma 3.3 holds for a constant C independent of initial data u_0. In particular, the estimate (3.6) holds with a constant C independent of $\mu \in (0, \mu^*)$ for any u_μ with $\mu \in (0,\mu^*)$ and any $t > 0$. By Fatou's lemma,

$$\int_\Omega u^*(\cdot,t)\, dx = \int_\Omega \lim_{\mu\nearrow\mu^*} u_\mu(\cdot,t)\, dx \le \liminf_{\mu\nearrow\mu^*} \int_\Omega u_\mu(\cdot,t)\, dx \le C \quad \forall\, t > 0$$

Hence u_{μ^*} exists globally in time as an L^1-solution.

Using the monotonicity of the sequence u_μ in μ, it follows from the monotone convergence theorem and Lemma 3.3 that

$$\begin{aligned} \int_0^t \int_\Omega u^*(x,s)\, dxds &= \int_0^t \int_\Omega \lim_{\mu\nearrow\mu^*} u_\mu(x,s)\, dxds \\ &= \lim_{\mu\nearrow\mu^*} \int_0^t \int_\Omega u_\mu(x,s)\, dxds \le Ct. \end{aligned}$$

Thus we obtain $\lim_{\mu\nearrow\mu^*} \|(u_\mu - u^*)\|_{L^1(\Omega\times(0,t))} = 0$ for all $t > 0$. Similarly, using (3.7) and monotone convergence theorem, we deduce that

$$\begin{aligned} \int_0^t \int_\Omega |x|^\sigma (u^*(x,s))^p(x,s)\, dxds &= \int_0^t \int_\Omega |x|^\sigma \Big(\lim_{\mu\nearrow\mu^*} u_\mu(x,s)\Big)^p dxds \\ &= \int_0^t \int_\Omega \lim_{\mu\nearrow\mu^*} |x|^\sigma u_\mu^p(x,s)\, dxds \\ &= \lim_{\mu\nearrow\mu^*} \int_0^t \int_\Omega |x|^\sigma u_\mu^p(x,s)\, dxds \\ &\le C(1+\lambda_1 t). \end{aligned}$$

Therefore, $|x|^\sigma (u^*)^p \in L^1(\Omega \times (0,t))$ for all $t > 0$. We can also check that

$$\lim_{\mu \nearrow \mu^*} \| |x|^\sigma (u_\mu^p - (u^*)^p) \|_{L^1(\Omega \times (0,t))} = 0 \quad \forall\, t > 0.$$

This is the condition (2.2).

In order to prove that u^* meets the condition (2.1), we consider the following auxiliary problem

$$\begin{cases} u_t = \Delta u + |x|^\sigma (u^*)^{p-1} u^*, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \mu^* g & x \in \Omega. \end{cases} \tag{3.9}$$

Recall that $|x|^\sigma (u^*)^{p-1} u^* \in L^1(\Omega \times (0,t))$ for all $t \in (0,\infty)$, provided that $N = 3$ and $p \geq \sigma + 1$.

By the general theory of L^1-semigroup of [1], the problem (3.9) admits an L^1-solution $v \in C([0,t]; L^1(\Omega))$ provided that $|x|^\sigma (u^*)^p \in L^1(\Omega \times (0,t))$. Furthermore, it is an L^1-contracting mapping. In the sequel, $\{e^{t\Delta}\}_{t \geq 0}$ denotes the semigroup generated by the heat operator with zero Dirichlet boundary condition. Then we have

$$\begin{aligned} v(t) &= e^{t\Delta} \mu^* g + \int_0^t e^{(t-s)\Delta} |x|^\sigma |u^*(s)|^{p-1} u^*(s)\, ds, \\ u_\mu(t) &= e^{t\Delta} \mu g + \int_0^t e^{(t-s)\Delta} |x|^\sigma |u_\mu(s)|^{p-1} u_\mu(s)\, ds. \end{aligned}$$

Thus, for all $\mu \in (0,\mu^*)$, we have

$$\begin{aligned} \|v(t) - u_\mu(t)\|_{L^1(\Omega)} &= \left\| e^{t\Delta}(\mu^* - \mu) g + \int_0^t e^{(t-s)\Delta} |x|^\sigma \big((u^*)^p(s) - u_\mu^p(s) \big)\, ds \right\|_{L^1(\Omega)} \\ &\leq (\mu^* - \mu) \|e^{t\Delta} g\|_{L^1(\Omega)} + \int_0^t \| e^{(t-s)\Delta} |x|^\sigma \big((u^*)^p - u_\mu^p \big) \|_{L^1(\Omega)}\, ds \\ &\leq (\mu^* - \mu) \|g\|_{L^1(\Omega)} + \int_0^t \| |x|^\sigma ((u^*)^p - u_\mu^p) \|_{L^1(\Omega)}\, ds. \end{aligned}$$

By letting $\mu \to \mu^*$, the right-hand side of the above inequality converges to 0. Hence $u^* = v$. This implies that u^* satisfies (2.1). Since v is L^1-global in time, we conclude that u^* is a minimal L^1-global solution.

Finally, we shall show that u^* is an L^∞ unbounded solution. Assume on the contrary that u^* is uniformly bounded. Then from the standard dynamical system argument with Lyapunov functional, the omega limit set $\omega(\mu^* g)$ is included in the set of nonnegative stationary solutions. Recall from [28] that (1.1) has no positive stationary solution, since $p > p_s = 5 + 2\sigma$. Thus $\omega(\mu^* g) = \{0\}$. This implies that $u^* \to 0$ as $t \to \infty$. On the other hand, by Lemma 3.2, the set A is an open subset of the set X. Then we can find $\mu > \mu^*$ such that $\mu g \in A$, a contradiction. Therefore, L^∞-norm of u^* diverges in finite or infinite time. This completes the proof of Proposition 2. □

Proposition 2 does not give us any information whether the solution exists globally in time or blows up in finite time. Next theorem is the answer to this question. The related result for the equation $u_t = \Delta u + u^p$ can be found in [9, 25] (see also [21] for more general results).

Theorem 2. *Let $\Omega = B_R$, $N = 3$ and $p > 5 + 2\sigma$. Let u^* be the function in Proposition 2. Then u^* blows up in finite time such that the origin is a blow-up point.*

Before starting the proof, let us recall the well-known zero number properties of parabolic equations [4]. We define

$$Z(f) = Z_{[0,R]}(f) := \sharp\{r \in [0,R], f(r) = 0\}$$

as the zero number of $f \in C([0,R])$ on $[0,R]$.

Lemma 3.4. *Let $V(r,t)$ be non-zero smooth radially symmetric solution of*

$$\begin{aligned} v_t = \Delta v + a(|x|,t)v, &\ \ x \in B_R,\ t_1 < t < t_2, \\ v \neq 0, &\ \ x \in \partial B_R,\ t_1 < t < t_2, \end{aligned}$$

where $a(|x|,t)$ is continuous on $B_R \times (t_1,t_2)$. Then

(i) *$Z(V(\cdot,t))$ is finite on (t_1,t_2).*
(ii) *$t \mapsto Z(V(\cdot,t))$ is monotone nonincreasing.*
(iii) *If $V_r(r^*,t^*) = V(r^*,t^*) = 0$ for some (r^*,t^*), then*

$$Z(V(\cdot,t)) > Z(V(\cdot,s)), \quad t_1 < t < t^* < s < t_2.$$

In the following, we shall denote the singular steady solution of

$$U_t = U_{rr} + \frac{N-1}{r}U_r + r^\sigma U^p, \quad r > 0, \tag{3.10}$$

by $\Phi^*(r) = c^* r^{-\beta}$, where $\beta := (\sigma+2)/(p-1)$ and $c^* := \beta(N-2-\beta)$. We can easily check that this is well-defined when $p > (N+\sigma)/(N-2)$.

Now we start to prove our main theorem.

Proof of Theorem 2. We divide our proof into two steps.

Step 1. We shall use the zero number argument as [9] to show that u^* blows up in finite time. For a contradiction, we suppose that the $U^*(r,t) = u^*(x,t)$ does not blow up in finite time. Hence U^* is smooth for all $t > 0$.

We first claim that

$$Z(U^*(\cdot,t) - \Phi^*(\cdot)) \geq 2 \tag{3.11}$$

for any $t > 0$. Otherwise, there exists $t_0 > 0$ such that $Z(U^*(\cdot,t_0) - \Phi^*(\cdot))$ is either 0 or 1. This means that $U^*(\cdot,t_0)$ and $\Phi^*(\cdot)$ have no intersection or one degenerate intersection. The former case implies that $Z(U^*(\cdot,\tilde{t}) - \Phi^*(\cdot)) = 0$ for all $\tilde{t} > t_0$, by using (ii) of Lemma 3.4. The latter case is also reduced to the former case by using (iii) of Lemma 3.4. By comparing $\Phi^*(|\cdot| + |x_0|)$ and $U^*(\cdot,t)$ for sufficiently small $|x_0|$, we have $U^*(|x|,t) \leq \Phi^*(|x| + |x_0|)$ for any $t \in (t_0,\infty)$, $x \in B_R$. Hence $U^*(r,t)$ is uniformly bounded for $r \in [0,R]$ and $t \geq t_0$. On the other hand, $p > p_s$ imply that 0 is the only nonnegative stationary solution. Therefore, $\lim_{t\to\infty} u^*(x,t) = 0$ for any $x \in B_R$. This contradicts Proposition 2. Hence (3.11) is established.

Next, it is known that, when $N = 3$ and $p > p_s := 5 + 2\sigma$, the equation (3.10) has the following backward self-similar solution U_m:

$$U_m(r,t) = (T-t)^{-\beta/2}\varphi_m(r/\sqrt{T-t}), \quad t < T, \tag{3.12}$$

for any $T > 0$, where φ_m is positive in $[0,\infty)$ such that

$$\varphi_m'' + \left(\frac{N-1}{y} - \frac{y}{2}\right)\varphi_m' + |y|^\sigma \varphi_m^p - \frac{\beta}{2}\varphi_m = 0,\ y > 0,$$
$$\varphi_m'(0) = 0, \quad \lim_{y\to\infty}[y^\beta \varphi_m(y)] = m$$

for a certain $m \in (0, c^*)$. See [6] for $\sigma \geq 0$ and [18, 19] for $\sigma = 0$.

We now consider the intersection between U^* and U_m. Fix any $t_1 \in (0,1)$. Since $U_r^*(R,t_1) < 0$, there is a positive constant $\varepsilon \in (0, R/2)$ such that $U_r^*(r,t_1) < 0$ for all $r \in [R-\varepsilon, R]$. Set

$$\gamma_1 := \min_{r\in[0,R-\varepsilon]} U^*(r,t_1)/2, \quad \gamma_2 := \max_{r\in[R-\varepsilon,R]} U_r^*(r,t_1).$$

Then $\gamma_1 > 0$ and $\gamma_2 < 0$. By (3.12), we can choose $T > 0$ sufficiently large so that $U_m(r,t_1) < \gamma_1$ for all $r > 0$ and $(U_m)_r(r,t_1) > \gamma_2$ for all $r \in [R-\varepsilon, R]$. On the other hand, from the boundary condition of (1.1), we have $U^*(R,t) = 0$, $U_m(R,t) > 0$ for any $t > 0$. This implies that $Z(U^*(\cdot,t_1) - U_m(\cdot,t_1)) = 1$. Combining this with Lemma 3.4, we have

$$Z(U^*(\cdot,t) - U_m(\cdot,t)) \leq 1, \qquad t \geq t_1. \tag{3.13}$$

Note that $U_m(r,T) = mr^{-\beta}$ for all $r > 0$. Recall $\Phi^*(r) = c^* r^{-\beta}$ and (3.11). Let r_1 and r_2 be the smallest and largest intersection points of U^* and Φ^* in $(0,R)$. By choosing $t_2 < T$ sufficiently close to T, it follows that there must be at least one intersection of U^* and U_m in $(0,r_1)$ and (r_2,R), respectively. This implies that $Z(U^*(\cdot,t_2) - U_m(\cdot,t_2)) \geq 2$, contradicting (3.13) and (ii) of Lemma 3.4. We conclude that u^* blows up in finite time, say at T.

Step 2. We prove that $x = 0$ is a blow-up point of u^*. Assume on the contrary that $x = 0$ is not a blow-up point. Also, for contradiction we assume that $r = R$ is a blow-up point of u^*. Recall (3.8) with $z(r,t) := ru^*(r,t)$ and it follows from the proof of Lemma 3.3 that $z_r < 0$ in $[r_0,R] \times [0,T_0)$ for some $r_0 \in (0,R)$, where $[0,T_0)$ is the maximal existence time interval of z. Note that $T_0 = T$. Then we have $z(r,t) \to \infty$ as $t \to T_0$ uniformly on $[c,d]$, where we take $c = r_0$ and $d = (r_0+R)/2$.

Following [7], we consider the function

$$J(r,t) := z_r(r,t) + \epsilon h(r) z^\gamma(r,t), \quad h(r) := \sin\frac{\pi(r-c)}{d-c},\ \gamma := \frac{1+p}{2},$$

where ϵ is a positive constant to be determined. By a simple computation, it is easy to see that J satisfies

$$J_t - J_{rr} - aJ \leq -\epsilon b h z^\gamma,$$

where

$$a := pr^{\sigma+1-p} z^{p-1} - 2\gamma\epsilon h' z^{\gamma-1},$$
$$b := (p-\gamma) r^{\sigma+1-p} z^{p-1} - 2\gamma\epsilon h' z^{\gamma-1} - [\pi/(d-c)]^2.$$

Since $1 < \gamma < p$, $p \geq \sigma + 1$, $R < \infty$ and $z(r,t) \to \infty$ as $t \to T_0$ uniformly on $[c,d]$, we can find $t_0 \in (0,T_0)$ such that $b > 0$ in $[c,d] \times [t_0,T_0)$. For this t_0, we can choose $\epsilon > 0$ small enough such that $J < 0$ on $[c,d] \times \{t_0\}$. Since $J < 0$ on $r = c, d$, it follows from the maximum principle that $J < 0$ in $[c,d] \times [t_0,T_0)$. This gives the inequality

$$\frac{z_r}{z^\gamma} < -\epsilon h \quad \text{on } [c,d] \times [t_0,T_0).$$

Integrating the above inequality from c to d for any $t \in [t_0, T_0)$ and letting $t \uparrow T_0$, we reach a contradiction. Therefore, $r = R$ is not a blow-up point of u^*.

Now, we can apply Theorem 1 to conclude that u^* blows up completely in finite time. But, this contradicts Proposition 2. Therefore, $x = 0$ is a blow-up point of u^* and we finish the proof of Theorem 2. □

4. Non blow-up at zero point of potential

Consider the following more general problem than (1.1):

$$\begin{cases} u_t = \Delta u + q(x)u^p & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \bar{\Omega}, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.1}$$

where Ω is bounded smooth domain, q is Hölder continuous in $\bar{\Omega}$, $q(x) \geq 0$, $q(x) \not\equiv 0$, $p > 1$ and $u_0 \geq 0$, $u_0 \not\equiv 0$ is a smooth function with $u_0|_{\partial\Omega} = 0$. We assume all zeros of $q(x)$ are included in Ω.

Assume that

$$u_t(x,0) \geq 0 \quad \text{for all} \quad x \in \Omega. \tag{4.2}$$

Note that the condition (4.2) is valid if we assume that

$$\Delta u_0 + q(x)u_0^p \geq 0 \quad \text{ in } \bar{\Omega}.$$

We shall prove that u blows up in finite time and satisfies

$$\|u(x,t)\|_{L^\infty(\Omega)} \leq K(T-t)^{-\frac{1}{p-1}} \tag{4.3}$$

for some $K = K(p, q, \Omega, T) > 0$. More precisely, we prove the following theorem.

Theorem 3. *Assume* (4.2) *holds. Then* $T < \infty$ *and* u *satisfies* (4.3) *for some* $K = K(p, q, \Omega, T) > 0$. *Moreover, any zero point of* $q(x)$ *is not a blow-up point.*

Proof. Define

$$J := u_t - \varepsilon u^p.$$

By a simple calculation, we have

$$J_t - \Delta J = q(x)f'(u)J + \varepsilon f''(u)|\nabla u|^2 \geq q(x)f'(u)J$$

with $f(u) = u^p$. Since $v := u_t$ is a nontrivial solution of

$$\begin{cases} v_t = \Delta v + q(x)f'(u)v, & x \in \Omega,\ t > 0, \\ v(x,0) = u_t(x,0), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

from the maximum principle and the Hopf lemma, $u_t > 0$ in $\Omega \times (0,T)$ and $\frac{\partial}{\partial\nu}u_t < 0$ on $\partial\Omega \times (0,T)$, where ν is the outward unit normal vector on the boundary. Set $t_0 = T/2$. Then we can choose $\varepsilon > 0$ small enough such that $u_t(x,t_0) \geq \varepsilon u^p(x,t_0)$ for all $x \in \Omega$. Thus we can easily check that $J \geq 0$ on the parabolic boundary of $\Omega \times (t_0,T)$ if $\varepsilon > 0$ is sufficiently small. It follows from the maximum principle that $J \geq 0$ in $\Omega \times (t_0,T)$. Consequently, we have

$$u_t - \varepsilon u^p \geq 0 \quad \text{in} \quad \Omega \times (t_0,T).$$

By integrating this inequality for t, we conclude

$$u^{1-p}(x,t) \geq (p-1)\varepsilon(T-t), \quad t \in (t_0, T).$$

This means that $T < \infty$ and (4.3) follows.

Next we show that any zero of $q(x)$ is not a blow-up point. The proof is by using a comparison argument. Let us define

$$w(x,t) = \frac{A}{[v(x) + (T-t)]^{\frac{1}{p-1}}},$$

where the constant $A > K$ and $v(x)$ will be determined later.

Let x_0 be any zero point of $q(x)$. We may assume that $\{x : |x - x_0| \leq 2r_0\} \subset \Omega$ for some $r_0 > 0$. Then we define

$$v(x) = \delta \cos^2 \Big(\frac{\pi |x - x_0|}{2r_0}\Big), \; B_0 := \{x : |x - x_0| \leq r_0\},$$

where δ is a positive constant. Note that $w(x,t) \geq u(x,t)$ for $x \in \partial B_0$ and $t \in (0,T)$, by (4.3) and using $A > K$. Also,

$$w(x,0) = \frac{A}{[v(x) + T]^{\frac{1}{p-1}}} \geq u_0(x), \quad |x - x_0| \leq r_0$$

if we take A sufficiently large.

The inequality

$$w_t - \Delta w - q(x) w^p \geq 0$$

is equivalent to

$$1 - (p-1)A^{p-1}q(x) + \Delta v(x) - \frac{p}{p-1}\frac{|\nabla v|^2}{v(x) + (T-t)} \geq 0$$

for all $(x,t) \in B_0 \times (0,T)$. We have this inequality if

$$1 - (p-1)A^{p-1}q(x) + \Delta v(x) - \frac{p}{p-1}\frac{|\nabla v|^2}{v(x)} \geq 0 \tag{4.4}$$

for all $x \in B_0$. It is easy to see that $\Delta v/\delta$ and $[|\nabla v|^2/v]/\delta$ are bounded (independent of δ) in B_0 for any positive constant δ. Furthermore, by fixing A and taking r_0 sufficiently small, we have $(p-1)A^{p-1}q(x) < 1/3$ for all $x \in B_0$. For these fixed A and r_0, we can take $\delta > 0$ sufficiently small so that the last two terms in the inequality (4.4) are bounded by $1/3$ in B_0. Hence (4.4) holds and, by the comparison principle, we conclude that

$$w(x,t) = \frac{A}{[v(x) + (T-t)]^{\frac{1}{p-1}}} \geq u(x,t), \qquad |x - x_0| \leq r_0, \; t \in (0,T).$$

In particular, $x = x_0$ is not a blow-up point of u. □

Remark 4.1. *The result of monotonicity in time implies the finite time blow-up for the homogeneous equation can be found in Theorem 23.5 of* [30]*. For non blow-up at any zero of potential $q(x)$, different from the argument of* [10]*, we construct supersolution that does not blow-up at any zero point of $q(x)$. This proof is much simpler than that in the work* [10] *mentioned above.*

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Department of Mathematics, National Taiwan Normal University, 88, S-4, Ting Chou Road, Taipei 11677, Taiwan
E-mail address: jsguo@ntnu.edu.tw

National Center for Theoretical Sciences, Taipei Office, 1, S-4, Roosevelt Road, Taipei 10617, Taiwan
E-mail address: shimojotw@gmail.com