CRITERA FOR DIRECT BLIND DECONVOLUTION
OF MIMO FIR SYSTEMS
DRIVEN BY WHITE SOURCE SIGNALS

Yuijiro Inouye1 and Ruey-wen Liu2

1Department of Electronic and Control Systems Engineering
Shimane University, 1060 Nishikawatsu, Matsue, Shimane 690-8504, Japan
inouye@riko.shimane-u.ac.jp
2Department of Electrical Engineering
University of Notre Dame, Notre Dame, IN 46556, USA
Liu.1@nd.edu

ABSTRACT
This paper addresses the blind deconvolution of multi-input–
multi-output (MIMO) FIR systems driven by white non-Gaussian
source signals. First, we present a weaker condition on source
signals than the so-called i.i.d. condition so that blind
deconvolution is possible. Then, under this condition, we
provide a necessary and sufficient condition for blind
deconvolution of MIMO FIR systems. Finally, based on this
result, we propose two maximization criteria for blind
deconvolution of MIMO FIR systems. These criteria are simple
enough to be implemented by adaptive algorithms.

1. INTRODUCTION
Consider a set of received signals that are linear (convolutive)
mixtures of a set of source signals. The objective of blind
deconvolution is to recover the source signals from the set of
received signals without the knowledge of the linear mixtures or
the LTI systems. The case when the mixture is instantaneous has
been well studied. For example, see [1] and the references
therein. Blind deconvolution has received increasing attention in
the past few years [2]–[7]. One way for achieving blind
deconvolution is first to blindly identify the channel system from
the channel outputs [8], and then to design an equalizer
accordingly. The other way of achieving blind deconvolution is
to directly design an equalizer from the equalizer outputs. For
example, see [2]–[7] and the references therein. This direct
approach is preferable, because it bypasses the process of blind
system identification and the order estimation of the channel
system that is usually needed for blind system identification.
Moreover, the computation becomes simpler, because the
dimension of the channel output is usually larger than that of the
equalizer output.

Most of the approaches assume that the sequence of source
signals is temporally i.i.d. For example see [2]–[5] and the
references therein. However, the condition of i.i.d. for source
signals is too restrictive for some applications. For example, in
digital communications, the information bearing sequences are
coded and hence are unlikely i.i.d. On the other hand, these
coded sequences are usually interleaved to encounter burst errors
and are usually considered to be uncorrelated. Therefore, it is
vitally important to weaken the temporary i.i.d. condition to the
temporary white condition.

In this paper, we first present a weaker condition on the source
signals than the i.i.d. condition so that blind deconvolution is
possible. Then under this weaker condition we provide a
necessary and sufficient condition for blind deconvolution of
MIMO FIR systems. Finally, based on this result, we propose
two maximization criteria for blind deconvolution of MIMO FIR
systems. These criteria are simple enough to be implemented by
adaptive algorithms.

This paper uses the following notation. Matrices are denoted by
boldface uppercase letters. Column vectors are denoted by
boldface lowercase letters. All others are scalars. The
superscript $T$ denotes the transpose of a matrix. The superscript
$*$ denotes the complex conjugate of a scalar or a matrix. The
$(i,j)$th component of a matrix $A$ is denoted by $a_{ij}$. Let $I$
denote an identity of an appropriate size. Let $\text{cum}(x_1,\cdots,x_n)$
denote the $n$th-order (or joint) cumulant of random variables
$x_1,\cdots,x_n$, which is defined as a coefficient of the Taylor
expansion of the natural logarithm of the joint characteristic
function of $x_1,\cdots,x_n$ [11].

2. PROBLEM FORMULATION
We consider an MIMO FIR system shown in Figure 1,

![Figure 1. A model for blind deconvolution.](image-url)
where,
\[ y(t) = \sum_{k=0}^{K-1} H(k) x(t-k) + n(t), \]  
(1)
\[ z(t) = \sum_{k=0}^{K-1} W(k) y(t-k), \]  
(2)
Note that no condition is imposed on \( H(0) \) nor \( H(K-1) \).
The equalizer output \( z(t) \) and the input \( x(t) \) are related by
\[ z(t) = G(z)x(t) + W(z)n(t), \]  
(3)
where \( G(z) := W(z)H(z) \).
The objective of blind deconvolution or source separation is to design an equalizer \( W(z) \) that recovers the original source signals only from the observations of the system outputs \( y(t) \)’s.

**Definition 1:** A scalar function \( g(z) \) of a complex variable \( z \) is said to be monomial if it can be represented as \( g(z) = cz^n \).

An LTI system with \( n \) inputs and \( n \) outputs is said to be transparent (or decoupled) if its transfer function matrix \( G(z) \) has a single nonzero monomial entry in each row and each column.

Note that an LTI system is transparent if and only if \( G(z) \) has a decomposition of the form
\[ G(z) = A(z)DP, \]  
(4)
where \( A(z) \) is a diagonal matrix with diagonal entries \( A_{ii}(z) = z^{l_i} \) (where \( l_i \) is a non-negative integer), \( D \) a regular constant diagonal matrix, and \( P \) a permutation matrix. Then, the blind deconvolution problem is formulated as follows: Design an equalizer \( W(z) \), so that the condition
\[ W(z)H(z) = A(z)DP, \]  
(5)
is satisfied, i.e., \( G(z) \) is transparent.

A channel system \( H(z) \) is said to be deconvolvable if there exists an equalizer \( W(z) \) so that the composite system \( G(z) \) is transparent. A necessary and sufficient condition for the existence of such equalizer \( W(z) \) is given by Massey and Sain [10].

**Theorem 1 (Massey-Sain Theorem):** Let \( H(z) \) be an \( m \times n \) matrix transfer function of an FIR channel system. Then a necessary and sufficient condition for \( H(z) \) to be deconvolvable is that the greatest common divisor (GCD) of all the minors of order \( n \) in \( H(z) \) is nonzero monomial, that is,
\[ \text{GCD} |d_j(z)| = z^{l_j} \]  
(6)
for some integer \( l_j \geq 0 \), where \( d_j(z) \)'s denote all the minors of order \( n \) in \( H(z) \).

To specify the source signals, we introduce some preliminary definitions and notions for stationary non-Gaussian vector-valued random processes as shown in [11].

**Definition 2:** Let \( \{s(t)\} \) be a complex-valued stationary random vector process with components \( \{s_i(t)\}, i = 1, \cdots, n \). Then the family of second-order cumulant sequences of \( \{s(t)\} \) is defined by \( c_{s_i,s_j}(\tau) := \text{cum} \{s_i(t),s_j(t+\tau)\} \) for \( i, j = 1, \cdots, n \) and \( \tau \in \mathbb{Z} \). In particular, the sequence \( \{c_{s_i,s_j}(\tau)\} \) is also called the cross-correlation of \( \{s_i(t)\} \) and \( \{s_j(t)\} \) for \( i \neq j \) and the auto-correlation of \( \{s_i(t)\} \) for \( i = j \), and \( \sigma^2_i(0) \) (denoted by \( \sigma^2_i \)) is called the variance of \( \{s_i(t)\} \).
The family of fourth-order cumulant sequences is defined by \( c_{s_i,s_j,s_k,s_l}(t_1,t_2,t_3,t_4) := \text{cum} \{s_i(t_1),s_j(t_1+t_2),s_k(t_1+t_3),s_l(t_1+t_4)\} \) for \( t_1, t_2, t_3, t_4 \in \mathbb{Z} \). In particular, the sequence \( \{c_{s_i,s_j,s_k,s_l}(t_1,t_2,t_3)\} \) is called the fourth-order auto- or cross-correlation of \( \{s_i(t)\}, \{s_j(t)\}, \{s_k(t)\}, \{s_l(t)\} \) depending on whether all the indices \( i, j, k, l \) are the same or not. Furthermore, \( c_{s_i,s_j,s_k,s_l}(0,0,0,0) \) is called the fourth-order cumulant or kurtosis of \( s_i(t) \) and denoted by \( \kappa^4_i \).

For notational simplicity, we denote the second- and fourth-order cumulants, \( c_{s_i,s_j}(t_1,t_2,t_3,t_4) \) and \( c_{s_i,s_j,s_k,s_l}(t_1,t_2,t_3,t_4) \), by \( c_{ij}(t) \) and \( c_{ijkl}(t_1,t_2,t_3,t_4) \), respectively, if it is clear from the context.

**Definition 3:** A process \( \{s(t)\} \) is said to be temporally uncorrelated if all the auto-correlations \( c_{s_i,s_j}(\tau), i = 1, \cdots, n \) are zero except at the origin \( \tau = 0 \) and it is said to be spatially uncorrelated if all the cross-correlations \( c_{s_i,s_j}(\tau), i \neq j \) are zero.

It is said to be second-order white if it is both temporally and spatially uncorrelated. Furthermore, it is temporally fourth-order uncorrelated if all the auto-correlations \( c_{s_i,s_j,s_k,s_l}(t_1,t_2,t_3,t_4) \), \( i = 1, \cdots, n \) are zero except at the origin \( t_1 = t_2 = t_3 = 0 \). It is spatially fourth-order uncorrelated if all the fourth-order cross-correlations \( c_{s_i,s_j,s_k,s_l}(t_1,t_2,t_3,t_4) \) (where \( t_1, t_2, t_3, t_4 \) are not all the same.) are zero. It is said to be fourth-order white, if it is temporally and spatially fourth-order uncorrelated.

Henceforth, we assume throughout the paper that

1) The matrices \( H(z) \) and \( W(z) \) are transfer functions representing FIR systems.

2) The vector sequence \( \{s(t)\} \) is a zero-mean random process satisfies the cumulant summability conditions of orders 2 and 4. In addition, assume that the kurtoses \( \kappa^4_i, i = 1, \cdots, n \), of all the components of \( s(t) \) are nonzero, which implies that it is a non-Gaussian process.

3) For the purpose of analysis, the noise is assumed to be zero, i.e., \( n(t) = 0 \), although the criteria presented in Section 4 can be applied on noisy cases.

We also consider the following two conditions:

(A1) The sequence \( \{s(t)\} \) is second-order white, and spatially fourth-order uncorrelated. In addition, assume that the kurtoses \( \kappa^4_i, i = 1, \cdots, n \), of all the components of \( s(t) \) are nonzero.
(A2) The sequence \( s(t) \) is second-order and fourth-order white.

3. NECESSARY AND SUFFICIENT CONDITIONS

Consider a composite system described by (1)–(3) with (A1).

We present a necessary and sufficient condition for blind deconvolution.

Theorem 2: Let \( H(z) \) be deconvolvable, and let \( \{s(t)\} \) satisfy (A1). Suppose an equalizer \( W(z) \) is used to make a composite system \( G(z) \). Then the composite system \( G(z) \) is transparent if and only if the output sequence \( \{z(t)\} \) is a second-order white and spatially fourth-order uncorrelated random process with nonzero variances \( \sigma_i^2 \neq 0, i = 1, \cdots, n \).

See [12] for the proof.

Remark 2.1: The condition on the signal source \( \{s(t)\} \) can hardly be further weakened. It is known that the second-order statistics alone is not sufficient for blind deconvolution. We have to use high order statistics. In (A1), only the fourth-order spatial statistics are added.

4. CRITERIA FOR MULTICHANNEL BLIND DECONVOLUTION

In this section, based on Theorem 2, we present optimization criteria for blind deconvolution of MIMO FIR systems. The assumption made on the source sequence \( \{s(t)\} \) is specified by (A1) or (A2) in the previous sections.

We shall present two criterion functions for multichannel blind deconvolution below according to (A1) and (A2), respectively. Under (A1), we consider the following maximization criterion (A):

Maximize \( J_A \) subject to the constraints \( c_{i,j}(\tau) = \delta(i-j)\delta(\tau) \) for all \( \tau \in Z \) and all \( i, j = 1, \cdots, n \), where \( J_A \) is defined by

\[
J_A := \sum_{i=1}^{n} \sum_{t_1,t_2 \in Z} |c_{i,i}(t_1, t_2, t_3)|^2
\]

Under (A2) we consider the following maximization criterion (B):

Maximize \( J_B \) subject to the constraints \( c_{i,j}(\tau) = \delta(i-j)\delta(\tau) \) for all \( \tau \in Z \) and all \( i, j = 1, \cdots, n \), where \( J_B \) is defined by

\[
J_B := \sum_{i=1}^{n} |c_{i,i}|^2
\]

We note that the constraints of the criterion (A) are the same as those of the criterion (B), and both of them require the equalizer output process \( \{z(t)\} \) to be normalized-white in the second-order sense. This is equivalent to the condition

\[
E(z(t + k)z^{\tau T}(t)) = I \delta(k) .
\]

In order to show the validity of the criteria (A) and (B), we require the following lemma. To this end, we present first the definition of paraunitary systems.

Definition 4: Let \( H(z) \) be an \( n \times n \) transfer function matrix representing an FIR system. Then it is said to be paraunitary if \( H(e^{-i\omega}) \) is unitary, that is, \( H(e^{-i\omega})H(e^{-i\omega})^{T} = I \) for any real \( \omega \). Also, the system itself is called paraunitary for simplicity.

Lemma 1: Let \( x(t) \) be a complex vector-valued stationary random process described by

\[
x(t) = H(z)s(t) ,
\]

where \( H(z) \) is an \( n \times n \) transfer function matrix of a paraunitary system and \( \{s(t)\} \) is a complex vector-valued stationary random process with zero mean. Then, it holds true that

\[
K_x = K_s ,
\]

where,

\[
K_x := \sum_{l_1,l_2,l_3=1}^{n} \sum_{t_1,t_2,t_3 \in Z} |c_{l_1,l_2,l_3}(t_1, t_2, t_3)|^2
\]

\[
K_s := \sum_{l_1,l_2,l_3=1}^{n} \sum_{t_1,t_2,t_3 \in Z} |c_{l_1,l_2,l_3}|^2
\]

See [12] for the proof.

It may be interesting to note from the above lemma that although paraunitary is a condition based on second statistics, it imposes a condition on higher-order statistics.

Using Theorem 2 and the above lemma, we can establish the following theorem, which shows the validity of the maximization criteria (A) and (B).

Theorem 3: Let \( H(z) \) be deconvolvable, and let \( \{s(t)\} \) satisfy (A1) or (A2)). Suppose an equalizer \( W(z) \) is used to make a composite system \( G(z) \). Then under (A1) or (A2) on \( \{s(t)\} \), the maximization criterion (A) or the maximization criterion (B) makes the composite system \( G(z) \) transparent.

Remark 3.1: Although criterion \( J_A \) is not found elsewhere, the criterion \( J_B \) has appeared in [3], but under a stronger condition on source signals, i.e., the i.i.d. assumption is presumed.

Remark 3.2: Numerical algorithms based on criterion (B) under the i.i.d. assumption have been proposed in [13]. It can be shown that these algorithms can also be applied to the case when source signals satisfy (A2).

Remark 3.2: The criteria (A) and (B) may be regarded as extensions of the Salvi-Weinstein (SW) criterion [14] for
single channel case to the multichannel case. It is widely known that the SW criterion has close connections with the constant modulus (CM) criterion and the mean square error (MSE) criterion. For a simple case, some results on their connections are reported by Gu and Tong [15].

**Proof of Theorem 3:** Without loss of generality, we may assume that the input process \( \{s(t)\} \) is normalized-white, and because (A1) or (A2), \( \{z(t)\} \) is normalized-white. Hence, \( G(z) \) is paraunitary. Thus applying Lemma 1 to \( G(z) \), we have

\[
K_z = K_s,
\]

On the other hand, \( K_z \) and \( K_s \) can be decomposed as

\[
K_z = K_z^{(a)} + K_z^{(c)},
\]

\[
K_s = K_s^{(a)} + K_s^{(c)},
\]

where \( K_z^{(a)} \) and \( K_s^{(a)} \) are the sums of all the absolute squares of all the fourth-order auto-cumulant sequences of \( \{z(t)\} \) and \( \{s(t)\} \), respectively, and \( K_z^{(c)} \) and \( K_s^{(c)} \) are the sums of all the fourth-order cross-cumulant sequences \( \{z(t)\} \) and \( \{s(t)\} \), respectively. Under (A1) or (A2), it holds true that \( K_z^{(c)} = 0 \), which implies from (12), (13), and (14),

\[
K_z^{(a)} + K_z^{(c)} = K_s^{(a)} (= \text{constant}).
\]

Now, consider (A1). Since \( K_z^{(c)} \) is non-negative and \( K_z^{(a)} = J_A \) from their definitions, the maximization \( J_A \) subject to the constraints in the criterion (A) implies \( K_z^{(a)} = 0 \), which means that the output process \( \{z(t)\} \) is spatially fourth-order uncorrelated. Based on Theorem 2, this concludes that the composite system \( G(z) \) is transparent, because \( \{z(t)\} \) is second-order white. Similarly, we can prove the case under (A2).

5. CONCLUSIONS

We considered the blind deconvolution of MIMO FIR systems driven by white non-Gaussian source signals. First, we found a weaker condition on source signals than the so-called i.i.d. condition so that blind deconvolution is possible. It was found that the condition is that the source signals are second-order white and spatially fourth-order uncorrelated. Then, under this condition, we provided a necessary and sufficient condition for blind deconvolution of MIMO FIR systems. It was shown that blind deconvolution is achieved if and only if the composite output signals are second-order white and spatially fourth-order uncorrelated. Finally, based on this result, we proposed two maximization criteria for blind deconvolution of multiuser-multichannel systems.

These criteria use only the second- and fourth-order statistics of the equalizer outputs. Therefore, we can directly use these criteria to recover the source signals without first using a channel identification process. In particular, the maximization criteria (A) and (B) require no information of, and hence are robust to, the order of the channel systems, but only a bound of the order. Numerical algorithms based on the maximization criterion (A) are being developed and will be reported in future work.

6. REFERENCES